ON THE CLASSIFICATION OF FINITE-DIMENSIONAL POINTED HOPF ALGEBRAS

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We dedicate this paper to our wives Sonia and Heidi for their longtime support

Abstract. We classify finite-dimensional complex Hopf algebras A which are pointed, that is, all of whose irreducible comodules are one-dimensional, and whose group of group-like elements $G(A)$ is abelian such that all prime divisors of the order of $G(A)$ are > 7. Since these Hopf algebras turn out to be deformations of a natural class of generalized small quantum groups, our result can be read as an axiomatic description of generalized small quantum groups.

Introduction

One of the very few general classification results for Hopf algebras is due to Milnor, Moore, Cartier and Kostant around 1963. It says that any cocommutative Hopf algebra over the complex numbers is a semidirect product of the universal enveloping algebra of a Lie algebra and a group algebra.

In the terminology of Sweedler's book [S] from 1969, cocommutative Hopf algebras (over an algebraically closed field) are examples of *pointed Hopf algebras*, that is, all their simple subcoalgebras are one-dimensional, or equivalently, all their simple comodules are one-dimensional. Thus duals of finite-dimensional pointed Hopf algebras are analogs of basic algebras in the theory of finite-dimensional algebras. A rich supply of examples of non-cocommutative pointed Hopf algebra was only found in the mid-eighties of the last century: The Drinfeld-Jimbo quantum groups $U_q(\mathfrak{g})$, $\mathfrak{g}$ a semisimple Lie algebra, and their multiparameter versions, as well as the finite-dimensional small quantum groups introduced by Lusztig a bit later are all pointed.

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In the present paper we assume that the ground-field k is algebraically closed of characteristic zero. If A is a Hopf algebra, we denote the group of group-like elements of A by

$$G(A) = \{g \in A \mid \Delta(g) = g \otimes g, \varepsilon(g) = 1\}.$$

We classify all finite-dimensional pointed Hopf algebras A over k with abelian group $G(A)$ such that the prime divisors of the order of $G(A)$ are > 7. We describe these Hopf algebras by generators and relations, and we show that they are the small quantum groups discovered by Lusztig and variations of them. Thus our results can be viewed as an axiomatic description of generalized small quantum groups.

Other types of Hopf algebras occur if prime divisors ≤ 7 of the order of the abelian group $G(A)$ are allowed. However, extending the methods of this paper it should be possible to describe their structure by some generalization of Cartan matrices. Finite-dimensional pointed Hopf algebras with non-abelian group $G(A)$ seem to be of a very different nature. Their structure is not understood.

We now give a brief overview of the classification program for finite-dimensional Hopf algebras. We remark that these Hopf algebras give rise to finite tensor categories in the sense of [EO] and thus classification results on finite-dimensional Hopf algebras should have applications in conformal field theory [Ga].

The classification splits into several very different parts according to the behaviour of the coradical. Recall that the coradical A_0 of a Hopf algebra A is the sum of all its simple subcoalgebras.

$$\text{Classification of fin.-dim. Hopf algebras } A \begin{cases} \nearrow \ \text{semisimple } (\Leftrightarrow A = A_0) \\ \searrow \ \text{non-semisimple} \begin{cases} \nearrow \ A_0 \text{ Hopf subalg. } \supseteq \text{pointed} \\ \searrow \ \text{other} \end{cases} \end{cases}$$

If $A = A_0$ then A is semisimple as an algebra. Semisimple Hopf algebras define examples of fusion categories. There are various important results on semisimple Hopf algebras, but there is at present no general strategy to classify these algebras.

Next assume that $A_0 \neq A$ and that the coradical A_0 is a Hopf subalgebra. If A is pointed then A_0 is a Hopf subalgebra, namely the group algebra $k[G(A)]$. The only general method for the classification of a class of Hopf algebras is the Lifting Method, developed in [AS1] and which works for Hopf algebras whose coradical is a Hopf subalgebra. Since the coradical is a semisimple Hopf algebra and the classification of semisimple Hopf algebras is still widely open, it is natural to concentrate on pointed Hopf algebras. The starting point of this method is an analog of the Milnor-Moore-Cartier-Kostant decomposition theorem on the level of the associated graded Hopf algebras. The enveloping algebra of a Lie algebra is replaced by a braided Hopf algebra which is generated by primitive elements.

In the case when the coradical A_0 of A is not a Hopf subalgebra very little is known. There are a few results on the classification of arbitrary Hopf algebras of a given dimension such as for dimension p or p^2 with prime p. For their proof only difficult ad hoc methods are used.

To formulate our main result we first describe the data $\mathcal{D}, \lambda, \mu$ we need to define the Hopf algebras of the class we are considering. We fix a finite abelian group Γ.

The datum $\mathcal{D}$. A *datum $\mathcal{D}$ of finite Cartan type* for Γ ,

$$\mathcal{D} = \mathcal{D}(\Gamma, (g_i)_{1\le i\le\theta}, (\chi_i)_{1\le i\le\theta}, (a_{ij})_{1\le i,j\le\theta}),$$

consists of elements $g_i \in \Gamma, \chi_i \in \widehat{\Gamma}, 1 \le i \le \theta$, and a Cartan matrix $(a_{ij})_{1\le i,j\le\theta}$ of finite type satisfying

$$q_{ij}q_{ji} = q_{ii}^{a_{ij}},\ q_{ii} \neq 1,\ \text{with } q_{ij} = \chi_j(g_i) \text{ for all } 1 \le i,j \le \theta. \tag{0.1}$$

The Cartan condition (0.1) implies in particular,

$$q_{ii}^{a_{ij}} = q_{jj}^{a_{ji}} \text{ for all } 1 \le i,j \le \theta. \tag{0.2}$$

The explicit classification of all data of finite Cartan type for a given finite abelian group Γ is a computational problem. But at least it is a finite problem since the size θ of the Cartan matrix is bounded by $2(\mathrm{ord}(\Gamma))^2$ by [AS2, 8.1], if Γ is an abelian group of odd order. For groups of prime order, all possibilities for $\mathcal{D}$ are listed in [AS2].

Let Φ be the root system of the Cartan matrix $(a_{ij})_{1\le i,j\le\theta}$, $\alpha_1, \dots, \alpha_\theta$ a system of simple roots, and $\mathcal{X}$ the set of connected components of the Dynkin diagram of Φ. Let $\Phi_J, J \in \mathcal{X}$, be the root system of the component J. We write $i \sim j$, if α_i and α_j are in the same connected component of the Dynkin diagram of Φ. For a positive root

$\alpha = \sum_{i=1}^{\theta} n_i\alpha_i, n_i \in \mathbb{N} = \{0, 1, 2, \dots\}$, for all i, we define

$$g_\alpha = \prod_{i=1}^{\theta} g_i^{n_i}, \chi_\alpha = \prod_{i=1}^{\theta} \chi_i^{n_i}.$$

We assume that the order of q_{ii} is odd for all i, and that the order of q_{ii} is prime to 3 for all i in a connected component of type G_2. Then it follows from (0.2) that the order N_i of q_{ii} is constant in each connected component J, and we define $N_J = N_i$ for all $i \in J$.

The parameter λ. Let $\lambda = (\lambda_{ij})_{1\le i<j\le\theta, i\not\sim j}$ be a family of elements in k satisfying the following condition for all $1 \le i < j \le \theta$, $i \not\sim j$: If $g_ig_j = 1$ or $\chi_i\chi_j \neq \varepsilon$, then $\lambda_{ij} = 0$.

The parameter μ. Let $\mu = (\mu_\alpha)_{\alpha\in\Phi^+}$ be a family of elements in k such that for all $\alpha \in \Phi_J^+, J \in \mathcal{X}$, if $g_\alpha^{N_J} = 1$ or $\chi_\alpha^{N_J} \neq \varepsilon$, then $\mu_\alpha = 0$.

Thus λ and μ are finite families of free parameters in k. We can normalize λ and assume that $\lambda_{ij} = 1$, if $\lambda_{ij} \neq 0$.

The Hopf algebra $u(\mathcal{D}, \lambda, \mu)$. The definition of $u(\mathcal{D}, \lambda, \mu)$ in Section 4.2 can be summarized as follows. In Definition 2.14 we associate to any μ and $\alpha \in \Phi^+$ an element $u_\alpha(\mu)$ in the group algebra $k[\Gamma]$. By construction, $u_\alpha(\mu)$ lies in the augmentation ideal of $k[g_i^{N_i} \mid 1 \le i \le \theta]$. The braided adjoint action $\mathrm{ad}_c(x_i)$ of x_i is defined in (1.14), and the root vectors x_α are explained in Section 2.1.

The Hopf algebra $u(\mathcal{D}, \lambda, \mu)$ is generated as an algebra by the group Γ, that is, by generators of Γ satisfying the relations of the group, and $x_1, \dots, x_\theta$, with the relations:

(*Action of the group*) $gx_ig^{-1} = \chi_i(g)x_i,$ for all $i,$ and all $g \in \Gamma$,

(*Serre relations*) $\mathrm{ad}_c(x_i)^{1-a_{ij}}(x_j) = 0,$ for all $i \neq j, i \sim j$,

(*Linking relations*) $\mathrm{ad}_c(x_i)(x_j) = \lambda_{ij}(1 - g_ig_j),$ for all $i < j, i \not\sim j$,

(*Root vector relations*) $x_\alpha^{N_J} = u_\alpha(\mu),$ for all $\alpha \in \Phi_J^+, J \in \mathcal{X}$.

The coalgebra structure is given by

$$\Delta(x_i) = g_i \otimes x_i + x_i \otimes 1, \quad \Delta(g) = g \otimes g, \text{ for all } 1 \le i \le \theta, g \in \Gamma.$$

Now we can formulate our main result.

Classification Theorem 0.1. *(1) Let $\mathcal{D}, \lambda$ and μ as above. Assume that q_{ii} has odd order for all i and that the order of q_{ii} is prime to 3 for all i in a connected component of type G_2. Then $u(\mathcal{D}, \lambda, \mu)$ is a* *pointed Hopf algebra of dimension* $\prod_{J\in\mathcal{X}} N_J^{|\Phi^+_J|}|\Gamma|$ *with group-like elements* $G(u(\mathcal{D},\lambda,\mu)) = \Gamma$.

(2) Let A *be a finite-dimensional pointed Hopf algebra with abelian group* $\Gamma = G(A)$. *Assume that all prime divisors of the order of* Γ *are* > 7. *Then* $A \cong u(\mathcal{D},\lambda,\mu)$ *for some* $\mathcal{D},\lambda,\mu$.

Moreover, in Theorem 7.2 we determine all isomorphisms between the Hopf algebras $u(\mathcal{D},\lambda,\mu)$.

Part (1) of Theorem 0.1 is shown in Theorem 4.5, and part (2) is a special case of Theorem 6.2.

In [AS4] we proved the Classification Theorem for groups of the form $(\mathbb{Z}/(p))^s, s \geq 1$, where p is a prime number > 17. In this special case, all the elements μ and $u_\alpha(\mu)$ are zero. In [AS1] we proved part (1) of Theorem 0.1 for Dynkin diagrams whose connected components are of type A_1, and in [AS5] for Dynkin diagrams of type A_n; in [D2] our construction was extended to Dynkin diagrams whose connected components are of type A_n for various n. In [BDR] the Hopf algebra $u(\mathcal{D},\lambda,\mu)$ was introduced for type B_2.

Our proof of Theorem 0.1 is based on [AS1, AS2, AS3, AS4, AS5], and on previous work on quantum groups in [dCK, dCP, L1, L2, L3, M1, Ro], in particular on Lusztig's theory of the small quantum groups. Another essential ingredient of our proof are the recent results of Heckenberger on Nichols algebras of diagonal type in [H1, H2, H3] which use Kharchenko's theory [K] of PBW-bases in braided Hopf algebras of diagonal type.

In [AS2, 1.4] we conjectured that any finite-dimensional pointed Hopf algebra (over an algebraically closed field of characteristic 0) is generated by group-like and skew-primitive elements. Our Classification Theorem and Theorem 6.2 confirm this conjecture for a large class of Hopf algebras.

Finally we note that the following analog of Cauchy's Theorem from group theory holds for the Hopf algebras $A = u(\mathcal{D},\lambda,\mu)$: If p is a prime divisor of the dimension of A, then A contains a group-like element of order p. We conjecture that Cauchy's Theorem holds for all finite-dimensional pointed Hopf algebras.

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1. Braided Hopf algebras

1.1. **Yetter-Drinfeld modules over abelian groups and the tensor algebra.** Let Γ be an abelian group, and $\widehat{\Gamma}$ the character group of all group homomorphisms from Γ to the multiplicative group $k^\times$ of the field k. The braided category ${}^{\Gamma}_{\Gamma}\mathcal{YD}$ of (left) Yetter-Drinfeld modules over Γ is the category of left $k[\Gamma]$-modules which are Γ-graded vector spaces $V = \bigoplus_{g\in\Gamma} V_g$ such that each homogeneous component V_g is stable under the action of Γ. Morphisms are Γ-linear maps $f : \bigoplus_{g\in\Gamma} V_g \to \bigoplus_{g\in\Gamma} W_g$ with $f(V_g) \subset W_g$ for all $g \in \Gamma$. The Γ-grading is equivalent to a left $k[\Gamma]$-comodule structure $\delta : V \to k[\Gamma] \otimes V$, where $\delta(v) = g \otimes v$ is equivalent to $v \in V_g$. We use a Sweedler notation $\delta(v) = v_{(-1)} \otimes v_{(0)}$ for all $v \in V$.

If $V = \bigoplus_{g\in\Gamma} V_g$ and $W = \bigoplus_{g\in\Gamma} W_g$ are in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$, the monoidal structure is given by the usual tensor product $V \otimes W$ with diagonal Γ-action $g(v \otimes w) = gv \otimes gw$, $v \in V, w \in W$, and Γ-grading $(V \otimes W)_g = \bigoplus_{ab=g} V_a \otimes W_b$ for all $g \in \Gamma$. The braiding in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$ is the isomorphism

$$c = c_{V,W} : V \otimes W \to W \otimes V$$

defined by $c(v \otimes w) = g \cdot w \otimes v$ for all $g \in \Gamma, v \in V_g,$ and $w \in W$. Thus each Yetter-Drinfeld module V defines a braided vector space $(V, c_{V,V})$.

If χ is a character of Γ and V a left Γ-module, we define

$$V^{\chi} := \{v \in V \mid g \cdot v = \chi(g)v \text{ for all } g \in \Gamma\}.$$

Let $\theta \geq 1$ be a natural number, $g_1, \dots, g_\theta \in \Gamma$, and $\chi_1, \dots, \chi_\theta \in \widehat{\Gamma}$. Let V be a vector space with basis $x_1, \dots, x_\theta$. V is an object in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$ by defining $x_i \in V_{g_i}^{\chi_i}$ for all i. Thus each x_i has degree g_i, and the group Γ acts on x_i via the character χ_i. We define

$$q_{ij} := \chi_j(g_i) \text{ for all } 1 \leq i, j \leq \theta.$$

The braiding on V is determined by the matrix (q_{ij}) since

$$c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i \text{ for all } 1 \leq i, j \leq \theta.$$

We will identify the tensor algebra $T(V)$ with the free associative algebra $k\langle x_1, \dots, x_\theta\rangle$. It is an algebra in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$, where a monomial

$$x = x_{i_1} x_{i_2} \cdots x_{i_n}, 1 \leq i_1, \dots, i_n \leq \theta,$$

has Γ-degree $g_{i_1} g_{i_2} \cdots g_{i_n}$ and the action of $g \in \Gamma$ on x is given by $g \rightharpoonup x = \chi_{i_1}\chi_{i_2}\cdots\chi_{i_n}(g)x$. $T(V)$ is a braided Hopf algebra in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$ with comultiplication

$$\Delta_{T(V)} : T(V) \to T(V) \underline{\otimes} T(V),\ x_i \mapsto x_i \otimes 1 + 1 \otimes x_i,\ 1 \leq i \leq \theta.$$

Here we write $T(V)\underline{\otimes}T(V)$ to indicate the braided algebra structure on the vector space $T(V) \otimes T(V)$, that is

$$(x \otimes y)(x' \otimes y') = x(g \rightharpoonup x') \otimes yy',$$

for all $x, x', y, y' \in T(V)$ and $y \in T(V)_g, g \in \Gamma$.

Let $I = \{1, 2, \dots, \theta\}$, and $\mathbb{Z}[I]$ the free abelian group of rank θ with basis $\alpha_1, \dots, \alpha_\theta$. Given the matrix (q_{ij}), we define the bilinear map

$$\mathbb{Z}[I] \times \mathbb{Z}[I] \to k^\times,\ (\alpha, \beta) \mapsto q_{\alpha\beta},\ \text{by } q_{\alpha_i\alpha_j} = q_{ij}, 1 \le i, j \le \theta. \tag{1.1}$$

We consider V as a Yetter-Drinfeld module over $\mathbb{Z}[I]$ with $x_i \in V_{\alpha_i}^{\psi_i}$ for all $1 \le i \le \theta$, where ψ_j is the character of $\mathbb{Z}[I]$ with

$$\psi_j(\alpha_i) = q_{ij} \text{ for all } 1 \le i, j \le \theta.$$

Thus $T(V) = k\langle x_1, \dots, x_\theta\rangle$ is also a braided Hopf algebra in ${}^{\mathbb{Z}[I]}_{\mathbb{Z}[I]}\mathcal{YD}$. The $\mathbb{Z}[I]$-degree of a monomial $x = x_{i_1}x_{i_2}\cdots x_{i_n}, 1 \le i_1, \dots, i_n \le \theta$, is $\sum_{i=1}^{\theta} n_i\alpha_i$, where for all i, n_i is the number of occurences of i in the sequence $(i_1, i_2, \dots, i_n)$. It follows for the action of $\mathbb{Z}[I]$ on homogeneous components that

$$\alpha \rightharpoonup x = q_{\alpha\beta}x \text{ for all } \alpha, \beta \in \mathbb{Z}[I], x \in T(V)_\beta. \tag{1.2}$$

The braiding on $T(V)$ as a Yetter-Drinfeld module over Γ or $\mathbb{Z}[I]$ is in both cases given by

$$c(x \otimes y) = q_{\alpha\beta}y \otimes x,\ \text{where } x \in T(V)_\alpha, y \in T(V)_\beta, \alpha, \beta \in \mathbb{Z}[I]. \tag{1.3}$$

The comultiplication of $T(V)$ as a braided Hopf algebra in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$ only depends on the matrix (q_{ij}), hence it coincides with the comultiplication of $T(V)$ as a coalgebra in ${}^{\mathbb{Z}[I]}_{\mathbb{Z}[I]}\mathcal{YD}$. In particular, the comultiplication of $T(V)$ is $\mathbb{Z}[I]$-graded.

1.2. **Bosonization and twisting.** Let R be a braided Hopf algebra in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$. We will use a Sweedler notation for the comultiplication

$$\Delta_R : R \to R \otimes R,\ \Delta_R(r) = r^{(1)} \otimes r^{(2)}.$$

For Hopf algebras A in the usual sense, we always use the Sweedler notation

$$\Delta : A \to A \otimes A,\ \Delta(a) = a_{(1)} \otimes a_{(2)}.$$

Then the smash product $A = R\#k[\Gamma]$ is a Hopf algebra in the usual sense (the bosonization of R). As vector spaces, $R\#k[\Gamma] = R \otimes k[\Gamma]$. Multiplication and comultiplication are defined by

$$(r\#g)(s\#h) = r(g\cdot s)\#gh,\ \Delta(r\#g) = r^{(1)}\#{r^{(2)}}_{(-1)}g \otimes {r^{(2)}}_{(0)}\#g. \tag{1.4}$$

Then the maps

$$\iota : k[\Gamma] \to R\#k[\Gamma],\ \text{and } \pi : R\#k[\Gamma] \to k[\Gamma]$$

with $\iota(g) = 1\#g$ and $\pi(r\#g) = \varepsilon(r)g$ for all $r \in R$, $g \in \Gamma$ are Hopf algebra maps with $\pi\iota = \mathrm{id}$.

For simplicity we will often write rg instead of $r\#g$ in $R\#k[\Gamma]$ for $r \in R, g \in \Gamma$. Thus $\Delta_{R\#k[\Gamma]}(r) = r^{(1)}{r^{(2)}}_{(-1)} \otimes {r^{(2)}}_{(0)}$.

Conversely, if A is a Hopf algebra in the usual sense with Hopf algebra maps $\iota : k[\Gamma] \to A$ and $\pi : A \to k[\Gamma]$ such that $\pi\iota = \mathrm{id}$, then

$$R = \{a \in A \mid (\mathrm{id} \otimes \pi)\Delta(a) = a \otimes 1\} \tag{1.5}$$

is a braided Hopf algebra in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$ in the following way. As an algebra, R is a subalgebra of A. The $k[\Gamma]$-coaction, Γ-action and comultiplication of R are defined by

$$\delta(r) = \pi(r_{(1)}) \otimes r_{(2)}, \quad g \rightharpoonup r = \iota(g) r \iota(g^{-1}) \tag{1.6}$$

and

$$\Delta_R(r) = \vartheta(r_{(1)}) \otimes r_{(2)}. \tag{1.7}$$

Here, $\Delta_A(r) = r_{(1)} \otimes r_{(2)}$, and ϑ is the map

$$\vartheta : A \to R, \ \vartheta(r) = r_{(1)}\iota(S(\pi(r_{(2)}))), \tag{1.8}$$

where S is the antipode of A. Then

$$R\#k[\Gamma] \to A, \ r\#g \mapsto r\iota(g), \ r \in R, g \in \Gamma, \tag{1.9}$$

is an isomorphism of Hopf algebras.

We recall the notion of *twisting* the algebra structure of an arbitrary Hopf algebra A, see for example [KS, 10.2.3]. Let $\sigma : A \otimes A \to k$ be a convolution invertible linear map, and a normalized 2-cocycle, that is, for all $x, y, z \in A$,

$$\sigma(x_{(1)}, y_{(1)})\sigma(x_{(2)}y_{(2)}, z) = \sigma(y_{(1)}, z_{(1)})\sigma(x, y_{(2)}z_{(2)}), \tag{1.10}$$

and $\sigma(x, 1) = \varepsilon(x) = \sigma(1, x)$. The Hopf algebra A_σ with twisted algebra structure is equal to A as a coalgebra, and has multiplication $\cdot_\sigma$ with

$$x \cdot_\sigma y = \sigma(x_{(1)}, y_{(1)})x_{(2)}y_{(2)}\sigma^{-1}(x_{(3)}, y_{(3)}) \text{ for all } x, y \in A. \tag{1.11}$$

In the situation $A = R\#k[\Gamma]$ above, let $\sigma : \Gamma \times \Gamma \to k^\times$ be a normalized 2-cocycle of the group Γ. Then σ extends to a 2-cocycle of the group algebra $k[\Gamma]$ and it defines a normalized and invertible 2-cocycle

$$\sigma_\pi = \sigma(\pi \otimes \pi)$$

of the Hopf algebra A. Since $k[\Gamma]$ is cocommutative, ι and π are Hopf algebra maps

$$\iota : k[\Gamma] \to A_{\sigma_\pi} \text{ and } \pi : A_{\sigma_\pi} \to k[\Gamma].$$

Hence the coinvariant elements

$$R_\sigma = \{a \in A_{\sigma_\pi} \mid (\mathrm{id} \otimes \pi)\Delta(a) = a \otimes 1\}$$

form a braided Hopf algebra in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$. As a $k[\Gamma]$-comodule, R_σ coincides with R, but R_σ and R have different action, multiplication and comultiplication. To simplify the formulas, we will treat ι as an inclusion map. Also we denote the new action by $\rightharpoonup_\sigma$.

In any braided Hopf algebra R with multiplication m and braiding $c : R \otimes R \to R \otimes R$ we define the *braided commutator* of elements $x, y \in R$ by

$$[x, y]_c = xy - mc(x \otimes y). \tag{1.12}$$

If $x \in R$ is a primitive element, then

$$(\mathrm{ad}_c x)(y) = [x, y]_c \tag{1.13}$$

denotes the *braided adjoint action* of x on R. For example, in the situation of the free algebra in Section 1.1 with braiding (1.3), we have for all x_i and $y = x_{j_1} \cdots x_{j_n}$,

$$(\mathrm{ad}_c x_i)(y) = x_i y - q_{ij_1} \cdots q_{ij_n} y x_i. \tag{1.14}$$

To formulate the next lemma we need one more notation. If V is a left C-comodule over a coalgebra C, then V is a right module over the dual algebra C^* by $v \leftharpoonup p = p(v_{(-1)})v_{(0)}$ for all $v \in V, p \in C^*$. In particular, if R is a braided Hopf algebra in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$, then the $k[\Gamma]$-coaction defines a left $k[\Gamma] \otimes k[\Gamma]$-comodule structure on $R \otimes R$, hence a right $(k[\Gamma] \otimes k[\Gamma])^*$-module structure on $R \otimes R$ denoted by $\leftharpoonup$.

Lemma 1.1. *Let Γ be an abelian group, $\sigma : \Gamma \times \Gamma \to k^\times$ a normalized 2-cocycle, R a braided Hopf algebra in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$, $g, h \in \Gamma, x \in R_g, y \in R_h$, and $r \in R$. Then*

1. *$x \cdot_\sigma y = \sigma(g, h)xy$.*
2. *$\Delta_{R_\sigma}(r) = \Delta_R(r) \leftharpoonup \sigma^{-1}$.*
3. *If $y \in R_h^\eta$ for some character $\eta \in \widehat{\Gamma}$, then*

$$g \rightharpoonup_\sigma y = \sigma(g, h)\sigma^{-1}(h, g)\eta(g)y, \tag{1.15}$$

and hence $[x, y]_{c_\sigma} = \sigma(g, h)[x, y]_c$.

Proof. Note that for all homogeneous elements $z \in R_s, s \in \Gamma$,

$$(\pi \otimes \mathrm{id} \otimes \pi)\Delta^2(z) = s \otimes z \otimes 1,$$

because of (1.5) and (1.6). This implies (1), and in (3) we obtain

$$g \cdot_\sigma y = \sigma(g, h)gy \text{ and } y \cdot_\sigma g = \sigma(h, g)yg.$$

Thus

$$g \cdot_\sigma y = \sigma(g, h)gy = \sigma(g, h)\eta(g)yg = \sigma(g, h)\eta(g)\sigma(h, g)^{-1}y \cdot_\sigma g$$

and (1.15) follows. In turn, (1.15) implies the last assertion in (3).

To prove (2), using the cocommutativity of the group algebra we compute

$$\begin{aligned}\Delta_{R_\sigma}(r) &= r_{(1)} \cdot_\sigma S(\pi(r_{(2)})) \otimes r_{(3)} \\ &= \sigma(\pi(r_{(1)}), S(\pi(r_{(5)})))\vartheta(r_{(2)})\sigma^{-1}(\pi(r_{(3)}), S(\pi(r_{(4)}))) \otimes r_{(6)}.\end{aligned}$$

On the other hand, $\Delta_R(r) = r_{(1)}S\pi(r_{(2)}) \otimes r_{(3)}$, hence $r^{(1)}{}_{(-1)}\otimes r^{(2)}{}_{(-1)}\otimes r^{(1)}{}_{(0)}\otimes r^{(2)}{}_{(0)} = \pi(r_{(1)}S(r_{(3)}))\otimes\pi(r_{(4)})\otimes\vartheta(r_{(2)})\otimes r_{(5)}$, and $\Delta_R(r) \leftharpoonup \sigma^{-1} = \sigma^{-1}(\pi(r_{(1)}S(r_{(3)})), \pi(r_{(4)}))\vartheta(r_{(2)}) \otimes r_{(5)}$. Hence the claim follows from the equality

$$\sigma(a, S(b_{(3)}))\sigma^{-1}(b_{(1)}, S(b_{(2)})) = \sigma^{-1}(aS(b_{(1)}), b_{(2)})$$

for all $a, b \in k[\Gamma]$. It is enough to check this equation for elements $a, b \in \Gamma$. Then the equality follows from the group cocycle condition, which implies $\sigma(g, g^{-1}) = \sigma(g^{-1}, g)$ for all $g \in \Gamma$. □

Part (3) of the previous lemma extends [AS4, Lemma 3.15] and [AS5, Lemma 2.12].

We now apply the twisting procedure to the braided Hopf algebra $T(V) \in {}^{\mathbb{Z}[I]}_{\mathbb{Z}[I]}\mathcal{YD}$.

Lemma 1.2. *Let $\theta \geq 1$, and let $(q_{ij})_{1\leq i,j\leq\theta}$, $(q'_{ij})_{1\leq i,j\leq\theta}$ be matrices with coefficients in k. As in Section 1.1 let V and $V' \in {}^{\mathbb{Z}[I]}_{\mathbb{Z}[I]}\mathcal{YD}$ with basis $x_1, \dots, x_\theta$ and $x'_1, \dots, x'_\theta$ respectively, where $x_i \in V^{\psi_i}_{\alpha_i}, x'_i \in V'^{\psi'_i}_{\alpha_i}$ with $\psi_j(\alpha_i) = q_{ij}, \psi'_j(\alpha_i) = q'_{ij}$ for all i, j. Then $T(V)$ and $T(V')$ are braided Hopf algebras in ${}^{\mathbb{Z}[I]}_{\mathbb{Z}[I]}\mathcal{YD}$ as in Section 1.1. Assume*

$$q_{ij}q_{ji} = q'_{ij}q'_{ji}, \text{ and } q_{ii} = q'_{ii} \text{ for all } 1 \leq i, j \leq \theta. \tag{1.16}$$

Then there is a 2-cocycle $\sigma : \mathbb{Z}[I] \times \mathbb{Z}[I] \to k^\times$ with

$$\sigma(\alpha, \beta)\sigma^{-1}(\beta, \alpha) = q_{\alpha\beta}q'^{-1}_{\alpha\beta} \text{ for all } \alpha, \beta \in \mathbb{Z}[I], \tag{1.17}$$

and a k-linear isomorphism $\varphi : T(V) \to T(V')$ with $\varphi(x_i) = x'_i$ for all i and such that for all $\alpha, \beta \in \mathbb{Z}[I], x \in T(V)_\alpha, y \in T(V)_\beta$ and $z \in T(V)$

1. *$\varphi(xy) = \sigma(\alpha, \beta)\varphi(x)\varphi(y)$.*
2. *$\Delta_{T(V')}(\varphi(z)) = (\varphi \otimes \varphi)(\Delta_{T(V)}(z)) \leftharpoonup \sigma$.*
3. *$\varphi([x, y]_c) = \sigma(\alpha, \beta)[\varphi(x), \varphi(y)]_{c'}$.*

Proof. Define σ as the bilinear map with $\sigma(\alpha_i, \alpha_j) = q_{ij}q'^{-1}_{ij}$ if $i \leq j$, and $\sigma(\alpha_i, \alpha_j) = 1$ if $i > j$ (see [AS5, Prop. 3.9]).

Let $\varphi : T(V) \to T(V')_\sigma$ be the algebra map with $\varphi(x_i) = x'_i$ for all i. Then φ is bijective since it follows from Lemma 1.1 (1) and the bilinearity of σ that for all monomials $x = x_{i_1}x_{i_2}\cdots x_{i_n}$ of length $n \geq 1$ with $x' = x'_{i_1}x'_{i_2}\cdots x'_{i_n}$,

$$\varphi(x) = \prod_{r<s} \sigma(\alpha_{i_r}, \alpha_{i_s})x'.$$

In particular, φ is $\mathbb{Z}[I]$-graded.

To see that φ is $\mathbb{Z}[I]$-linear, let $\alpha, \beta \in \mathbb{Z}[I]$ and $x \in T(V)_\beta$. By (1.2) and Lemma 1.1 (3),

$$\alpha \rightharpoonup x = q_{\alpha\beta}x, \text{ and } \alpha \rightharpoonup_\sigma \varphi(x) = \sigma(\alpha,\beta)\sigma^{-1}(\beta,\alpha)q'_{\alpha\beta}\varphi(x),$$

and $\varphi(\alpha \rightharpoonup x) = \alpha \rightharpoonup_\sigma \varphi(x)$ follows by (1.17).

Since the elements x_i and x'_i are primitive we now see that φ is an isomorphism of braided Hopf algebras. Then the claim follows from Lemma 1.1. □

2. Serre relations and root vectors

2.1. Datum of finite Cartan type and root vectors.

Definition 2.1. A *datum of Cartan type*

$$\mathcal{D} = \mathcal{D}(\Gamma, (g_i)_{1\leq i\leq\theta}, (\chi_i)_{1\leq i\leq\theta}, (a_{ij})_{1\leq i,j\leq\theta})$$

consists of an abelian group Γ, elements $g_i \in \Gamma, \chi_i \in \widehat{\Gamma}, 1 \leq i \leq \theta$, and a generalized Cartan matrix (a_{ij}) of size θ satisfying

$$q_{ij}q_{ji} = q_{ii}^{a_{ij}},\ q_{ii} \neq 1, \text{ with } q_{ij} = \chi_j(g_i) \text{ for all } 1 \leq i,j \leq \theta. \tag{2.1}$$

We call θ the *rank* of $\mathcal{D}$. A datum $\mathcal{D}$ of Cartan type will be called of *finite Cartan type* if (a_{ij}) is of finite type.

Example 2.2. A Cartan datum $(I, \cdot)$ in the sense of Lusztig [L3, 1.1.1] defines a datum of Cartan type for the free abelian group ZI with $g_i = \alpha_i, \chi_i = \psi_i, 1 \leq i \leq \theta$, as in Section 1.1, where

$$q_{ij} = v^{d_i a_{ij}}, d_i = \frac{i\cdot i}{2}, a_{ij} = 2\frac{i\cdot j}{i\cdot i} \text{ for all } 1 \leq i,j \leq \theta.$$

In Example 2.2, $d_i a_{ij} = i \cdot j$ is the symmetrized Cartan matrix, and $q_{ij} = q_{ji}$ for all $1 \leq i,j \leq \theta$. In general, the matrix (q_{ij}) of a datum of Cartan type is not symmetric, but by Lemma 1.2 we can reduce to the symmetric case by twisting.

We fix a finite abelian group Γ and a datum

$$\mathcal{D} = \mathcal{D}(\Gamma, (g_i)_{1\leq i\leq\theta}, (\chi_i)_{1\leq i\leq\theta}, (a_{ij})_{1\leq i,j\leq\theta})$$

of finite Cartan type. The Weyl group $W \subset \operatorname{Aut}(\mathbb{Z}[I])$ of (a_{ij}) is generated by the reflections $s_i : \mathbb{Z}[I] \to \mathbb{Z}[I]$ with $s_i(\alpha_j) = \alpha_j - a_{ij}\alpha_i$ for all i, j. The root system is $\Phi = \cup_{i=1}^{\theta} W(\alpha_i)$, and

$$\Phi^+ = \Big\{ \alpha \in \Phi \mid \alpha = \sum_{i=1}^{\theta} n_i\alpha_i, n_i \geq 0 \text{ for all } 1 \leq i \leq \theta \Big\}$$

denotes the set of positive roots with respect to the basis of simple roots $\alpha_1, \dots, \alpha_\theta$. Let p be the number of positive roots.

For $\alpha = \sum_{i=1}^{\theta} n_i\alpha_i \in \mathbb{Z}[I], n_i \in \mathbb{Z}$ for all i, we define

$$g_\alpha = g_1^{n_1} g_2^{n_2} \cdots g_\theta^{n_\theta} \text{ and } \chi_\alpha = \chi_1^{n_1}\chi_2^{n_2}\cdots\chi_\theta^{n_\theta}. \tag{2.2}$$

Hence for all $\alpha, \beta \in \mathbb{Z}[I]$,

$$q_{\alpha\beta} = \chi_\beta(g_\alpha), \tag{2.3}$$

where $q_{\alpha\beta}$ is given by (1.1).

In this section, we assume that the Dynkin diagram of (a_{ij}) is *connected*. In this case we say that $\mathcal{D}$ is connected. We assume for all $1 \leq i \leq \theta$,

$$q_{ii} \text{ has odd order, and} \tag{2.4}$$

$$\text{the order of } q_{ii} \text{ is prime to 3, if } (a_{ij}) \text{ is of type } G_2. \tag{2.5}$$

Then it follows from Lemma 2.3 that the elements q_{ii} have the same order in $k^\times$. We define

$$N = \text{ order of } q_{ii}, 1 \leq i \leq \theta. \tag{2.6}$$

Lemma 2.3. *Let $\mathcal{D}$ be a connected datum of finite Cartan type with Cartan matrix (a_{ij}) and assume (2.4) and (2.5). Then there are integers $d_i \in \{1, 2, 3\}, 1 \leq i \leq \theta$, and $q \in k$ such that for all $1 \leq i, j \leq \theta$,*

$$q_{ii} = q^{2d_i}, \quad d_i a_{ij} = d_j a_{ji},$$

and the order of q is odd, and if the Cartan matrix of $\mathcal{D}$ is of type G_2, then the order of q is prime to 3.

Proof. By (2.1), $q_{ii}^{a_{ij}} = q_{jj}^{a_{ji}}$ for all $1 \leq i, j \leq \theta$. It then follows from the list of Cartan matrices with connected Dynkin diagram that in each case there is an index h such that $q_{ii} = (q_{hh})^{d_i}$ for all $1 \leq i \leq \theta$, where the $d_i \in \{1, 2, 3\}$ symmetrize (a_{ij}). Indeed this is obvious in the simply laced case, and in the notation of [B] with $l = \theta$ take $h = 1$ for B_l, $h = l$ for C_l, and $h = 2$ for F_4 and G_2. By (2.4) we let q be a square root of q_{hh} of odd order. Then by (2.5) the order of q is prime to 3 if the Cartan matrix of $\mathcal{D}$ is of type G_2. □

We fix a reduced decomposition of the longest element

$$w_0 = s_{i_1} s_{i_2} \cdots s_{i_p}$$

of W in terms of the simple reflections. Then

$$\beta_l = s_{i_1} \cdots s_{i_{l-1}}(\alpha_{i_l}), 1 \leq l \leq p,$$

is a convex ordering of the positive roots.

Definition 2.4. Let $V = V(\mathcal{D})$ be a vector space with basis $x_1, \ldots, x_\theta$, and let $V \in {}^{\Gamma}_{\Gamma}\mathcal{YD}$ by $x_i \in V^{\chi_i}_{g_i}$ for all $1 \leq i \leq \theta$. Then $T(V)$ is a braided Hopf algebra in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$ as in Section 1.1. Let

$$R(\mathcal{D}) = T(V)/((\mathrm{ad}_c x_i)^{1-a_{ij}}(x_j) \mid 1 \leq i \neq j \leq \theta)$$

be the quotient Hopf algebra in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$.

It is well-known that the elements $(\mathrm{ad}_c x_i)^{1-a_{ij}}(x_j), 1 \leq i \neq j \leq \theta$, are primitive in the free algebra $T(V)$ (see for example [AS2, A.1]), hence they generate a Hopf ideal. By abuse of language, we denote the images of the elements x_i in $R(\mathcal{D})$ again by x_i.

In the situation of Example 2.2, Lusztig [L2] defined root vectors x_α in $R(\mathcal{D}) = U^+$ for each positive root α using the convex ordering of the positive roots. As noted in [AS4], these root vectors can be seen to be iterated braided commutators of the elements $x_1, \ldots, x_\theta$ with respect to the braiding given by the matrix $(v^{d_i a_{ij}})$. This follows for example from the inductive definition of the root vectors in [Ri].

In the case of our general braiding given by (q_{ij}) we define root vectors $x_\alpha \in R(\mathcal{D})$ for each $\alpha \in \Phi^+$ by the same iterated braided commutator of the elements $x_1, \ldots, x_\theta$ as in Lusztig's case but with respect to the general braiding.

Definition 2.5. Let $K(\mathcal{D})$ be the subalgebra of $R(\mathcal{D})$ generated by the elements $x_\alpha^N, \alpha \in \Phi^+$.

Theorem 2.6. *Let $\mathcal{D}$ be a connected datum of finite Cartan type, and assume* (2.4), (2.5).

(1) *The elements*

$$x_{\beta_1}^{a_1} x_{\beta_2}^{a_2} \cdots x_{\beta_p}^{a_p}, a_1, a_2, \ldots, a_p \geq 0,$$

form a basis of $R(\mathcal{D})$.

(2) *$K(\mathcal{D})$ is a braided Hopf subalgebra of $R(\mathcal{D})$.*

(3) *For all $\alpha, \beta \in \Phi^+, [x_\alpha, x_\beta^N]_c = 0$, that is,*

$$x_\alpha x_\beta^N = q_{\alpha\beta}^N x_\beta^N x_\alpha.$$

Proof. (a) In the situation of Example 2.2, the elements in (1) form Lusztig's PBW-basis of U^+ over $\mathbb{Z}[v, v^{-1}]$ by [L2, 5.7].

(b) Now we assume that the braiding has the form $(q_{ij} = q^{d_i a_{ij}})$, where $(d_i a_{ij})$ is the symmetrized Cartan matrix, and q is a non-zero element in k of odd order, and whose order is prime to 3 if the Dynkin diagram of (a_{ij}) is G_2. Then (1) follows from Lusztig's result by extension of scalars, and (2) is shown in [dCP, 19.1] (for another proof see [M2, 3.1]). The algebra $K(\mathcal{D})$ is commutative since it is a subalgebra of the commutative algebra Z_0 of [dCP, 19.1]. This proves (3) since $q^N = 1$, hence $\chi_\beta^N(g_\alpha) = 1$.

(c) In the situation of a general braiding matrix $(q_{ij})_{1\le i,j\le\theta}$ assumed in the theorem, we apply Lemma 2.3 and define a matrix $(q'_{ij})_{1\le i,j\le\theta}$ by $q'_{ij} = q^{d_i a_{ij}}$ for all i, j. Then $q_{ij}q_{ji} = q'_{ij}q'_{ji}$, and $q_{ii} = q'_{ii}$ for all $1 \le i, j \le \theta$. Thus by part (b) of the proof, (1),(2) and (3) hold for the braiding (q'_{ij}), and hence by Lemma 1.2 for (q_{ij}). □

2.2. **The Hopf algebra $K(\mathcal{D})\#k[\Gamma]$.** We assume the situation of Section 2.1. By Theorem 2.6 (2), $K(\mathcal{D})$ is a braided Hopf algebra in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$, and the smash product $K(\mathcal{D})\#k[\Gamma]$ is a Hopf algebra in the usual sense. We want to describe all Hopf algebra maps

$$K(\mathcal{D})\#k[\Gamma] \to k[\Gamma]$$

which are the identity on the group algebra $k[\Gamma]$.

Definition 2.7. For any $1 \le l \le p$ and $a = (a_1, a_2, \dots, a_p) \in \mathbb{N}^p$ we define

$$\begin{aligned}
h_l &= g_{\beta_l}^N,\\
\eta_l &= \chi_{\beta_l}^N,\\
z_l &= x_{\beta_l}^N,\\
z^a &= z_1^{a_1} z_2^{a_2} \cdots z_p^{a_p} \in K(\mathcal{D}),\\
h^a &= h_1^{a_1} h_2^{a_2} \cdots h_p^{a_p} \in \Gamma,\\
\eta^a &= \eta_1^{a_1} \eta_2^{a_2} \cdots \eta_p^{a_p} \in \widehat{\Gamma},\\
\underline{a} &= a_1\beta_1 + a_2\beta_2 + \cdots + a_p\beta_p \in \mathbb{Z}[I].
\end{aligned}$$

For $\alpha = \sum_{i=1}^{\theta} n_i\alpha_i \in \mathbb{Z}[I], n_i \in \mathbb{Z}$ for all i, we call $\mathrm{ht}(\alpha) = \sum_{i=1}^{\theta} n_i$ the *height* of α. Let $e_l = (\delta_{kl})_{1\le k\le p} \in \mathbb{N}^p$, where $\delta_{kl} = 1$ if $k = l$ and $\delta_{kl} = 0$ if $k \ne l$.

Note that for all $a, b, c \in \mathbb{N}^p$,

$$h^a = h^b h^c, \ \eta^a = \eta^b \eta^c, \text{ if } \underline{a} = \underline{b} + \underline{c}, \tag{2.7}$$

$$\mathrm{ht}(\underline{b}) < \mathrm{ht}(\underline{a}), \text{ if } \underline{a} = \underline{b} + \underline{c} \text{ and } c \neq 0. \tag{2.8}$$

As explained in Section 1.1, we view $T(V)$ as a braided Hopf algebra in ${}^{\mathbb{Z}[I]}_{\mathbb{Z}[I]}\mathcal{YD}$. Then the quotient Hopf algebra $R(\mathcal{D})$ and its Hopf subalgebra $K(\mathcal{D})$ are braided Hopf algebras in ${}^{\mathbb{Z}[I]}_{\mathbb{Z}[I]}\mathcal{YD}$. In particular, the comultiplication $\Delta_{K(\mathcal{D})} : K(\mathcal{D}) \to K(\mathcal{D}) \otimes K(\mathcal{D})$ is $\mathbb{Z}[I]$-graded. By construction, for any $\alpha \in \Phi^+$, the root vector x_α in $R(\mathcal{D})$ is $\mathbb{Z}[I]$-homogeneous of $\mathbb{Z}[I]$-degree α. Thus $x_\alpha \in R(\mathcal{D})^{\chi_\alpha}_{g_\alpha}$, and for all $a \in \mathbb{N}^p$, z^a has $\mathbb{Z}[I]$-degree $N\underline{a}$, and

$$z^a \in K(\mathcal{D})^{\eta^a}_{h^a}. \tag{2.9}$$

By Theorem 2.6 the elements $z^a g$ with $a \in \mathbb{N}^p, g \in \Gamma$, form a basis of $K(\mathcal{D})\#k[\Gamma]$, and it follows that for all $a, b = (b_i), c = (c_i) \in \mathbb{N}^p$,

$$z^b z^c = \gamma_{b,c} z^{b+c}, \text{ where } \gamma_{b,c} = \prod_{k>l} \eta_l(h_k)^{b_k c_l}, \tag{2.10}$$

$$h^a z^b = \eta^b(h^a) z^b h^a \text{ in } R\#k[\Gamma]. \tag{2.11}$$

Lemma 2.8. *For any $0 \neq a \in \mathbb{N}^p$ there are uniquely determined scalars $t^a_{b,c} \in k, 0 \neq b, c \in \mathbb{N}^p$, such that*

$$\Delta_{K(\mathcal{D})}(z^a) = z^a \otimes 1 + 1 \otimes z^a + \sum_{b,c \neq 0, \underline{b}+\underline{c}=\underline{a}} t^a_{b,c}\, z^b \otimes z^c. \tag{2.12}$$

Proof. Since $\Delta_{K(\mathcal{D})}$ is $\mathbb{Z}[I]$-graded, $\Delta_{K(\mathcal{D})}(z^a)$ is a linear combination of elements $z^b \otimes z^c$ where $\underline{b} + \underline{c} = \underline{a}$. Hence

$$\Delta_{K(\mathcal{D})}(z^a) = x \otimes 1 + 1 \otimes y + \sum_{b,c \neq 0, \underline{b}+\underline{c}=\underline{a}} t^a_{b,c}\, z^b \otimes z^c,$$

where x, y are elements in $K(\mathcal{D})$. By applying the augmentation ε it follows that $x = y = z^a$. □

We now define recursively a family of elements u^a in $k[\Gamma]$ depending on parameters μ_a which behave like the elements z^a with respect to comultiplication.

Lemma 2.9. *Let $n \geq 1$. Let $(\mu_b)_{0 \neq b \in \mathbb{N}^p, \mathrm{ht}(\underline{b}) < n}$ be a family of elements in k, and let $(u^b)_{0 \neq b \in \mathbb{N}^p, \mathrm{ht}(\underline{b}) < n}$ be a family of elements in $k[\Gamma]$. Assume for all $0 \neq b \in \mathbb{N}^p, \mathrm{ht}(\underline{b}) < n$, that*

$$u^b = \mu_b(1 - h^b) + \sum_{d,e \neq 0, \underline{d}+\underline{e}=\underline{b}} t^b_{d,e}\, \mu_d u^e, \tag{2.13}$$

$$\Delta(u^b) = h^b \otimes u^b + u^b \otimes 1 + \sum_{d,e\neq 0, \underline{d}+\underline{e}=\underline{b}} t^b_{d,e}\, u^d h^e \otimes u^e. \tag{2.14}$$

Let $a \in \mathbb{N}^p$ with $\operatorname{ht}(\underline{a}) = n$, and $u^a \in k[\Gamma]$. Then the following statements are equivalent:

$$u^a = \mu_a(1 - h^a) + \sum_{b,c\neq 0, \underline{b}+\underline{c}=\underline{a}} t^a_{b,c}\, \mu_b u^c \text{ for some } \mu_a \in k. \tag{2.15}$$

$$\Delta(u^a) = h^a \otimes u^a + u^a \otimes 1 + \sum_{b,c\neq 0, \underline{b}+\underline{c}=\underline{a}} t^a_{b,c}\, u^b h^c \otimes u^c. \tag{2.16}$$

Proof. If $n = 1$, the equivalence between (2.15) and (2.16) is well-known and easy to see. The point of the Lemma is the inductive construction of the u^a's. Let

$$v_a = u^a - \sum_{b,c\neq 0, \underline{b}+\underline{c}=\underline{a}} t^a_{b,c}\, \mu_b u^c.$$

Then u^a can be written as in (2.15) if and only if $\Delta(v_a) = h^a \otimes v_a + v_a \otimes 1$. Hence it is enough to prove that

$$\Delta(v_a) - h^a \otimes v_a - v_a \otimes 1 = \Delta(u^a) - h^a \otimes u^a - u^a \otimes 1 - \sum_{b,c\neq 0, \underline{b}+\underline{c}=\underline{a}} t^a_{b,c}\, u^b h^c \otimes u^c.$$

We compute

$$\begin{aligned}
&\Delta(v_a) - h^a \otimes v_a - v_a \otimes 1 = \\
&\quad = \Delta(u^a) - \sum_{b,c\neq 0, \underline{b}+\underline{c}=\underline{a}} t^a_{b,c}\, \mu_b \Delta(u^c) - h^a \otimes v_a - v_a \otimes 1 \\
&\quad = \Delta(u^a) - h^a \otimes u^a - u^a \otimes 1 + \sum_{b,c\neq 0, \underline{b}+\underline{c}=\underline{a}} t^a_{b,c}\, \mu_b (h^a \otimes u^c - h^c \otimes u^c) \\
&\quad - \sum_{\substack{b,c,f,g\neq 0 \\ \underline{b}+\underline{c}=\underline{a}, \underline{f}+\underline{g}=\underline{c}}} t^a_{b,c}\, t^c_{f,g}\, \mu_b u^f h^g \otimes u^g,
\end{aligned}$$

using the definition of v_a in the first equation, and the formula for $\Delta(u^c)$ from (2.14) in the second equation. Note that the term

$$\sum_{b,c\neq 0, \underline{b}+\underline{c}=\underline{a}} t^a_{b,c}\, \mu_b u^c \otimes 1$$

cancels. Hence we have to show that

$$\sum_{\substack{b,c,f,g\neq 0\\ \underline{b}+\underline{c}=\underline{a},\underline{f}+\underline{g}=\underline{c}}} t^a_{b,c}\, t^c_{f,g}\, \mu_b u^f h^g \otimes u^g =$$

$$= \sum_{b,c\neq 0,\underline{b}+\underline{c}=\underline{a}} t^a_{b,c}(\mu_b h^a \otimes u^c - \mu_b h^c \otimes u^c + u^b h^c \otimes u^c).$$

Since for all $b,c \neq 0, \underline{b}+\underline{c}=\underline{a}$, we have $h^a = h^b h^c$, it follows that

$$\mu_b h^a \otimes u^c - \mu_b h^c \otimes u^c + u^b h^c \otimes u^c = (\mu_b(h^b-1)+u^b)h^c \otimes u^c.$$

Using the formula for u^b from (2.13), we finally have to prove

$$\sum_{\substack{b,c,f,g\neq 0\\ \underline{b}+\underline{c}=\underline{a},\underline{f}+\underline{g}=\underline{c}}} t^a_{b,c}\, t^c_{f,g}\, \mu_b u^f h^g \otimes u^g = \sum_{\substack{b,c,d,e\neq 0\\ \underline{b}+\underline{c}=\underline{a},\underline{d}+\underline{e}=\underline{b}}} t^a_{b,c}\, t^b_{d,e}\, \mu_d u^e h^c \otimes u^c.$$

This last equality follows from the coassociativity of $K(\mathcal{D})$. Indeed, from

$$(\mathrm{id}\otimes \Delta_{K(\mathcal{D})})\Delta_{K(\mathcal{D})}(z^a) = (\Delta_{K(\mathcal{D})}\otimes \mathrm{id})\Delta_{K(\mathcal{D})}(z^a)$$

we obtain with (2.12) after cancelling several terms

$$\sum_{\substack{b,c,f,g\neq 0\\ \underline{b}+\underline{c}=\underline{a},\underline{f}+\underline{g}=\underline{c}}} t^a_{b,c}\, t^c_{f,g}\, z^b \otimes z^f \otimes z^g = \sum_{\substack{b,c,d,e\neq 0\\ \underline{b}+\underline{c}=\underline{a},\underline{d}+\underline{e}=\underline{b}}} t^a_{b,c}\, t^b_{d,e}\, z^d \otimes z^e \otimes z^c.$$

Thus mapping $z^r \otimes z^s \otimes z^t, r,s,t \neq 0, \mathrm{ht}(\underline{r}), \mathrm{ht}(\underline{s}), \mathrm{ht}(\underline{t}) < n$, onto $\mu_r u^s h^t \otimes u^t$ proves the claim. Here we are using that the elements z^a are linearly independent by Theorem 2.6. □

Let $K(\mathcal{D})\#k[\Gamma]$ be the Hopf algebra corresponding to the braided Hopf algebra $K(\mathcal{D})$ by (1.4). Thus by definition and Lemma 2.8, for all $0 \neq a \in \mathbb{N}^p$,

$$\Delta_{K(\mathcal{D})\#k[\Gamma]}(z^a) = h^a \otimes z^a + z^a \otimes 1 + \sum_{b,c\neq 0,\underline{b}+\underline{c}=\underline{a}} t^a_{b,c}\, z^b h^c \otimes z^c. \tag{2.17}$$

For all $n \geq 0$, let $K(\mathcal{D})_n$ be the vector subspace spanned by all elements $z^a, a \in \mathbb{N}^p, \mathrm{ht}(\underline{a}) \leq n$. Then $K(\mathcal{D})_n\#k[\Gamma] \subset K(\mathcal{D})\#k[\Gamma]$ is a subcoalgebra.

In the next Lemma we describe all coalgebra maps

$$\varphi : K(\mathcal{D})_n\#k[\Gamma] \to k[\Gamma] \text{ with } \varphi|\Gamma = \mathrm{id}.$$

Note that such a coalgebra map is given by a family of elements $\varphi(z^a) =: u^a, 0 \neq a \in \mathbb{N}^p, \mathrm{ht}(\underline{a}) \leq n$, such that (2.16) holds for all $0 \neq a, \mathrm{ht}(\underline{a}) \leq n$. It follows by induction on $\mathrm{ht}(\underline{a})$ from Lemma 2.9 with (2.15) that $\varepsilon(u^a) = 0$ for all a.

Lemma 2.10. *Let $n \geq 1$.*

(1) *Let $(\mu_a)_{0 \neq a \in \mathbb{N}^p, \mathrm{ht}(\underline{a}) \leq n}$ be a family of elements in k such that for all a, if $h^a = 1$, then $\mu_a = 0$. Define the family $(u^a)_{0 \neq a \in \mathbb{N}^p, \mathrm{ht}(\underline{a}) \leq n}$ by induction on $\mathrm{ht}(\underline{a})$ by (2.15). Then the map $\varphi : K(\mathcal{D})_n \# k[\Gamma] \to k[\Gamma]$ given by $\varphi|\Gamma = \mathrm{id}$,*

$$\varphi(z^a g) = u^a g, 0 \neq a \in \mathbb{N}^p, \mathrm{ht}(\underline{a}) \leq n, g \in \Gamma,$$

is a coalgebra map.

(2) *The map defined in (1) from the set of all $(\mu_a)_{0 \neq a \in \mathbb{N}^p, \mathrm{ht}(\underline{a}) \leq n}$ such that for all a, if $h^a = 1$, then $\mu_a = 0$, to the set of all coalgebra maps φ with $\varphi|\Gamma = \mathrm{id}$ is bijective.*

Proof. This follows from Lemma 2.9 by induction on $\mathrm{ht}(\underline{a})$. Note that the coefficient μ_a in (2.15) is uniquely determined if we define $\mu_a = 0$ if $h^a = 1$. □

Definition 2.11. Let $n \geq 1$. A coalgebra map

$$\varphi : K(\mathcal{D})_n \# k[\Gamma] \to k[\Gamma] \text{ with } \varphi|\Gamma = \mathrm{id}$$

is called a *partial Hopf algebra map*, if for all $x, y \in K(\mathcal{D})_n \# k[\Gamma]$ with $xy \in K(\mathcal{D})_n \# k[\Gamma]$, we have $\varphi(xy) = \varphi(x)\varphi(y)$.

Lemma 2.12. *Let $n \geq 1$, and $\varphi : K(\mathcal{D})_n \# k[\Gamma] \to k[\Gamma]$ a coalgebra map, $(\mu_a)_{0 \neq a \in \mathbb{N}^p, \mathrm{ht}(\underline{a}) \leq n}$ the family of scalars corresponding to φ by Lemma 2.10, and $u^a = \varphi(z^a)$ for all $a \in \mathbb{N}^p$ with $\mathrm{ht}(\underline{a}) \leq n$. Then the following are equivalent:*

(1) *φ is a partial Hopf algebra map.*
(2) *For all $0 \neq a = (a_1, \dots, a_p) \in \mathbb{N}^p$ with $\mathrm{ht}(\underline{a}) \leq n$,*
 (a) *$u^a = \prod_{a_l > 0} u_l^{a_l}$, where for all $1 \leq l \leq p, u_l = u^{e_l}$, if $a_l > 0$,*
 (b) *if $\eta^a \neq \varepsilon$, then $\mu_a = 0$, and $u^a = 0$.*
(3) (a) *As (2) (a).*
 (b) *For all $1 \leq l \leq p$ with $\mathrm{ht}(\underline{e_l}) \leq n$, if $\eta_l \neq \varepsilon$, then $u^{e_l} = 0$.*

Proof. (1) ⇒ (2): If φ is a partial Hopf algebra map, then (a) follows immediately, and to prove (b), let $0 \neq a \in \mathbb{N}^p, \mathrm{ht}(\underline{a}) \leq n$, and $g \in \Gamma$, with $\eta^a \neq \varepsilon$. Then

$$\varphi(gz^a) = \eta^a(g) u^a g = u^a g,$$

since $gz^a = \eta^a(g) z^a g$ by (2.11). Thus $u^a = 0$, and it follows by induction on $\mathrm{ht}(\underline{a})$ from (2.15) that $\mu_a = 0$, since for all $0 \neq b, c \in \mathbb{N}^p$ with $\mathrm{ht}(\underline{b}) + \mathrm{ht}(\underline{c}) = \mathrm{ht}(\underline{a})$, $\eta^b \neq \varepsilon$, or $\eta^c \neq \varepsilon$.

(2) ⇒ (3) is trivial. (3) ⇒ (1): The coalgebra map φ is a partial Hopf algebra map if and only if for all $b, c \in \mathbb{N}^p$ with $\mathrm{ht}(\underline{b}) + \mathrm{ht}(\underline{c}) \leq n$,

and $g, h \in \Gamma$,

$$\varphi(z^b g z^c h) = u^b g u^c h.$$

By (2.10) and (2.11), $z^b g z^c h = \eta^c(g)\gamma_{b,c} z^{b+c} gh$. Thus (1) is equivalent to

$$\eta^c(g)\gamma_{b,c} u^{b+c} = u^b u^c \text{ for all } b, c \in \mathbb{N}^p, \operatorname{ht}(\underline{b}) + \operatorname{ht}(\underline{c}) \leq n, g \in \Gamma. \tag{2.18}$$

Let $b, c \in \mathbb{N}^p, \operatorname{ht}(\underline{b}) + \operatorname{ht}(\underline{c}) \leq n, g \in \Gamma$. By (a),

$$u^{b+c} = u^b u^c = \prod_{b_l + c_l > 0} u_l^{b_l + c_l}.$$

To prove (2.18) assume that $u^b u^c \neq 0$. Then $u_l \neq 0$ for all l with $c_l > 0$. Hence by (b), $\eta_l = \varepsilon$ for all l with $c_l > 0$, and $\eta^c(g) = 1, \gamma_{b,c} = 1$. □

To formulate the main result of this section, we define $M(\mathcal{D})$ as the set of all families $(\mu_l)_{1 \leq l \leq p}$ of elements in k satisfying the following condition for all $1 \leq l \leq p$: If $h_l = 1$ or $\eta_l \neq \varepsilon$, then $\mu_l = 0$.

Theorem 2.13. (1) *Let* $\mu = (\mu_l)_{1 \leq l \leq p} \in M(\mathcal{D})$. *Then there is exactly one Hopf algebra map*

$$\varphi_\mu : K(\mathcal{D}) \# k[\Gamma] \to k[\Gamma], \ \varphi|\Gamma = \mathrm{id}$$

such that the family $(\mu_a)_{0 \neq a \in \mathbb{N}^p}$ *associated to* φ_μ *by Lemma 2.10 satisfies* $\mu_{e_l} = \mu_l$ *for all* $1 \leq l \leq p$.

(2) *The map* $\mu \mapsto \varphi_\mu$ *defined in* (1) *from* $M(\mathcal{D})$ *to the set of all Hopf algebra homomorphisms* $\varphi : K(\mathcal{D}) \# k[\Gamma] \to k[\Gamma]$ *with* $\varphi|\Gamma = \mathrm{id}$ *is bijective.*

Proof. (1) We proceed by induction on n to construct partial Hopf algebra maps on $K(\mathcal{D})_n \# k[\Gamma]$, the case $n = 0$ being trivial. We assume that we are given a partial Hopf algebra map

$$\varphi : K(\mathcal{D})_{n-1} \# k[\Gamma] \to k[\Gamma], \ n \geq 1,$$

such that $\mu_{e_l} = \mu_l$ for all $1 \leq l \leq p$ with $\operatorname{ht}(\underline{e_l}) \leq n - 1$. Here $(\mu_a)_{0 \neq a \in \mathbb{N}^p, \operatorname{ht}(\underline{a}) \leq n-1}$ is the family of scalars associated to φ by Lemma 2.10. We define $u^b = \varphi(z^b)$ for all $0 \neq b, \operatorname{ht}(\underline{b}) \leq n - 1$. It is enough to show that there is exactly one partial Hopf algebra map

$$\psi : K(\mathcal{D})_n \# k[\Gamma] \to k[\Gamma]$$

extending φ, and such that $\mu_{e_l} = \mu_l$ for all l with $\operatorname{ht}(\underline{e_l}) \leq n$.

Let $a \in \mathbb{N}^p$ with $\operatorname{ht}(\underline{a}) = n$. To define $\psi(z^a) =: u^a$ we distinguish two cases.

If $a = e_l$ for some $1 \leq l \leq p$, we define

$$u^a = \mu_l(1 - h^a) + \sum_{b,c \neq 0, \underline{b} + \underline{c} = \underline{a}} t^a_{b,c} \mu_b u^c. \tag{2.19}$$

Then (2.16) holds by Lemma 2.9.

If $a = (a_1, \dots, a_l, 0, \dots, 0), a_l \geq 1, 1 \leq l \leq p$, and $a \neq e_l$, then $a = r + s$, where $0 \neq r, s = e_l$. We define $u^a = u^r u^s$. To see that u^a satisfies (2.16), using (2.17) we write

$$\Delta(z^c) = h^c \otimes z^c + z^c \otimes 1 + T(c), \text{ for all } 0 \neq c \in \mathbb{N}^p.$$

Since $z^r z^s = z^a$ because of (2.10) (note that $\gamma_{r,s} = 1$ in this case) we see that $\Delta(z^r)\Delta(z^s) = h^a \otimes z^a + z^a \otimes 1 + T(r, s)$, where

$$\begin{aligned} T(r,s) &= h^r z^s \otimes z^r + z^r h^s \otimes z^s \\ &\quad + (h^r \otimes z^r + z^r \otimes 1)T(s) + T(r)(h^s \otimes z^s + z^s \otimes 1) + T(r)T(s), \end{aligned}$$

and $T(r, s) = T(a)$. Since φ on $K(\mathcal{D})_{n-1}\#k[\Gamma]$ is a coalgebra map,

$$\Delta(u^c) = h^c \otimes u^c + u^c \otimes 1 + (\varphi \otimes \varphi)(T(c)),$$

for all $0 \neq c \in \mathbb{N}^p$ with $\operatorname{ht}(\underline{c}) \leq n - 1$. In particular,

$$\Delta(u^r)\Delta(u^s) = h^a \otimes u^a + u^a \otimes 1 + (\varphi \otimes \varphi)(T(r, s)).$$

Thus $\Delta(u^a) = h^a \otimes u^a + u^a \otimes 1 + (\varphi \otimes \varphi)(T(a))$, that is, u^a satisfies (2.16).

Thus the extension of φ defined by $\psi(z^a g) = u^a g$ for all $g \in \Gamma, a \in \mathbb{N}^p, \operatorname{ht}(\underline{a}) = n$ is a coalgebra map.

To prove that the extension ψ is a partial Hopf algebra map, we check condition (3) in Lemma 2.12. Since the restriction of ψ to $K(\mathcal{D})_{n-1}\#k[\Gamma]$ is a partial Hopf algebra map, (3) (a) is satisfied. To prove (3)(b), let $1 \leq l \leq p$ with $\operatorname{ht}(\underline{e_l}) = n$, $a = e_l$, and assume $\eta_l \neq \varepsilon$. Then for all $0 \neq b, c \in \mathbb{N}^p$ with $\underline{b} + \underline{c} = \underline{a}$, we have $\eta^b \neq \varepsilon$ or $\eta^c \neq \varepsilon$. Since φ is a Hopf algebra map, it follows from Lemma 2.12 that $\mu_b = 0$ or $u^c = 0$. By assumption, $\mu_l = 0$. Hence by (2.19), $u^a = 0$.

This proves (1) since the uniqueness of the extension follows from Lemma 2.9 and Lemma 2.10.

(2) By Lemma 2.10, the map $\mu \mapsto \varphi_\mu$ is injective. To prove surjectivity, let $\varphi : K(\mathcal{D})\#k[\Gamma] \to k[\Gamma]$ be a Hopf algebra map with $\varphi|\Gamma = \text{id}$. By Lemma 2.10, φ is defined by a family $(\mu_a)_{0 \neq a \in \mathbb{N}^p}$ of scalars. By (1), φ is determined by the values $\mu_{e_l}, 1 \leq l \leq p$. □

Definition 2.14. For any $\mu \in M(\mathcal{D})$ and $1 \leq l \leq p$, let φ_μ be the Hopf algebra map defined in Theorem 2.13, and

$$u_l(\mu) = \varphi_\mu(z_l) \in k[\Gamma].$$

If α is a positive root in Φ^+ with $\alpha = \beta_l$, we define $u_\alpha(\mu) = u_l(\mu)$.

Note that by (2.15), each $u_\alpha(\mu)$ lies in the augmentation ideal of $k[g_i^N \mid 1 \leq i \leq \theta]$.

3. Linking

3.1. **Notations.** In this Section we fix a finite abelian group Γ, and a datum $\mathcal{D} = \mathcal{D}(\Gamma, (g_i)_{1\le i\le\theta}, (\chi_i)_{1\le i\le\theta}, (a_{ij})_{1\le i,j\le\theta})$ of finite Cartan type. We follow the notations of the previous Section, in particular,

$$q_{ij} = \chi_j(g_i) \text{ for all } i, j.$$

For all $1 \le i, j \le \theta$ we write $i \sim j$ if i and j are in the same connected component of the Dynkin diagram of (a_{ij}). Let $\mathcal{X} = \{I_1, \dots, I_t\}$ be the set of connected components of $I = \{1, 2, \dots, \theta\}$. We assume for all $1 \le i \le \theta$

(3.1) q_{ii} has odd order, and

(3.2) the order of q_{ii} is prime to 3, if i lies in a component G_2.

For all $J \in \mathcal{X}$, let N_J be the common order of $q_{ii}, i \in J$.

As in Section 2.2, for all $J \in \mathcal{X}$ we choose a reduced decomposition of the longest element $w_{0,J}$ of the Weyl group W_J of the root system Φ_J of $(a_{ij})_{i,j\in J}$. Then for all $J, K \in \mathcal{X}$, $w_{0,J}$ and $w_{0,K}$ commute in the Weyl group W of the root system Φ of $(a_{ij})_{1\le i,j\le\theta}$, and

$$w_0 = w_{0,I_1} w_{0,I_2} \cdots w_{0,I_t}$$

gives a reduced representation of the longest element of W. For all $J \in \mathcal{X}$, let p_J be the number of positive roots in Φ_J^+, and

$$\Phi_J^+ = \{\beta_{J,1}, \dots, \beta_{J,p_J}\}$$

the corresponding convex ordering. Then

$$\Phi^+ = \{\beta_{I_1,1}, \dots, \beta_{I_1,p_{I_1}}, \dots, \beta_{I_t,1}, \dots, \beta_{I_t,p_{I_t}}\}$$

is the convex ordering corresponding to the reduced representation of $w_0 = w_{0,I_1} w_{0,I_2} \cdots w_{0,I_t}$. We also write

$$\Phi^+ = \{\beta_1, \dots, \beta_p\}, \; p = \sum_{J\in\mathcal{X}} p_J,$$

for this ordering.

In Section 2.1 we have defined root vectors x_α in the free algebra $k\langle x_1, \dots, x_\theta\rangle$ for each positive root in $\Phi_J^+ \subset \Phi, J \in \mathcal{X}$.

We recall a notion from [AS4].

Definition 3.1. A family $\lambda = (\lambda_{ij})_{1\le i<j\le\theta,\, i\not\sim j}$ of elements in k is called a *family of linking parameters for* $\mathcal{D}$ if the following condition is satisfied for all $1 \le i, j \le \theta$, $i \not\sim j$:

(3.3) If $g_i g_j = 1$ or $\chi_i\chi_j \ne \varepsilon$, then $\lambda_{ij} = 0$.

Vertices $1 \le i, j \le \theta$ are called *linkable* if $i \not\sim j$, $g_i g_j \ne 1$ and $\chi_i\chi_j = \varepsilon$.

It is useful to formally extend the notion of linking parameters by

$$\lambda_{ji} = -q_{ji}\lambda_{ij} \text{ for all } 1 \le i < j \le \theta,\ i \not\sim j. \tag{3.4}$$

Assume in addition that $\operatorname{ord}(q_{ii}) > 3$ for all i. We remark that any vertex i is linkable to at most one vertex j, and if i, j are linkable, then $q_{ii} = q_{jj}^{-1}$ [AS4, Section 5.1].

The free algebra $k\langle x_1, \dots, x_\theta\rangle$ is a braided Hopf algebra in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$ as explained in Section 1.1. Then $k\langle x_1, \dots, x_\theta\rangle \# k[\Gamma]$ is a Hopf algebra as in 1.2.

3.2. **The Hopf algebra $U(\mathcal{D}, \lambda)$.** We assume the situation of Section 3.1.

Definition 3.2. Let $\lambda = (\lambda_{ij})_{1\le i,j\le\theta,\, i\not\sim j}$ be a family of linking parameters for $\mathcal{D}$. Let $U(\mathcal{D}, \lambda)$ be the quotient Hopf algebra of $k\langle x_1, \dots, x_\theta\rangle \# k[\Gamma]$ modulo the ideal generated by

$$\operatorname{ad}_c(x_i)^{1-a_{ij}}(x_j), \text{ for all } 1 \le i, j \le \theta, i \sim j, i \ne j, \tag{3.5}$$

$$x_i x_j - q_{ij} x_j x_i - \lambda_{ij}(1 - g_i g_j), \text{ for all } 1 \le i < j \le \theta,\ i \not\sim j. \tag{3.6}$$

In (3.6) we can add the redundant elements

$$x_j x_i - q_{ji} x_i x_j - \lambda_{ji}(1 - g_j g_i), 1 \le i < j \le \theta,\ i \not\sim j.$$

Thus the definition of $U(\mathcal{D}, \lambda)$ does not depend on the ordering of the index set.

We denote the images of x_i and $g \in \Gamma$ in $U(\mathcal{D}, \lambda)$ again by x_i and g. The elements in (3.5) and (3.6) are skew-primitive. Hence $U(\mathcal{D}, \lambda)$ is a Hopf algebra with

$$\Delta(x_i) = g_i \otimes x_i + x_i \otimes 1,\ 1 \le i \le \theta.$$

In part (1) of the next theorem we adapt the method of proof of [AS4, Section 5.3] to find a basis of the infinite-dimensional Hopf algebra $U(\mathcal{D}, \lambda)$ in terms of the root vectors. In part (2) we prove a crucial skew-commutativity relation for the root vectors.

Theorem 3.3. *Let Γ be a finite abelian group, and $\mathcal{D}$ a datum of finite Cartan type satisfying* (3.1) *and* (3.2). *Let λ be a family of linking parameters for $\mathcal{D}$. Then*

(1) *The elements*

$$x_{\beta_1}^{a_1} x_{\beta_2}^{a_2} \cdots x_{\beta_p}^{a_p} g,\ a_1, a_2, \dots, a_p \ge 0, g \in \Gamma,$$

form a basis of the vector space $U(\mathcal{D}, \lambda)$.

(2) *Let $J \in \mathcal{X}$ and $\alpha \in \Phi^+, \beta \in \Phi^+_J$. Then $[x_\alpha, x_\beta^{N_J}]_c = 0$, that is,*

$$x_\alpha x_\beta^{N_J} = q_{\alpha\beta}^{N_J} x_\beta^{N_J} x_\alpha.$$

Proof. We proceed by induction on the number t of connected components.

If I is connected, (1) and (2) follow from Theorem 2.6.

If $t > 1$, let $I_1 = \{1, 2, \dots, \widetilde{\theta}\}$, $1 \le \widetilde{\theta} < \theta$. For all $1 \le i \le \widetilde{\theta}$, let l_i be the least common multiple of the orders of g_i and χ_i, $1 \le i \le \widetilde{\theta}$. Let $\widetilde{\Gamma} = \langle \widetilde{g}_1, \dots, \widetilde{g}_{\widetilde{\theta}} \mid \widetilde{g}_i\widetilde{g}_j = \widetilde{g}_j\widetilde{g}_i, \widetilde{g}_i^{l_i} = 1 \text{ for all } 1 \le i, j \le \widetilde{\theta}\rangle$, and for all $1 \le j \le \widetilde{\theta}$ let $\widetilde{\chi}_j$ be the character of $\widetilde{\Gamma}$ with $\widetilde{\chi}_j(\widetilde{g}_i) = \chi_j(g_i)$ for all $1 \le i \le \widetilde{\theta}$. Then we define

$$\mathcal{D}_1 = \mathcal{D}(\widetilde{\Gamma}, (\widetilde{g}_i)_{1\le i\le\widetilde{\theta}}, (\widetilde{\chi}_i)_{1\le i\le\widetilde{\theta}}, (a_{ij})_{1\le i,j\le\widetilde{\theta}}),$$
$$\mathcal{D}_2 = \mathcal{D}(\Gamma, (g_i)_{\widetilde{\theta}<i\le\theta}, (\chi_i)_{\widetilde{\theta}<i\le\theta}, (a_{ij})_{\widetilde{\theta}<i,j\le\theta}),$$

and $\lambda_2 = (\lambda_{ij})_{\widetilde{\theta}<i<j\le\theta, i\nsim j}$. Let $U = U(\mathcal{D}_1)$ (with empty family of linking parameters) with generators $x_1, \dots, x_{\widetilde{\theta}}$, and $\widetilde{g} \in \widetilde{\Gamma}$, and $A = U(\mathcal{D}_2, \lambda_2)$ with generators $y_{\widetilde{\theta}+1}, \dots, y_\theta$, and $g \in \Gamma$.

It is shown in [AS4, Lemma 5.19] that there are algebra maps γ_i, $(\varepsilon, \gamma_i)-$derivations δ_i and a Hopf algebra map φ,

$$\gamma_i : A \to k,\ \delta_i : A \to k,\ \varphi : U \to (A^0)^{\text{cop}},\ 1 \le i \le \widetilde{\theta},$$

such that for all $1 \le i \le \widetilde{\theta} < j \le \theta$,

$$\gamma_i|\Gamma = \chi_i, \quad \gamma_i(y_j) = 0,$$
$$\delta_i|\Gamma = 0, \quad \delta_i(y_j) = \lambda_{ji},$$
$$\varphi(\widetilde{g}_i) = \gamma_i, \quad \varphi(x_i) = \delta_i.$$

Then $\sigma : U \otimes A \otimes U \otimes A \to U \otimes A$, defined for all $u, v \in U, a, b \in A$ by

$$\sigma(u \otimes a, v \otimes b) = \varepsilon(u)\tau(v, a)\varepsilon(b),\ \tau(v, a) = \varphi(v)(a),$$

is a 2-cocycle on the tensor product Hopf algebra of U and A, and $(U \otimes A)_\sigma$ is the Hopf algebra with twisted multiplication defined in (1.11). Multiplication in $(U \otimes A)_\sigma$ is given for all $u, v \in U, a, b \in A$ by

$$(u \otimes a) \cdot_\sigma (v \otimes b) = u\tau(v_{(1)}, a_{(1)})v_{(2)} \otimes a_{(2)}\tau^{-1}(v_{(3)}, a_{(3)})b, \tag{3.7}$$

with $\tau^{-1}(u, a) = \varphi(u)(S^{-1}(a))$.

The group-like elements $\widetilde{g}_i \otimes g_i^{-1}$, $1 \le i \le \widetilde{\theta}$, are central in $(U \otimes A)_\sigma$, and as in the last part of the proof of [AS4, Theorem 5.17] it can be seen that the map $(U \otimes A)_\sigma \to U(\mathcal{D}, \lambda)$,

$$x_i \otimes 1 \mapsto x_i, \quad \widetilde{g}_i \otimes 1 \mapsto g_i, \quad 1 \otimes y_j \mapsto x_j, \quad 1 \otimes g \mapsto g$$

for all $1 \le i \le \widetilde{\theta} < j \le \theta$, $g \in \Gamma$, induces an isomorphism of Hopf algebras

$$(U \otimes A)_\sigma/(\widetilde{g}_i \otimes g_i^{-1} - 1 \otimes 1 \mid 1 \le i \le \widetilde{\theta}) \cong U(\mathcal{D}, \lambda). \tag{3.8}$$

Let $p_1 = p_{I_1}$. By induction and Theorem 2.6, the elements

$$x_{\beta_1}^{a_1} \cdots x_{\beta_{p_1}}^{a_{p_1}} \widetilde{g} \otimes y_{\beta_{p_1+1}}^{a_{p_1+1}} \cdots y_{\beta_p}^{a_p} g, \ a_1, \dots, a_p \geq 0, \widetilde{g} \in \widetilde{\Gamma}, g \in \Gamma,$$

are a basis of $U \otimes A$. It follows from (3.7) that for all $p_1 < l \leq p$ and $1 \leq i \leq \widetilde{\theta}$,

$$(1 \otimes y_{\beta_l}) \cdot_\sigma (\widetilde{g}_i \otimes 1) = \chi_i(g_{\beta_l}) \widetilde{g}_i \otimes y_{\beta_l}.$$

Hence

$$(x_{\beta_1}^{a_1} \cdots x_{\beta_{p_1}}^{a_{p_1}} \otimes y_{\beta_{p_1+1}}^{a_{p_1+1}} \cdots y_{\beta_p}^{a_p}) \cdot_\sigma (\widetilde{g} \otimes g), \ a_1, \dots a_p \geq 0, \widetilde{g} \in \widetilde{\Gamma}, g \in \Gamma,$$

is a basis of $(U \otimes A)_\sigma$.

Let $P = \{\widetilde{g} \otimes g \in (U \otimes A)_\sigma \mid \widetilde{g} \in \widetilde{\Gamma}, g \in \Gamma\}$, and let $\widetilde{P} \subset P$ be the subgroup generated by $\widetilde{g}_i \otimes g_i^{-1}$, $1 \leq i \leq \widetilde{\theta}$. Then

$$\Gamma \to P/\widetilde{P}, \ g \mapsto \overline{1 \otimes g},$$

is a group isomorphism. By (3.8), $(U \otimes A)_\sigma \otimes_{k[P]} k[P/\widetilde{P}] \cong U(\mathcal{D}, \lambda)$. Hence

$$x_{\beta_1}^{a_1} x_{\beta_2}^{a_2} \cdots x_{\beta_p}^{a_p} g, \ a_1, a_2, \dots, a_p \geq 0, g \in \Gamma,$$

is a basis of $U(\mathcal{D}, \lambda)$.

To prove (2), let $J = I_1, N = N_J, \widetilde{\theta} < i \leq \theta$ and $\beta \in \Phi_J^+$. We first show that

$$(1 \otimes y_i) \cdot_\sigma (x_\beta^N \otimes 1) = \chi_\beta^N(g_i)(x_\beta^N \otimes 1) \cdot_\sigma (1 \otimes y_i) \tag{3.9}$$

in $(U \otimes A)_\sigma$. We use the notations of Section 2.2 for $\mathcal{D}_1$ with

$$z^a = x_\beta^N, \text{ where } \beta = \beta_l, a = e_l \text{ for some } 1 \leq l \leq p_1.$$

By (2.17)

$$\Delta_U(x_\beta^N) = \widetilde{g}_\beta^N \otimes x_\beta^N + x_\beta^N \otimes 1 + \sum_{b,c \neq 0, \underline{b} + \underline{c} = \beta} t_{b,c}^a z^b h^c \otimes z^c.$$

Since $\Delta(y_i) = g_i \otimes y_i + y_i \otimes 1$, and

$$\Delta^2(y_i) = g_i \otimes g_i \otimes y_i + g_i \otimes y_i \otimes 1 + y_i \otimes 1 \otimes 1,$$

we have for all $u \in U$ by (3.7)

$$\begin{aligned} (1 \otimes y_i) \cdot_\sigma (u \otimes 1) &= \varphi(u_{(1)})(g_i) u_{(2)} \otimes g_i \varphi(u_{(3)})(S^{-1}(y_i)) \\ &+ \varphi(u_{(1)})(g_i) u_{(2)} \otimes y_i \varphi(u_{(3)})(1) \\ &+ \varphi(u_{(1)})(y_i) u_{(2)} \otimes 1 \varphi(u_{(3)})(1). \end{aligned}$$

It follows from the definition of φ that

$$\varphi(x_\gamma)(g) = 0 \text{ for all } \gamma \in \Phi_J^+, g \in \Gamma.$$

Hence to compute $(1 \otimes y_i) \cdot_\sigma (u \otimes 1)$ with $u = x_\beta^N$, we only need to take into account the term $\widetilde{g}_\beta^N \otimes x_\beta^N \otimes 1$ of $\Delta^2(x_\beta^N)$, and we obtain

$$\begin{aligned}(1 \otimes y_i) \cdot_\sigma (x_\beta^N \otimes 1) &= \varphi(\widetilde{g}_\beta^N)(y_{i(1)}) x_\beta^N \otimes y_{i(2)} \varphi(1)(S^{-1}(y_{i(3)})) \\ &= \varphi(\widetilde{g}_\beta^N)(y_{i(1)}) x_\beta^N \otimes y_{i(2)} \\ &= \varphi(\widetilde{g}_\beta^N)(g_i) x_\beta^N \otimes y_i + \varphi(\widetilde{g}_\beta^N)(y_i) x_\beta^N \otimes 1 \\ &= \chi_\beta^N(g_i)(x_\beta^N \otimes 1) \cdot_\sigma (1 \otimes y_i),\end{aligned}$$

since $\varphi(\widetilde{g}_\beta^N)|\Gamma = \chi_\beta^N$ and $\varphi(\widetilde{g}_\beta^N)(y_i) = 0$ by the definition of φ.

From (3.8) and (3.9) we see that for all simple roots $\alpha \in \Phi_K^+$ with $I_1 \neq K \in \mathcal{X}$ and all roots $\beta \in \Phi_J^+$ with $J = I_1$

$$x_\alpha x_\beta^{N_J} = \chi_\beta^{N_J}(g_\alpha) x_\beta^{N_J} x_\alpha \tag{3.10}$$

in $U(\mathcal{D}, \lambda)$. Since the root vectors x_α are homogeneous, (3.10) holds for all $\alpha \in \Phi_K^+, K \neq I_1$, and $\beta \in \Phi_{I_1}^+$. Since $U(\mathcal{D}, \lambda)$ and the root vectors $x_\alpha, \alpha \in \Phi^+$, do not depend on the order of the connected components, we can reorder the connected components and obtain (3.10) for all positive roots α, β lying in different connected components. For roots in the same connected component, (3.10) follows from Theorem 2.6. □

4. Finite-dimensional quotients

4.1. **A general criterion.** In this section we prove a generalized version of Theorem [AS5, 6.24].

Let Γ be an abelian group, A an algebra containing the group algebra $k[\Gamma]$ as a subalgebra and $p \geq 1$. We assume

$$y_1, \dots, y_p \in A, h_1, \dots, h_p \in \Gamma, \psi_1, \dots, \psi_p \in \widehat{\Gamma}, \text{ and } N_1, \dots, N_p \geq 1,$$

such that

$$g y_l = \psi_l(g) y_l g, \text{ for all } 1 \leq l \leq p, g \in \Gamma, \tag{4.1}$$

$$y_k y_l^{N_l} = \psi_l^{N_l}(h_k) y_l^{N_l} y_k, \text{ for all } 1 \leq k, l \leq p, \tag{4.2}$$

$$y_1^{a_1} \cdots y_p^{a_p} g, \ a_1, \cdots, a_p \geq 0, g \in \Gamma, \text{ form a basis of } A. \tag{4.3}$$

Let $\mathbb{T} = \{t = (t_1, \dots, t_p) \in \mathbb{N}^p \mid 0 \leq t_l < N_l \text{ for all } 1 \leq l \leq p\}$. For all $a = (a_1, \dots, a_p) \in \mathbb{N}^p$, we define

$$\begin{aligned} y^a &= y_1^{a_1} \cdots y_p^{a_p}, \\ \psi^a &= \psi_1^{a_1} \cdots \psi_p^{a_p}, \\ aN &= (a_1 N_1, \dots, a_p N_p). \end{aligned}$$

Hence any element $v \in A$ can be written as

$$v = \sum_{t\in\mathbb{T},a\in\mathbb{N}^p} y^t y^{aN} v_{t,a},\ v_{t,a} \in k[\Gamma] \text{ for all } t \in \mathbb{T}, a \in \mathbb{N}^p, \tag{4.4}$$

where the coefficients $v_{t,a} \in k[\Gamma]$ are uniquely determined.

In [AS5] we assumed that $A = R\#k[\Gamma]$, and the subalgebra R of A generated by $y_1, \dots, y_p$ had the basis $y_1^{a_1} \cdots y_p^{a_p}, a_1, \dots, a_p \geq 0$. To see that [AS5, Theorem 6.24] extends to the more general situation considered here we first prove a more general version of [AS5, Lemma 6.23].

We need the following commutation rules for the generators of A. For all $a, b \in \mathbb{N}^p, 1 \leq l \leq p$,

$$y^{aN} y_l = y_l y^{aN} \psi^{aN}(h_l^{-1}), \tag{4.5}$$

$$y^{bN} y^{aN} = y^{(a+b)N} \psi^{aN}(g(b)), \tag{4.6}$$

where $g(b) = (g_1(b), \dots, g_p(b)) \in \Gamma^p$ is a family of elements in Γ depending on b, and $\psi^{aN}(g(b)) = \psi^{a_1 N_1}(g_1(b)) \cdots \psi^{a_p N_p}(g_p(b))$. Both equations (4.5) and (4.6) follow from (4.2). To prove (4.6) we write

$$\begin{aligned} y^{bN} y^{aN} &= y_1^{b_1 N_1} \cdots y_p^{b_p N_p} y_1^{a_1 N_1} \cdots y_p^{a_p N_p} \\ &= y_1^{b_1 N_1} \cdots y_{p-1}^{b_{p-1} N_{p-1}} y_1^{a_1 N_1} \cdots y_{p-1}^{a_{p-1} N_{p-1}} y_p^{(a_p + b_p) N_p} \\ &\times (\psi_1^{a_1 N_1} \cdots \psi_{p-1}^{a_{p-1} N_{p-1}})(h_p^{b_p N_p}) \end{aligned}$$

and continue in this way.

For any character $\psi \in \widehat{\Gamma}$ let $\widetilde{\psi} : k[\Gamma] \to k[\Gamma]$ be the algebra map with $\widetilde{\psi}(g) = \psi(g) g$ for all $g \in \Gamma$. Thus

$$\widehat{\Gamma} \to \mathrm{Aut}(k[\Gamma]),\ \psi \mapsto \widetilde{\psi},$$

is a group homomorphism. Then for all $1 \leq l \leq p, a \in \mathbb{N}^p$ and $v \in k[\Gamma]$ it follows from (4.1) that

$$v y_l = y_l \widetilde{\psi}_l(v), \tag{4.7}$$

$$v y^{aN} = y^{aN} \widetilde{\psi^{aN}}(v). \tag{4.8}$$

Lemma 4.1. *Let $u_l \in k[\Gamma], 1 \leq l \leq p$, be a family of central elements in A and assume for all $1 \leq l \leq p$ that $u_l = 0$ whenever $\psi_l^{N_l} \neq \varepsilon$. Let M be a free right $k[\Gamma]$-module with basis $m(t), t \in \mathbb{T}$, and define a right $k[\Gamma]$-linear map by*

$$\varphi : A \to M,\ y^t y^{aN} \mapsto m(t) u^a \text{ for all } t \in \mathbb{T}, a \in \mathbb{N}^p,$$

where $u^a = u_1^{a_1} \cdots u_p^{a_p}$ for all $a = (a_1, \dots, a_p) \in \mathbb{N}^p$. Then the kernel of φ is a right ideal of A.

Proof. Let $z \in A$ be an element of the kernel of φ. We have to show that $\varphi(zy_l) = 0$ for all $1 \leq l \leq p$. We fix an arbitrary $1 \leq l \leq p$ and by (4.4) we have basis representations

$$z = \sum_{s \in \mathbb{T}, a \in \mathbb{N}^p} y^s y^{aN} v_{s,a}, \tag{4.9}$$

$$y^s y_l = \sum_{t \in \mathbb{L}, b \in \mathbb{N}^p} y^t y^{bN} w^s_{t,b} \text{ for all } s \in \mathbb{T}, \tag{4.10}$$

where the $v_{s,a}$ and the $w^s_{t,b}$ are elements in $k[\Gamma]$.

To compute zy_l we multiply (4.9) with y_l and then use (4.7), (4.5), (4.10), (4.8) and (4.6) to obtain

$$\begin{aligned} zy_l &= \sum_{s \in \mathbb{T}, a \in \mathbb{N}^p} y^s y^{aN} v_{s,a} y_l \\ &= \sum_{\substack{s,t \in \mathbb{L} \\ a,b \in \mathbb{N}^p}} y^t y^{(a+b)N} \psi^{aN}(g(b)) \widetilde{\psi^{aN}}(w^s_{t,b}) \psi^{aN}(h_l^{-1}) \widetilde{\psi}_l(v_{s,a}). \end{aligned}$$

We note that $u_l^{a_l} \psi_l^{a_l N_l}(g) = u_l^{a_l}$ for all $1 \leq l \leq p, a_l \in \mathbb{N}$ and $g \in \Gamma$, since $u_l = 0$ whenever $\psi_l^{N_l} \neq \varepsilon$. Thus for all $a, b \in \mathbb{N}^p, s, t \in \mathbb{L}$

$$u^a \psi^{aN}(g(b)) = u^a, \; u^a \psi^{aN}(h_l^{-1}) = u^a, \tag{4.11}$$

$$u^a \widetilde{\psi^{aN}}(w^s_{t,b}) = u^a w^s_{t,b}. \tag{4.12}$$

Since u^a is central in A it follows from (4.3) that

$$u^a \widetilde{\psi}_l(v_{s,a}) = \widetilde{\psi}_l(u^a v_{s,a}). \tag{4.13}$$

Hence

$$\begin{aligned} \varphi(zy_l) &= \sum_{\substack{s,t \in \mathbb{T} \\ a,b \in \mathbb{N}^p}} m(t) u^{a+b} \psi^{aN}(g(b)) \widetilde{\psi^{aN}}(w^s_{t,b}) \psi^{aN}(h_l^{-1}) \widetilde{\psi}_l(v_{s,a}) \\ &= \sum_{\substack{s,t \in \mathbb{T} \\ b \in \mathbb{N}^p}} m(t) u^b w^s_{t,b} \widetilde{\psi}_l(\sum_{a \in \mathbb{N}^p} u^a v_{s,a}), \end{aligned}$$

where the second equality follows from (4.11), (4.12) and (4.13). This proves our claim since $\varphi(z) = 0$, and therefore

$$\sum_{a \in \mathbb{N}^p} u^a v_{s,a} = 0 \text{ for all } s \in \mathbb{T}.$$

□

Theorem 4.2. *Assume the situation above. Let $u_l \in k[\Gamma], 1 \leq l \leq p$, be a family of elements in the group algebra. Then the following are equivalent:*

(1) *The residue classes of* $y^t g,\ t \in \mathbb{T}, g \in \Gamma$, *form a basis of the quotient algebra* $A/(y_l^{N_l} - u_l \mid 1 \le l \le p)$.
(2) *For all* $1 \le l \le p$, u_l *is central in* A, *and if* $\psi_l^{N_l} \neq \varepsilon$, *then* $u_l = 0$.

Proof. As in the proof of [AS5, Theorem 6.24] this follows from Lemma 4.1. □

4.2. **The Hopf algebra $u(\mathcal{D}, \lambda, \mu)$.** Let Γ be a finite abelian group, and $\mathcal{D} = \mathcal{D}(\Gamma, (g_i)_{1\le i\le\theta}, (\chi_i)_{1\le i\le\theta}, (a_{ij})_{1\le i,j\le\theta})$ a datum of finite Cartan type. We assume the situation of Section 3.1.

Definition 4.3. A family $\mu = (\mu_\alpha)_{\alpha\in\Phi^+}$ of elements in k is called a *family of root vector parameters for* $\mathcal{D}$ if the following condition is satisfied for all $\alpha \in \Phi_J^+, J \in \mathcal{X}$: If $g_\alpha^{N_J} = 1$ or $\chi_\alpha^{N_J} \neq \varepsilon$, then $\mu_\alpha = 0$.

Let μ be a family of root vector parameters for $\mathcal{D}$. For all $J \in \mathcal{X}$, and $\alpha \in \Phi_J^+$, we define

$$\pi_J(\mu) = (\mu_\beta)_{\beta\in\Phi_J^+}, \text{ and } u_\alpha(\mu) = u_\alpha(\pi_J(\mu)), \tag{4.14}$$

where $u_\alpha(\pi_J(\mu))$ is introduced in Definition 2.14. Let λ be a family of linking parameters for $\mathcal{D}$. Then we define

$$u(\mathcal{D}, \lambda, \mu) = U(\mathcal{D}, \lambda)/(x_\alpha^{N_J} - u_\alpha(\mu) \mid \alpha \in \Phi_J^+, J \in \mathcal{X}). \tag{4.15}$$

By abuse of language we still write x_i and g for the images of x_i and $g \in \Gamma$ in $u(\mathcal{D}, \lambda, \mu)$. For all $1 \le l \le p$, we define $N_l = N_J$, if $\beta_l \in \Phi_J^+, J \in \mathcal{X}$.

Lemma 4.4. *Let* $\mathcal{D}, \lambda$ *and* μ *as above, and* $\alpha \in \Phi^+$. *Then* $u_\alpha(\mu)$ *is central in* $U(\mathcal{D}, \lambda)$.

Proof. Let $\alpha \in \Phi_J^+$, where $J \in \mathcal{X}$, and $N = N_J$. To simplify the notation, we assume $J = I_1 = \{1, 2, \dots, \widetilde{\theta}\}$, and $\Phi_J^+ = \{\beta_1, \beta_2, \dots, \beta_{p_1}\}$. We apply the results and notations of Section 2.2 to the connected component I_1. Let $\widetilde{\mu} = (\mu_l)_{1\le l\le p_1}, \mathcal{D}_1 = \mathcal{D}(\Gamma, (g_i)_{1\le i\le\widetilde{\theta}}, (\chi_i)_{1\le i\le\widetilde{\theta}}, (a_{i,j})_{1\le i,j\le\widetilde{\theta}})$, and

$$\varphi_{\widetilde{\mu}} : K(\mathcal{D}_1)\#k[\Gamma] \to k[\Gamma]$$

the Hopf algebra map defined by $\widetilde{\mu}$ in Theorem 2.13. As in Section 2.2 we define $u^a = \varphi_{\widetilde{\mu}}(z^a)$ for all $0 \neq a \in \mathbb{N}^{p_1}$. Thus $u_\alpha(\mu) = u^{e_l}$ where $\alpha = \beta_l$.

We show by induction on $\mathrm{ht}(\underline{a})$ that

$$x_i u^a = u^a x_i \text{ for all } 1 \le i \le \theta, 0 \neq a \in \mathbb{N}^{p_1}. \tag{4.16}$$

To prove (4.16) we can assume that $u^a \neq 0$. By (2.15) it suffices to show that

$$x_i h^a = h^a x_i \text{ for all } 1 \leq i \leq \theta, 0 \neq a \in \mathbb{N}^{p_1} \text{ with } u^a \neq 0. \tag{4.17}$$

Recall that $h^a = g_{\beta_1}^{Na_1} \cdots g_{\beta_{p_1}}^{Na_{p_1}}$ for $a = (a_1, \dots, a_{p_1})$.

Let $1 \leq i \leq \theta$ and $0 \neq a \in \mathbb{N}^{p_1}$ with $u^a \neq 0$. Let $1 \leq l \leq p_1$ and $\beta_l = \sum_{j=1}^{\widetilde{\theta}} n_j \alpha_j$, where $n_j \in \mathbb{N}$ for all $1 \leq j \leq \widetilde{\theta}$. Then by definition, $g_{\beta_l} = \prod_{1 \leq j \leq \widetilde{\theta}} g_j^{n_j}$, and $\chi_{\beta_l} = \prod_{1 \leq j \leq \widetilde{\theta}} \chi_j^{n_j}$. Hence

$$\chi_i(g_{\beta_l}^N)\chi_{\beta_l}^N(g_i) = \prod_{1 \leq j \leq \widetilde{\theta}} q_{ii}^{a_{ij} N n_j} = 1,$$

since $q_{ii}^N = 1$, if $i \in I_1$, and $a_{ij} = 0$, if $i \notin I_1$. Since $u^a \neq 0$, it follows from Lemma 2.12 that $\chi_{\beta_l}^N = \varepsilon$ for all $1 \leq l \leq p_1$ with $a_l > 0$. Hence $\chi_i(g_{\beta_l}^N) = 1$ for all l with $a_l > 0$. This implies (4.17) since

$$h^a x_i = \chi_i(h^a) x_i h^a.$$

□

Theorem 4.5. *Let $\mathcal{D}$ be a datum of finite Cartan type satisfying* (3.1) *and* (3.2). *Let λ and μ be families of linking and root vector parameters for $\mathcal{D}$. Then $u(\mathcal{D}, \lambda, \mu)$ is a quotient Hopf algebra of $U(\mathcal{D}, \lambda)$ with group-like elements $G(u(\mathcal{D}, \lambda, \mu)) \cong \Gamma$, and the elements*

$$x_{\beta_1}^{a_1} x_{\beta_2}^{a_2} \cdots x_{\beta_p}^{a_p} g, \ 0 \leq a_l < N_l, \ 1 \leq l \leq p, \ g \in \Gamma$$

form a basis of $u(\mathcal{D}, \lambda, \mu)$. In particular,

$$\dim u(\mathcal{D}, \lambda, \mu) = \prod_{J \in \mathcal{X}} N_J^{|\Phi_J^+|} |\Gamma|.$$

Proof. By Theorem 3.3, the elements

$$x_{\beta_1}^{a_1} x_{\beta_2}^{a_2} \cdots x_{\beta_p}^{a_p} g, \ 0 \leq a_l, \ 1 \leq l \leq p, \ g \in \Gamma$$

are a basis of $U(\mathcal{D}, \lambda)$. We want to apply Theorem 4.2 to $U(\mathcal{D}, \lambda)$ and

$$y_l = x_{\beta_l}, \ h_l = g_{\beta_l}, \ \psi_l = \chi_{\beta_l}, \ u_l = u_{\beta_l}(\mu), \ 1 \leq l \leq p.$$

Conditions (4.2), (4.3) are satisfied by Theorem 3.3. To check the conditions in Theorem 4.2 (2) we apply for each connected component $J \in \mathcal{X}$ the results of Section 2.2 with

$$\eta_l = \chi_{\beta_l}^{N_l}, \ 1 \leq l \leq p, \ \beta_l \in \Phi_J^+.$$

Since φ_μ is a Hopf algebra map by Theorem 2.13 it follows from Lemma 2.12 that $u_{\beta_l}(\mu) = 0$ if $\chi_{\beta_l}^{N_l} \neq \varepsilon$. By Lemma 4.4, $u_{\beta_l}(\mu)$ is central in $U(\mathcal{D}, \lambda)$. Hence the claim about the basis of $u(\mathcal{D}, \lambda, \mu)$ follows from Theorem 4.2.

We now show that $u(\mathcal{D}, \lambda, \mu)$ is a Hopf algebra. Let $J \in \mathcal{X}$. We denote the restriction of $\mathcal{D}$ to the connected component J by $\mathcal{D}_J$. By Theorem 2.13, the map $\varphi_\mu : K(\mathcal{D}_J)\#k[\Gamma] \to k[\Gamma]$ is a Hopf algebra homomorphism. The kernel of φ_μ is generated by all $x_\alpha^{N_J} - u_\alpha(\mu)$ with $\alpha \in \Phi_J^+$. Hence the elements $x_\alpha^{N_J} - u_\alpha(\mu), \alpha \in \Phi_J^+$, generate a Hopf ideal in $K(\mathcal{D}_J)\#k[\Gamma]$ and in $U(\mathcal{D}, \lambda)$.

The Hopf algebra $u(\mathcal{D}, \lambda, \mu)$ is generated by the skew-primitive elements $x_1, \dots, x_\theta$ and the image of Γ. Hence $G(u(\mathcal{D}, \lambda, \mu)) \cong \Gamma$. □

For explicit examples of the Hopf algebras $u(\mathcal{D}, \lambda, \mu)$ see [AS5, Section 6] for type $A_n, n \geq 1$, and [BDR] for type B_2. In these papers, and for these types, the elements $u_\alpha(\mu)$ are precisely written down. It is an interesting problem to find an explicit algorithm describing the elements $u_\alpha(\mu)$ for any connected Dynkin diagram.

5. The associated graded Hopf algebra

5.1. **Nichols algebras.** To determine the structure of a given pointed Hopf algebra, we proceed as in [AS1] and study the associated graded Hopf algebra.

Let A be a pointed Hopf algebra with group of group-like elements $G(A) = \Gamma$. Let

$$A_0 = k[\Gamma] \subset A_1 \subset \cdots \subset A, \ A = \cup_{n\geq 0} A_n$$

be the coradical filtration of A. We define the associated graded Hopf algebra [M, 5.2.8] by

$$\operatorname{gr}(A) = \oplus_{n\geq 0} A_n/A_{n-1}, \ A_{-1} = 0.$$

Then $\operatorname{gr}(A)$ is a pointed Hopf algebra with the same dimension and coradical as A. The projection map $\pi : \operatorname{gr}(A) \to k[\Gamma]$ and the inclusion $\iota : k[\Gamma] \to \operatorname{gr}(A)$ are Hopf algebra maps with $\pi\iota = \mathrm{id}_{k[\Gamma]}$. Let

$$R = \{x \in \operatorname{gr}(A) \mid (\mathrm{id} \otimes \pi)\Delta(x) = x \otimes 1\} \tag{5.1}$$

be the algebra of $k[\Gamma]$-coinvariant elements. Then $R = \oplus_{n\geq 0} R(n)$ is a graded Hopf algebra in ${}^\Gamma_\Gamma\mathcal{YD}$, and by (1.9)

$$\operatorname{gr}(A) \cong R\#k[\Gamma]. \tag{5.2}$$

Let $V = P(R) \in {}^\Gamma_\Gamma\mathcal{YD}$ be the Yetter-Drinfeld module of primitive elements in R. We call its braiding

$$c : V \otimes V \to V \otimes V$$

the *infinitesimal braiding of A*.

Let $\mathfrak{B}(V)$ be the subalgebra of R generated by V. Thus $B = \mathfrak{B}(V)$ is the *Nichols algebra* of V [AS2], that is,

(5.3) $B = \oplus_{n\geq 0} B(n)$ is a graded Hopf algebra in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$,

(5.4) $B(0) = k1, \; B(1) = V$,

(5.5) $B(1) = P(B)$,

(5.6) B is generated as an algebra by B(1).

$\mathfrak{B}(V)$ only depends on the vector space V with its Yetter-Drinfeld structure (see the discussion in [AS5, Section 2]). As an algebra and coalgebra, $\mathfrak{B}(V)$ only depends on the braided vector space (V, c).

We assume in addition that A is finite-dimensional and Γ is abelian. Then there are $g_1, \dots, g_\theta \in \Gamma$, $\chi_1, \dots, \chi_\theta \in \widehat{\Gamma}$ and a basis $x_1, \dots, x_\theta$ of V such that $x_i \in V_{g_i}^{\chi_i}$ for all $1 \leq i \leq \theta$. We call

$$(q_{ij} = \chi_j(g_i))_{1\leq i,j\leq\theta}$$

the *infinitesimal braiding matrix of* A.

A braiding matrix $(q_{ij})_{1\leq i,j\leq\theta}$ whose entries q_{ij} are roots of unity is of *Cartan type* if $q_{ii} \neq 1$ for all $1 \leq i \leq \theta$, and if there are integers $a_{ij}, 1 \leq i,j \leq \theta$, such that for all $1 \leq i,j \leq \theta$

$$q_{ij}q_{ji} = q_{ii}^{a_{ij}}.$$

We can assume that $a_{ii} = 2$ for all $1 \leq i \leq \theta$, and

$$-\mathrm{ord}(q_{ii}) < a_{ij} \leq 0 \text{ for all } 1 \leq i,j \leq \theta.$$

Then the matrix (a_{ij}) is uniquely determined. It is a generalized Cartan matrix and is called the *Cartan matrix of* (q_{ij}) [AS2].

The first step to classify pointed Hopf algebras is the computation of the Nichols algebra. We begin with the description of Nichols algebras of Yetter-Drinfeld modules of finite Cartan type.

Theorem 5.1. *Let $\mathcal{D} = \mathcal{D}(\Gamma, (g_i)_{1\leq i\leq\theta}, (\chi_i)_{1\leq i\leq\theta}, (a_{ij})_{1\leq i,j\leq\theta})$ be a datum of finite Cartan type with finite abelian group Γ. Assume* (3.1) *and* (3.2)*. Let $V \in {}^{\Gamma}_{\Gamma}\mathcal{YD}$ be a vector space with basis $x_1, \dots, x_\theta$ and $x_i \in V_{g_i}^{\chi_i}$ for all $1 \leq i \leq \theta$. Then $\mathfrak{B}(V)$ is the quotient algebra of $T(V)$ modulo the ideal generated by the elements*

(5.7) $\mathrm{ad}_c(x_i)^{1-a_{ij}}(x_j)$ *for all* $1 \leq i,j \leq \theta, i \neq j$,

(5.8) $x_\alpha^{N_J}$ *for all* $\alpha \in \Phi_J^+, J \in \mathcal{X}$.

Proof. Using results of Lusztig [L1],[L2], Rosso [Ro] and Müller [M1] and twisting we proved this theorem in [AS4, Theorem 4.5] assuming in addition that $\mathrm{ord}(q_{ij})$ is odd for all $1 \leq i,j \leq \theta, i \neq j$. By Lemma

2.3 the proof of [AS4, Theorem 4.5] works without this additional assumption. □

Corollary 5.2. *Assume the situation of Theorem 5.1, and let λ and μ be linking and root vector parameters for $\mathcal{D}$. Then*

$$\operatorname{gr}(u(\mathcal{D},\lambda,\mu)) \cong u(\mathcal{D},0,0) \cong \mathfrak{B}(V)\#k[\Gamma].$$

Proof. Let $A = u(\mathcal{D},\lambda,\mu)$. There is a well-defined Hopf algebra map

$$u(\mathcal{D},0,0) \to \operatorname{gr}(u(\mathcal{D},\lambda,\mu)),$$

mapping $x_i, 1 \le i \le \theta$, onto the residue class of x_i in A_1/A_0, and $g \in \Gamma$ onto g. Since $\dim(u(\mathcal{D},0,0)) = \dim(u(\mathcal{D},\lambda,\mu)) = \dim(\operatorname{gr}(u(\mathcal{D},\lambda,\mu))$ by Theorem 4.5, it follows that $u(\mathcal{D},0,0) \cong \operatorname{gr}(u(\mathcal{D},\lambda,\mu))$. By Theorem 5.1, $u(\mathcal{D},0,0) \cong \mathfrak{B}(V)\#k[\Gamma]$. □

In [AS2] and [AS4] we determined the structure of finite-dimensional Nichols algebras assuming that V is of Cartan type and satisfies some more assumptions in the case of small orders (≤ 17) of the diagonal elements q_{ii}. Recent results of Heckenberger [H1], [H2], [H3] together with Theorem 5.1 allow to prove the following very general structure theorem on Nichols algebras.

Theorem 5.3. *Let Γ be a finite abelian group, and $V \in {}^{\Gamma}_{\Gamma}\mathcal{YD}$ a Yetter-Drinfeld module such that $\mathfrak{B}(V)$ is finite-dimensional. Choose a basis $x_i \in V$ with $x_i \in V_{g_i}^{\chi_i}, g_i \in \Gamma, \chi_i \in \widehat{\Gamma}$, for all $1 \le i \le \theta$. For all $1 \le i,j \le \theta$, define $q_{ij} = \chi_j(g_i)$, and assume*

(5.9) $\operatorname{ord}(q_{ii})$ *is odd,*

(5.10) $\operatorname{ord}(q_{ii})$ *is prime to 3 if $q_{il}q_{li} \in \{q_{ii}^{-3}, q_{ll}^{-3}\}$ for some l,*

(5.11) $\operatorname{ord}(q_{ii}) > 3$.

Then there is a datum $\mathcal{D} = \mathcal{D}(\Gamma, (g_i)_{1\le i\le\theta}, (\chi_i)_{1\le i\le\theta}, (a_{ij})_{1\le i,j\le\theta})$ of finite Cartan type such that

$$\mathfrak{B}(V)\#k[\Gamma] \cong u(\mathcal{D},0,0).$$

Proof. Since $\mathfrak{B}(V)\#k[\Gamma]$ is finite-dimensional, $q_{ii} \ne 1$ for all $1 \le i \le \theta$ by [AS1, Lemma 3.1].

For all $1 \le i,j \le \theta, i \ne j$, let V_{ij} be the vector subspace of V spanned by x_i, x_j. Then $\mathfrak{B}(V_{ij})$ is isomorphic to a subalgebra of $\mathfrak{B}(V)$, hence it is finite-dimensional. Heckenberger [H1], [H2] classified finite-dimensional Nichols algebras of rank 2. By (5.9) and (5.11) it follows from the list in [H1, Theorem 4] that V_{ij} is of finite Cartan type, that is, there are $a_{ij}, a_{ji} \in \{0,-1,-2,-3\}$ with $a_{ij}a_{ji} \in \{0,1,2,3\}$, and

$$q_{ij}q_{ji} = q_{ii}^{a_{ij}} = q_{jj}^{a_{ji}}.$$

Thus $(q_{ij})_{1\le i,j\le\theta}$ is of Cartan type with generalized Cartan matrix (a_{ij}). In [H3, Theorem 4] Heckenberger extended part (ii) of [AS2, Theorem 1.1] (where we had to exclude some small primes) and showed that a diagonal braiding (q_{ij}) of a braided vector space V is of finite Cartan type if it is of Cartan type and $\mathfrak{B}(V)$ is finite-dimensional. Hence (a_{ij}) is a Cartan matrix of finite type, and the claim follows from Theorem 5.1. □

5.2. **Generation in degree one.** We generalize our results in [AS4, Section 7]. Let A be a finite-dimensional pointed Hopf algebra with Γ, V, and R as in Section 5.1. To prove that $\mathfrak{B}(V) = R$, we dualize. Let $S = R^*$ the dual Hopf algebra in ${}^\Gamma_\Gamma\mathcal{YD}$ as in [AS2, Lemma 5.5]. Then $S = \oplus_{n\ge0} S(n)$ is a graded Hopf algebra in ${}^\Gamma_\Gamma\mathcal{YD}$, and by [AS2, Lemma 5.5], R is generated in degree one, that is, $\mathfrak{B}(V) = R$, if and only if $P(S) = S(1)$. The dual vector space $S(1)$ of $V = R(1)$ has the same braiding (q_{ij}) (with respect to the dual basis) as V. Our strategy to show $P(S) = S(1)$ is to identify S as a Nichols algebra. In the next Lemma we use [H1, H2] to prove a very general version of [AS4, Lemma 7.2].

Lemma 5.4. *Let $\mathcal{D} = \mathcal{D}(\Gamma, (g_i)_{1\le i\le\theta}, (\chi_i)_{1\le i\le\theta}, (a_{ij})_{1\le i,j\le\theta})$ be a datum of finite Cartan type with finite abelian group Γ. Let $S = \oplus_{n\ge0} S_n$ be a finite-dimensional graded Hopf algebra in ${}^\Gamma_\Gamma\mathcal{YD}$ with $S(0) = k1$, and let $x_1, \dots, x_\theta$ be a basis of $S(1)$ with $x_i \in S(1)^{\chi_i}_{g_i}$ for all $1 \le i \le \theta$. Assume* (5.9) *and*

$$\operatorname{ord}(q_{ii}) > 7 \text{ for all } 1 \le i \le \theta. \tag{5.12}$$

Then

$$\operatorname{ad}_c(x_i)^{1-a_{ij}}(x_j) = 0 \text{ for all } 1 \le i, j \le \theta,\ i \ne j. \tag{5.13}$$

Proof. We first note that the Nichols algebra of the primitive elements $P(S) \in {}^\Gamma_\Gamma\mathcal{YD}$ is finite-dimensional. This can be seen by looking at $\operatorname{gr}(S\#k[\Gamma])$.

Assume that there are $1 \le i, j \le \theta, i \ne j$, with $\operatorname{ad}_c(x_i)^{1-a_{ij}}(x_j) \ne 0$. We define

$$y_1 = x_i, \quad y_2 = \operatorname{ad}_c(x_i)^{1-a_{ij}}(x_j).$$

By [AS2, A.1], y_2 is a primitive element. Since y_1, y_2 are non-zero elements of different degree, they are linearly independent. We know that the Nichols algebra of $W = ky_1 + ky_2$ is finite-dimensional, since $\mathfrak{B}(P(S))$ is finite-dimensional. We denote

$$h_1 = g_i,\ h_2 = g_i^{1-a_{ij}} g_j \in \Gamma, \text{ and } \eta_1 = \chi_i,\ \eta_2 = \chi_i^{1-a_{ij}}\chi_j \in \widehat{\Gamma}.$$

Thus $y_i \in S_{h_i}^{\eta_i}, 1 \leq i \leq 2$. Let $(Q_{ij} = \eta_j(h_i))_{1\leq i,j\leq 2}$ be the braiding matrix of y_1, y_2. We compute

$$Q_{11} = q_{ii}, \quad Q_{22} = q_{ii}^{1-a_{ij}} q_{jj}, \quad Q_{12}Q_{21} = q_{ii}^{2-a_{ij}}.$$

By assumption, the order of $Q_{11} = q_{ii}$ is odd and > 3. Since $\mathfrak{B}(W)$ is finite-dimensional, $Q_{22} \neq 1$ by [AS1, Lemma 3.1]. Thus Q_{22} has odd order, since the orders of q_{ii}, q_{jj} are odd. By checking Heckenberger's list in [H1, Theorem 4], and thanks to [H2], we see that the braiding (Q_{ij}) is of finite Cartan type or that we are in case (T3) with

$$Q_{12}Q_{21} = Q_{11}^{-1}.$$

Hence there exists $A_{12} \in \{0, -1, -2, -3\}$ with

$$Q_{12}Q_{21} = Q_{11}^{A_{12}}.$$

Since $Q_{12}Q_{21} = q_{ii}^{2-a_{ij}}$, and $Q_{11} = q_{ii}$, it follows that the order of q_{ii} divides $2 - a_{ij} - A_{12} \in \{2, 3, 4, 5, 6, 7, 8\}$. This is a contradiction since the order of q_{ii} is odd and > 7. □

The next theorem is one of the main results of this paper.

Theorem 5.5. *Let A be a finite-dimensional pointed Hopf algebra with abelian group $G(A) = \Gamma$ and infinitesimal braiding matrix $(q_{ij})_{1\leq i,j\leq\theta}$. Assume* (5.9), (5.10) *and* (5.12). *Then A is generated by group-like and skew-primitive elements, that is,*

$$R = \mathfrak{B}(V),$$

where R is defined by (5.1), *and $V = R(1)$.*

Proof. We argue as in the proof of [AS4, Theorem 7.6]. Let $S = R^*$ be the dual Hopf algebra in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$. Then $S(1) = R(1)^*$ has the same braiding (q_{ij}) as $R(1)$ with respect to the dual basis (x_i) of the corresponding basis of $R(1)$. By Theorem 5.3 (q_{ij}) is of finite Cartan type. By Lemma 5.4 the Serre relations (5.7) hold for the elements x_i. Then the root vector relations (5.8) follow by [AS4, Lemma 7.5]. Hence $S \cong \mathfrak{B}(S(1))$ by Theorem 5.1, and $S(1) = P(S)$. By duality, R is a Nichols algebra. □

6. Lifting

From Section 5 we know a presentation of $\mathrm{gr}(A)$ by generators and relations under the assumptions of Theorems 5.3 and 5.5. To lift this presentation to A we need the following formulation of [AS1, Lemma 5.4] which is a consequence of the theorem of Taft and Wilson [M, Theorem 5.4.1]. Here it is crucial that the group is abelian.

Lemma 6.1. *Let A be a finite-dimensional pointed Hopf algebra with abelian group $G(A) = \Gamma$. Write* $\operatorname{gr}(A) \cong R\#k[\Gamma]$ *as in* (5.2)*, and let $V = R(1)$ with basis $x_i \in V_{g_i}^{\chi_i}, g_i \in \Gamma, \chi_i \in \widehat{\Gamma}, 1 \leq i \leq \theta$. Let $A_0 \subset A_1$ be the first two terms of the coradical filtration of A. Then*

(6.1) $\quad \oplus_{g,h\in\Gamma,\varepsilon\neq\chi\in\widehat{\Gamma}} P_{g,h}^{\chi}(A) \xrightarrow{\cong} A_1/A_0 \xleftarrow{\cong} V\#k[\Gamma].$

(6.2) *For all* $g \in \Gamma, P_{g,1}(A)^{\varepsilon} = k(1-g),$ *and if* $\varepsilon \neq \chi \in \widehat{\Gamma},$ *then*

(6.3) $P_{g,1}(A)^{\chi} \neq 0 \iff g = g_i, \chi = \chi_i,$ *for some* $1 \leq i \leq \theta$.

We can now prove our main structure theorem.

Theorem 6.2. *Let A be a finite-dimensional pointed Hopf algebra with abelian group $G(A) = \Gamma$ and infinitesimal braiding matrix $(q_{ij})_{1\leq i,j\leq\theta}$. Assume* (5.9)*,* (5.10) *and* (5.12)*. Then*

$$A \cong u(\mathcal{D}, \lambda, \mu),$$

where $\mathcal{D} = \mathcal{D}(\Gamma, (g_i)_{1\leq i\leq\theta}, (\chi_i)_{1\leq i\leq\theta}, (a_{ij})_{1\leq i,j\leq\theta})$ is a datum of finite Cartan type, and λ and μ are families of linking and root vector parameters for $\mathcal{D}$.

Proof. By Theorems 5.3 and 5.5, there is a datum $\mathcal{D}$ of finite Cartan type such that $\operatorname{gr}(A) \cong u(\mathcal{D}, 0, 0)$. By Lemma 6.1, for all $1 \leq i \leq \theta$ we can choose

$$a_i \in P(A)_{g_i,1}^{\chi_i} \text{ corresponding to } x_i \text{ in (6.1).}$$

We have shown in Theorem [AS4, 6.8] that

$$\operatorname{ad}_c(a_i)^{1-a_{ij}}(a_j) = 0, \text{ for all } 1 \leq i,j \leq \theta, i \sim j, i \neq j,$$
$$a_i a_j - q_{ij} a_j a_i - \lambda_{ij}(1 - g_i g_j) = 0, \text{ for all } 1 \leq i < j \leq \theta,\ i \not\sim j,$$

for some family λ of linking parameters. Thus there is a homomorphism of Hopf algebras

$$\varphi : U(\mathcal{D}, \lambda) \to A,\ \varphi|\Gamma = \operatorname{id}_\Gamma,\ \varphi(x_i) = a_i, \text{ for all } 1 \leq i \leq \theta.$$

By Theorem 5.5, φ is surjective.

We now use the notation of Section 2.2 and show that

(6.4) $\qquad \varphi(x_\alpha^{N_J}) \in k[\Gamma]$ for all $\alpha \in \Phi_J^+, J \in \mathcal{X}$.

We fix $J \in \mathcal{X}$ with $p = |\Phi_J^+|$, and show by induction on $\operatorname{ht}(\underline{a})$ that

(6.5) $\qquad \varphi(z^a) \in k[\Gamma]$ for all $a \in \mathbb{N}^p$.

Let $0 \neq a \in \mathbb{N}^p$. Since φ is a Hopf algebra map, we see from (2.17) that

$$\Delta(\varphi(z^a)) = h^a \otimes \varphi(z^a) + \varphi(z^a) \otimes 1 + w,$$

where by induction

$$w = \sum_{b,c\neq o,\underline{b}+\underline{c}=\underline{a}} t^a_{b,c}\, \varphi(z^b)h^c \otimes \varphi(z^c) \in k[\Gamma] \otimes k[\Gamma].$$

In particular, $\varphi(z^a) \in A_1$ by definition of the coradical filtration. We multiply this equation with $g \otimes g, g \in \Gamma$, from the left and $g^{-1} \otimes g^{-1}$ from the right. Since $gz^a g^{-1} = \eta^a(g)z^a$, we obtain $w = \eta^a(g)w$ for all $g \in \Gamma$.

Suppose $\eta^a \neq \varepsilon$. Then $w = 0$, and $\varphi(z^a) \in P^{\eta^a}_{h^a,1}$. Then $\varphi(z^a) = 0$ by Lemma 6.1 (6.3), since $\chi_l(g_l) \neq 1$ for all $1 \leq l \leq \theta$, but $\eta^a(h^a) = 1$ by the Cartan condition (see the proof of [AS2, Lemma 7.5] for a similar computation).

If $\eta^a = \varepsilon$, then $\varphi(z^a) \in A_1^\varepsilon = k[\Gamma]$ by Lemma 6.1 (6.2).

This proves (6.5) and (6.4). Then we conclude for each $J \in \mathcal{X}$ from Theorem 2.13 that the map

$$K(\mathcal{D}_J)\#k[\Gamma] \to U(\mathcal{D}, \lambda) \xrightarrow{\varphi} A$$

has the form φ_{μ^J} for some family of scalars μ^J as in Theorem 2.13 for the connected component J. Define $\mu = (\mu_\alpha)_{\alpha\in\Phi+}$ by $\mu_\alpha = \mu^J_\alpha$ for all $\alpha \in \Phi^+_J$. Then μ is a family of root vector parameters for $\mathcal{D}$, and the elements $u_\alpha(\mu) \in k[\Gamma]$ are defined in (4.14) for each $J \in \mathcal{X}$ and $\alpha \in \Phi^+_J$. It follows that $\varphi(x^{N_J}_\alpha) = u_\alpha(\mu) = \varphi(u_\alpha(\mu))$ for all $J \in \mathcal{X}, \alpha \in \Phi^+_J$. Thus φ factorizes over $u(\mathcal{D}, \lambda, \mu)$. Since

$$\dim(A) = \dim(\mathrm{gr}(A)) = \dim(u(\mathcal{D}, 0, 0)) = \dim(u(\mathcal{D}, \lambda, \mu))$$

by Theorem 4.5, φ induces an isomorphism $u(\mathcal{D}, \lambda, \mu) \cong A$. □

Corollary 6.3. *Let A be a finite-dimensional pointed Hopf algebra with abelian group $G(A) = \Gamma$ satisfying the assumptions of Theorem 6.2. Then for each prime divisor p of the dimension of A there is a group-like element of order p in A.*

Proof. This follows from Theorems 6.2 and 4.5. □

We note that the analog of Cauchy's theorem in group theory is false for arbitrary, non-pointed Hopf algebras. Let A be a finite-dimensional Hopf algebra with only trivial group-like elements, such as the dual of the group algebra of a finite group G with $G = [G, G]$. Then A does not contain any Hopf subalgebra of prime dimension, since any Hopf algebra of prime dimension is a group algebra by Zhu's theorem [Z].

Cauchy's theorem for semisimple Hopf algebras in a version conjectured by Etingof and Gelaki was recently shown in [KSZ]: Each prime divisor of a semisimple Hopf algebra divides the exponent of the Hopf algebra.

7. Isomorphism classes

In this last section we determine all isomorphisms between the Hopf algebras $u(\mathcal{D}, \lambda, \mu)$ in terms of some universal constants. We explicitly computed these constants for connected components of type A in [AS7].

For convenience we introduce a normalization condition for Cartan matrices and their root systems. Let $(a'_{ij})_{1\leq i,j\leq\theta}$ and $(a_{ij})_{1\leq i,j\leq\theta}$ be Cartan matrices of finite type. A *diagram isomorphism* between (a'_{ij}) and (a_{ij}) is a permutation τ of $\{1, 2, \dots, \theta\}$ with $a'_{ij} = a_{\tau(i),\tau(j)}$ for all $1 \leq i, j \leq \theta$. We choose from each isomorphism class of connected Cartan matrices of finite type one representative. The chosen representatives are called *standard* Cartan matrices. We fix a reduced representation of the longest element in the Weyl group and the corresponding ordering of the positive roots of the standard matrices as described in Section 2.1.

From now on we assume that any Cartan datum $\mathcal{D}$ satisfies the following *additional normalizing condition*:

> The Cartan matrix $(a_{ij})_{1\leq i,j\leq\theta}$ of $\mathcal{D}$ is a block diagonal matrix, and each matrix on the diagonal is one of the standard connected Cartan matrices. Moreover for each connected component J of I we fix the same order of the positive roots as for the chosen representative.

Thus up to a shift of indices we can identify the Cartan matrix of any connected component with a standard Cartan matrix. We use the fixed ordering of the positive roots in each component to define the root vectors of $\mathcal{D}$.

Note that any diagram isomorphism induces an isomorphism of the corresponding Nichols algebras. Hence up to Hopf algebra isomorphisms we can assume by the proof of Theorem 6.2 that the normalizing condition is satisfied for the Hopf algebras $u(\mathcal{D}, \lambda, \mu)$.

In the next definition we extend the notation for the linking parameters by (3.4).

Definition 7.1. Let Γ, Γ' be abelian groups and let

$$\mathcal{D} = \mathcal{D}(\Gamma, (g_i)_{1\leq i\leq\theta}, (\chi_i)_{1\leq i\leq\theta}, (a_{ij})_{1\leq i,j\leq\theta}),$$
$$\mathcal{D}' = \mathcal{D}(\Gamma', (g'_i)_{1\leq i\leq\theta'}, (\chi'_i)_{1\leq i\leq\theta'}, (a'_{ij})_{1\leq i,j\leq\theta'})$$

be Cartan data of finite type satisfying (3.1) and (3.2). Assume $\theta = \theta'$. Let λ and λ' be linking parameters, and μ and μ' root vector parameters for $\mathcal{D}$ and $\mathcal{D}'$.

Let $\varphi : \Gamma' \to \Gamma$ be a group isomorphism, $\sigma \in S_\theta$ a permutation and $(s_i)_{1\leq i\leq\theta}$ a family of non-zero elements in k. The triple $(\varphi, \sigma, (s_i))$ is called an *isomorphism* from $(\mathcal{D}', \lambda', \mu')$ to $(\mathcal{D}, \lambda, \mu)$ if the following five conditions are satisfied:

$$\varphi(g_i') = g_{\sigma(i)} \text{ for all } 1 \leq i \leq \theta. \tag{7.1}$$

$$\chi_i' = \chi_{\sigma(i)}\varphi \text{ for all } 1 \leq i \leq \theta. \tag{7.2}$$

$$a_{ij}' = a_{\sigma(i),\sigma(j)} \text{ for all } 1 \leq i, j \leq \theta. \tag{7.3}$$

$$\lambda_{ij}' = s_i s_j \lambda_{\sigma(i)\sigma(j)} \text{ for all } 1 \leq i, j \leq \theta, i \nsim j \tag{7.4}$$

To formulate the fifth condition we have to introduce more notations for the connected components J of $\mathcal{D}$.

(1) Let $q_J = (\chi_j(g_i))_{i,j\in J}$ be the braiding matrix of the restriction $\mathcal{D}_J$ of $\mathcal{D}$ to J.
(2) Using (7.3) we identify the root systems of J and of $\sigma^{-1}(J)$ with the corresponding root system of the standard Cartan matrix. Then the restriction of σ to $\sigma^{-1}(J)$ becomes a diagram automorphism σ_J of the corresponding standard Cartan matrix.
(3) For any $\beta \in \Phi_J^+$ let $u_\beta'(\mu')$ and $u_\beta(\mu)$ be the elements in the group algebras $k[\Gamma']$ and $k[\Gamma]$ defined in (4.14). For any family $a = (a_\beta)_{\beta\in\Phi_J^+}$ of natural numbers $a_\beta \geq 0$ we define the product $u(\mu)^a = \prod_{\beta\in\Phi_J^+} u_\beta(\mu)^{a_\beta}$.
(4) For any $\alpha \in \Phi_J^+$ with $\alpha = \sum_{i\in J} n_i\alpha_i, n_i \geq 0$ for all $1 \leq i \leq \theta$, let $s_\alpha = \prod_{i\in J} s_i^{n_i}$.
(5) For any $\alpha \in \Phi_J^+$ and any family $a = (a_\beta)_{\beta\in\Phi_J^+}$ of natural numbers $a_\beta \geq 0$ let $t^a_{\alpha,q_J,\sigma_J}$ be the element in k defined below in Theorem 7.5 applied to $\mathcal{D}_J$.

Then the last condition is the following identity in the group algebra $k[\Gamma]$:

$$\varphi(u_\alpha'(\mu')) = s_\alpha^{N_J} \sum_a t^a_{\alpha,q_J,\sigma_J} u(\mu)^a \text{ for all } \alpha \in \Phi_J^+, J \in \mathcal{X}. \tag{7.5}$$

Finally let $\operatorname{Isom}((\mathcal{D}', \lambda', \mu'), (\mathcal{D}, \lambda, \mu))$ be the set of all isomorphisms from $(\mathcal{D}', \lambda', \mu')$ to $(\mathcal{D}, \lambda, \mu)$.

For Hopf algebras A', A we denote by $\operatorname{Isom}(A', A)$ the set of all Hopf algebra isomorphisms from A' to A.

We now can state the main result of this section.

Theorem 7.2. *Let $\mathcal{D}$ and $\mathcal{D}'$ be Cartan data of finite type with finite abelian groups Γ and Γ' and rank θ and θ'. Assume that $\mathcal{D}$ and $\mathcal{D}'$ satisfy* (3.1), (3.2). *In addition assume the following condition on the braiding matrix of $\mathcal{D}$:*

$$\operatorname{ord}(q_{ii}) > 4 \textit{ for all } 1 \leq i \leq \theta. \tag{7.6}$$

Let λ and λ' be linking parameters, and μ and μ' root vector parameters for $\mathcal{D}$ and $\mathcal{D}'$.

If the Hopf algebras $u(\mathcal{D}', \lambda', \mu')$ and $u(\mathcal{D}, \lambda, \mu)$ are isomorphic, then $\theta' = \theta$. Assume $\theta' = \theta$. Then the map

$$\mathrm{Isom}((\mathcal{D}', \lambda', \mu'), (\mathcal{D}, \lambda, \mu)) \to \mathrm{Isom}(u(\mathcal{D}', \lambda', \mu'), u(\mathcal{D}, \lambda, \mu))$$

given by $(\varphi, \sigma, (s_i)) \mapsto F$, where $F(x_i') = s_i x_{\sigma(i)}$ and $F(g') = \varphi(g')$ for all $1 \le i \le \theta$ and $g' \in \Gamma'$, is bijective.

Before we begin with the proof of Theorem 7.2 we need some preparations.

First we see that condition (7.3) is in most cases redundant.

Lemma 7.3. *In the situation of Definition 7.1 assume $\theta' = \theta$, (7.1), (7.2) and (7.6). Then (7.3) holds.*

Proof. For all $1 \le i, j \le \theta$ let $q'_{ij} = \chi'_j(g'_i)$, $q_{ij} = \chi_j(g_i)$. Then for all i, j (7.1) and (7.2) imply that $q'_{ij} = q_{\sigma(i)\sigma(j)}$. Hence $a_{ij} = a'_{\sigma(i)\sigma(j)}$, since $q_{ii}^{a_{ij}} = q_{ii}^{a'_{\sigma(i)\sigma(j)}}$, and $a_{ij} - a'_{\sigma(i)\sigma(j)} \in \{0, \pm1, \pm2, \pm3\}$. □

We need an extra information in the situation of Theorem 2.6.

Lemma 7.4. *Let $\mathcal{D}$ be a connected Cartan datum of finite type with root system Φ, finite abelian group Γ and $N = \mathrm{ord}(q_{ii})$ for all i. Assume (3.1) and (3.2). By Theorem 5.1 there is a canonical projection $\pi : R(\mathcal{D}) \to \mathfrak{B}(V)$ whose kernel is the ideal generated by all $x_\alpha^N, \alpha \in \Phi^+$. We denote the coalgebra structure of $R(\mathcal{D})$ by $\Delta(x) = x^{(1)} \otimes x^{(2)}$ for all $x \in R(\mathcal{D})$. Then*

$$K(\mathcal{D}) = R(\mathcal{D})^{\mathrm{co}\pi} = \{x \in R(\mathcal{D}) \mid x^{(1)} \otimes \pi(x^{(2)}) = x \otimes 1\}.$$

Proof. This follows by bosonization from the corresponding result for pointed Hopf algebras in [Ma]. □

In the following theorem we define the constants in (7.5).

Theorem 7.5. *Let $\mathcal{D}(\Gamma, (g_i)_{1\le i\le\theta}, (\chi_i)_{1\le i\le\theta}, (a_{ij})_{1\le i,j\le\theta})$ be a connected Cartan datum of finite type with root system Φ, finite abelian group Γ, and $N = \mathrm{ord}(q_{ii}), 1 \le i \le \theta$. Assume (3.1) and (3.2). Assume that $(a_{ij})_{1\le i,j\le\theta}$ is a standard Cartan matrix, and let σ be a diagram automorphism of $(a_{ij})_{1\le i,j\le\theta}$. Define*

$$\mathcal{D}^\sigma = \mathcal{D}(\Gamma, (g'_i)_{1\le i\le\theta}, (\chi'_i)_{1\le i\le\theta}, (a_{ij})_{1\le i,j\le\theta})$$

with $g'_i = g_{\sigma(i)}, \chi'_i = \chi_{\sigma(i)}$ for all $1 \le i \le \theta$. Let $V \in {}^\Gamma_\Gamma\mathcal{YD}$ with basis $x_i \in V_{g_i}^{\chi_i}$, and $V^\sigma \in {}^\Gamma_\Gamma\mathcal{YD}$ with basis $x_i^\sigma \in (V^\sigma)_{g'_i}^{\chi'_i}$ for all $1 \le i \le \theta$.

Then there is an algebra map

$$F^\sigma : R(\mathcal{D}^\sigma) \to R(\mathcal{D}), x_i^\sigma \mapsto x_{\sigma(i)} \text{ for all } 1 \le i \le \theta.$$

For each $\alpha \in \Phi^+$, *and each family* $a = (a_\beta)_{\beta \in \Phi^+}$ *of natural numbers* $a_\beta \ge 0$ *there are uniquely determined elements* $t_\alpha^a \in k$ *depending on the braiding matrix of* $\mathcal{D}$ *and the diagram automorphism* σ *such that*

$$F^\sigma(x_\alpha^\sigma)^N = \sum_a t_\alpha^a z^a,$$

where $z^a = x_{\beta_1}^{a_1 N} \cdots x_{\beta_p}^{a_p N}$ *as in Definition 2.7 with the fixed ordering* $\beta_1, \dots, \beta_p$ *of the positive roots, and where* x_α^σ *denotes the root vector of* α *in* $R(\mathcal{D}_\sigma)$.

Proof. The Cartan condition is satisfied for $\mathcal{D}^\sigma$ since σ is a diagram automorphism. The linear map

$$f^\sigma : V^\sigma \to V, x_i^\sigma \mapsto x_{\sigma(i)}, 1 \le i \le \theta,$$

is an isomorphism of Yetter-Drinfeld modules. Then f^σ induces isomorphisms $F^\sigma : R(\mathcal{D}^\sigma) \to R(\mathcal{D})$ and $\overline{F^\sigma} : \mathfrak{B}(V^\sigma) \to \mathfrak{B}(V)$. Let $\pi : R(V) \to \mathfrak{B}(V)$ and $\pi^\sigma : R(V^\sigma) \to \mathfrak{B}(V^\sigma)$ be the natural projections. Since $K(\mathcal{D}) = R(\mathcal{D})^{\mathrm{co}\pi}$ and $K(\mathcal{D}^\sigma) = R(\mathcal{D}^\sigma)^{\mathrm{co}\pi^\sigma}$ by Lemma 7.4, it follows that F^σ maps $K(\mathcal{D}^\sigma)$ into $K(\mathcal{D})$. This proves the claim by Theorem 2.6. □

The meaning of the elements $F^\sigma(x_\alpha^\sigma)$ in the previous theorem can be explained as follows. Let x_α be represented as iterated skew-commutator of simple root vectors $x_{i_1}, \dots, x_{i_s}$ in this order. Then $F^\sigma(x_\alpha^\sigma)$ is the same iterated skew-commutator of the sequence $x_{\sigma(i_1)}, \dots, x_{\sigma(i_s)}$.

As an example, take the Dynkin diagram A_2 with the non-simple root $\alpha = \alpha_1 + \alpha_2$ and the diagram automorphism σ with $\sigma(1) = 2, \sigma(2) = 1$. Then $x_\alpha = x_1 x_2 - q_{12} x_2 x_1$ and $F^\sigma(x_\alpha^\sigma) = x_2 x_1 - q_{21} x_1 x_2$.

Finally we note

Lemma 7.6. *Let* $\mathcal{D} = \mathcal{D}(\Gamma, (g_i)_{1 \le i \le \theta}, (\chi_i)_{1 \le i \le \theta}, (a_{ij})_{1 \le i,j \le \theta})$ *be a datum of finite Cartan type and assume* (7.6). *Then for all* $1 \le i, j \le \theta, i \ne j$, $g_i \ne g_j$ *or* $\chi_i \ne \chi_j$.

Proof. Assume there are $i \ne j$ with $g_i = g_j, \chi_i = \chi_j$. Then $q_{ii} = q_{jj}$, and $q_{ii}^2 = q_{ii}^{a_{ij}} = q_{ii}^{a_{ji}}$. Hence in contradiction to our assumption we have $q_{ii}^{2 - a_{ij}} = 1$ for $a_{ij} \in \{0, -1, -2\}$, and $q_{ii}^{a_{ij} - a_{ji}} = 1$ for $a_{ij} = -3$. □

We can now prove Theorem 7.2:

Proof. Assume that there is a Hopf algebra isomorphism

$$F : A' = u(\mathcal{D}', \lambda', \mu') \to A = u(\mathcal{D}, \lambda, \mu).$$

Then F preserves the coradical filtration and induces an isomorphism $A'_0 = k[\Gamma'] \cong A_0 = k[\Gamma]$, given by a group isomorphism $\varphi : \Gamma' \to \Gamma$, and by Corollary 5.2 an isomorphism

$$A'_1 = k[\Gamma'] \oplus \bigoplus_{\substack{g'\in\Gamma',\\ 1\le i\le\theta'}} kx'_i g' \cong A_1 = k[\Gamma] \oplus \bigoplus_{\substack{g\in\Gamma,\\ 1\le i\le\theta}} kx_i g.$$

Hence it follows from Lemma 7.6 (see [AS2, 6.3]) that $\theta = \theta'$, and that there are a permutation $\sigma \in S_\theta$ and elements $0 \neq s_i \in k, 1 \le i \le \theta$ such that (7.1) and (7.2) hold, and $F(x'_i) = s_i x_{\sigma(i)}$ for all $1 \le i \le \theta$. Then Lemma 7.3 implies (7.3), and $F([x'_i, x'_j]_{c'}) = s_i s_j [x_{\sigma(i)}, x_{\sigma(j)}]_c$ for all $1 \le i \le \theta$. Now (7.4) follows from the linking relations.

To establish (7.5) we fix a connected component J of $\mathcal{D}$, and we identify J and $J' = \sigma^{-1}(J)$ with the index set $\{1, \dots, \theta_J\}$ of the corresponding standard Cartan matrix. Then the restriction of σ to J' becomes the diagram automorphism σ_J. Let $V_J \in {}^{\Gamma}_{\Gamma}\mathcal{YD}$ with basis $x_i \in (V_J)^{\chi_i}_{g_i}, i \in J$, and $V'_J \in {}^{\Gamma'}_{\Gamma'}\mathcal{YD}$ with basis $x'_i \in (V'_J)^{\chi'_i}_{g'_i}, i \in J$. Let $f^{\sigma_J}_J : V^{\sigma}_J \to V_J$ be the map of Theorem 7.5 for $\mathcal{D}_J$ and V_J instead of $\mathcal{D}$ and V.

We define linear maps $f_J : V'_J \to V_J, x'_i \mapsto s_i x_{\sigma(i)}, 1 \le i \le \theta_J$, and $f'_J : V'_J \to V^{\sigma}_J, x'_i \mapsto s_i x^{\sigma_J}_i, 1 \le i \le \theta_J$. Then $f_J = f^{\sigma_J}_J f'_J$. The maps f_J and f'_J are Γ'-linear and Γ-colinear, where the action of Γ' on V_J and the coaction of Γ on V'_J are defined via φ. Hence by (7.1)–(7.3) they induce algebra maps $F_J : R(\mathcal{D}'_{J'}) \to R(\mathcal{D}_J)$ and $F'_J : R(\mathcal{D}'_{J'}) \to R(\mathcal{D}^{\sigma_J}_J)$, and $F_J = F^{\sigma_J}_J F'_J$. By Theorem 7.5 we then obtain for any $\alpha \in \Phi^+_J$

$$F_J(x'_\alpha)^{N_J} = s^{N_J}_\alpha F^{\sigma_J}_J(x^{\sigma_J}_\alpha)^{N_J} = s^{N_J}_\alpha \sum_a t^a_{\alpha,q_J,\sigma_J} z^a. \tag{7.7}$$

The canonical maps

$$\pi_J : R(\mathcal{D}_J) \to u(\mathcal{D}, \lambda, \mu) \text{ and } \pi_{J'} : R(\mathcal{D}'_{J'}) \to u(\mathcal{D}', \lambda', \mu')$$

map root vectors to root vectors. Since $F\pi_{J'} = \pi_J F_J$, we see from (7.7) that for all $\alpha \in \Phi^+_J$

$$F\pi_{J'}({x'}^{N_J}_\alpha) = \pi_J F_J({x'}^{N_J}_\alpha) = s^{N_J}_\alpha \sum_a t^a_{\alpha,q_J,\sigma_J} u(\mu)^a.$$

On the other hand ${x'}^{N_J}_\alpha = {u'}_\alpha(\mu')$ in $u(\mathcal{D}', \lambda', \mu')$, hence

$$F\pi_{J'}({x'}^{N_J}_\alpha) = \varphi(u'_\alpha(\mu')),$$

and (7.5) follows.

It is easy to see that conversely any isomorphism $(\varphi, \sigma, (s_i))$ defines an isomorphism of Hopf algebras, and that two such triples coincide if they define the same Hopf algebra map. $\square$

We remark that the situation greatly simplifies if the diagram automorphism σ_J in Definition 7.1 is the identity. This happens in particular if the Dynkin diagram of $(a_{ij})_{i,j\in J}$ is not of Type A, D or E_6. In this case it follows from the inductive definition of the $u_\alpha(\mu)$ that (7.5) is equivalent to

$$\mu'_\alpha = s_\alpha^{N_J}\mu_\alpha \text{ for all } \alpha \in \Phi_J^+, J \in \mathfrak{X}. \tag{7.8}$$

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