Pointed Hopf algebras with finite simple group of Lie type

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Joint work (in progress) with
Giovanna Carnovale and Gastón Andrés García.

Plan of the talk.

I. Problems.

II. Nichols algebras.

III. Racks versus Yetter-Drinfeld modules.

IV. Nichols algebras associated to conjugacy classes in a finite simple group of Lie type.

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(4). *In the new results of this talk, the finite group G is* simple of Lie type *and the finite rack X is a conjugacy class in G.*

II. Nichols algebras.

INPUT: braided vector space (W, c)	$\rightarrow$	OUTPUT: Nichols algebra $\mathfrak{B}(W)$

(W, c) braided vector space: $c \in GL(W \otimes W)$

$(c \otimes \mathrm{id})(\mathrm{id} \otimes c)(c \otimes \mathrm{id}) = (\mathrm{id} \otimes c)(c \otimes \mathrm{id})(\mathrm{id} \otimes c)$

Define $c_i \in \mathrm{Aut}(W^{\otimes n})$ by

$$c_i = \mathrm{id}_{W^{\otimes(i-1)}} \otimes c \otimes \mathrm{id}_{W^{\otimes(n-i-1)}}.$$

Then $c_1, \dots, c_{n-1}$ satisfy the relations of the braid group. Thus we have a representation $\rho_n : \mathbb{B}_n \to \mathrm{Aut}(W^{\otimes n})$.

Definition. (Nichols, 1978; Woronowicz, 1988). If (W, c) is a braided vector space, then the Nichols algebra is

$$\mathfrak{B}(W) = \oplus_{n \in \mathbb{N}_0} \mathfrak{B}^n(W),$$

where

$$\mathfrak{B}^n(W) = T^n(W) / \ker \sum_{\sigma \in \mathbb{S}_n} \rho_n(S(\sigma)).$$

Here $S : \mathbb{S}_n \to \mathbb{B}_n$ is the Matsumoto section (of sets), preserving the length.

Motivation. A.-Schneider, 1998: essential tool in the classification of pointed Hopf algebras over $\mathbb{C}$.

H pointed Hopf algebra with group

$$G = \{x \in H - 0 : \Delta(x) = x \otimes x\}.$$

$\rightsquigarrow$ (W, c) braided vector space of special type (a Yetter-Drinfeld module over G)

$\rightsquigarrow$ Nichols algebra $\mathfrak{B}(W)$; $\dim \mathfrak{B}(W) < \dim H$

A Yetter-Drinfeld module W over a finite group (not necessarily abelian) G is semisimple:

$$W = V_1 \oplus \cdots \oplus V_\theta,$$

where each V_i is irreducible and

$$c(V_i \otimes V_j) = V_j \otimes V_i.$$

Irreducibles: $\mathcal{C}$ a conjugacy class in G; fix $s \in \mathcal{C}$; let (ρ, V) irred. repr. of G^s.

$$M(\mathcal{C}, \rho) = \mathrm{Ind}_{G^s}^{G} = \mathbb{C}\mathcal{C} \otimes V.$$

Suggestion. Split the research on $\mathfrak{B}(W)$ into two parts:

- To study $\mathfrak{B}(V)$ for all V irreducible, i. e. for all $M(\mathcal{C},\rho)$.

- Assuming the knowledge of the Nichols algebras $\mathfrak{B}(V_i)$, to describe the Nichols algebra $\mathfrak{B}(W)$ as a "gluing" of the various Nichols subalgebras $\mathfrak{B}(V_i)$ along a generalized Dynkin diagram.

N. A., I. Heckenberger and H.-J. Schneider, The Nichols algebra of a semisimple Yetter-Drinfeld module, Amer. J. Math. 132 (2010), no. 6, 14931547.

I. Heckenberger and H.-J. Schneider, Nichols algebras over groups with finite root system of rank two I. J. Algebra 324 (2010), no. 11, 30903114.

I. Heckenberger and L. Vendramin, Nichols algebras over groups with finite root system of rank two II. arXiv:1302.0213v1.

III. Racks versus Yetter-Drinfeld modules.

A rack is a pair $(X, \triangleright)$, X a set, $\triangleright$ operation on X s. t.:

- $x \triangleright \_\_$ is a bijection for all $x \in X$,
- $x \triangleright (y \triangleright z) = (x \triangleright y) \triangleright (x \triangleright z)$ for all $x, y, z \in X$.

Main example: X a conjugacy class in G, $x \triangleright y := xyx^{-1}$.

All racks in this talks are conjugacy classes in finite groups.

Let X be a rack, $n \in \mathbb{N}$ and $q : X \times X \to \mathbf{GL}(n, \mathbb{C})$.

- q is a *2-cocycle* $\iff$ $q_{x \triangleright y, x \triangleright z} q_{x,z} = q_{x, y \triangleright z} q_{y,z}$ for all $x, y, z \in X$.

- $V = \mathbb{C}X \otimes \mathbb{C}^n$, $c^q : V \otimes V \to V \otimes V$, $c^q(e_x v \otimes e_y w) = e_{x \triangleright y} q_{x,y}(w) \otimes e_x v$, $x, y \in X$, $v, w \in \mathbb{C}^n$.

Then:

q is a 2-cocycle $\iff$ c^q is a solution of the braid equation.

In such case, we write $\mathfrak{B}(X, q) = \mathfrak{B}(V)$.

$$\boxed{\begin{array}{c}\text{Decide when}\\ \dim \mathfrak{B}(\mathcal{C},\rho) < \infty \text{ for all } G\end{array}} \rightsquigarrow\!\!\!\leftarrow \boxed{\begin{array}{c}\text{Decide when}\\ \dim \mathfrak{B}(X,q) < \infty\\ \text{for all } X, q\end{array}}$$

We know a short list of pairs (X,q) with $\dim \mathfrak{B}(X,q) < \infty \dots$

$$\boxed{\begin{array}{c}\text{Decide when}\\ \dim \mathfrak{B}(\mathcal{C},\rho) = \infty \text{ for all } G\end{array}} \rightsquigarrow\!\!\!\leftarrow \boxed{\begin{array}{c}\text{Decide when}\\ \dim \mathfrak{B}(X,q) = \infty\\ \text{for all } X, q\end{array}}$$

N. A., M. Graña. From racks to pointed Hopf algebras. Adv. Math. 178 (2), 177–243 (2003).

Definition. A rack X *collapses* when $\dim \mathfrak{B}(X,q) = \infty$ for all cocycles q (for all n).

Criterium of type D. A rack X is *of type D* when it has a decomposable subrack $Y = R \coprod S$ with elements $r \in R$, $s \in S$ such that $r \triangleright (s \triangleright (r \triangleright s)) \neq s$.

If $\mathcal{O}$ is a conjugacy class in a finite group G, then TFAE:

- The rack $\mathcal{O}$ is of type D.
- There exist r, $s \in \mathcal{O}$ such that $\mathcal{O}_r^{\langle r,s \rangle} \neq \mathcal{O}_s^{\langle r,s \rangle}$ and $(rs)^2 \neq (sr)^2$.

Theorem. A rack of type D collapses.

A., F. Fantino, Graña, Vendramin. Finite-dimensional pointed Hopf algebras with alternating groups are trivial. Ann. Mat. Pura Appl. (2011).

Proof. Based on results in [AHS, HS] (Weyl groupoid).

Criterium of type F. A rack X is *of type F* when it has a family of mutually disjoint subracks $(R_a)_{a\in A}$ such that

- $R_a \triangleright R_b = R_b$ for all $a, b \in A$;
- for all $a \neq b \in A$, there are $r_a \in R_a$, $r_b \in R_b$ such that $r_a \triangleright r_b \neq r_b$ (here the r_a's might be different for different b's);
- A has four elements.

Theorem. A rack of type F collapses.

Proof. Based on results in [AHS](Weyl groupoid) and Cuntz-Heckenberger.

Criterium of type C. A rack X is *of type C* when it has a decomposable subrack $Y = R \coprod S$, where $|R| > 6$ or $|S| > 6$, with elements $r \in R$, $s \in S$ such that $r \triangleright s \neq s$.

Theorem. A rack of type C collapses.

Proof. Based on recent results by Heckenberger-Vendramin.

If a rack X projects onto a rack of type D (resp. F, resp. C), then X is also of type D (resp. F, resp. C).

An indecomposable rack always admits a rack epimorphism onto a *simple* rack. Thus, any indecomposable rack having a quotient simple rack of type D collapses. Hence it is natural to ask:

Determine all simple racks of type D (resp. F, resp. C).

The classification of simple racks is known; non-trivial conjugacy classes in finite simple groups are simple racks.

A rack X is *cthulhu* when it is neither of type C, nor D, nor F. See `http://en.wikipedia.org/wiki/Cthulhu` for spelling and pronunciation.
X is *sober* if every subrack is either abelian or indecomposable.

IV. Nichols algebras associated to conjugacy classes in a finite simple group of Lie type.

Let p be a prime, $m \in \mathbb{N}$, $q = p^m$ and $\mathbb{F}_q$ the field with q elements.

$\diamond$ Let $\mathbb{G}$ be a simple (simply connected) algebraic group defined over $\mathbb{F}_q$. A Steinberg endomorphism $F : \mathbb{G} \to \mathbb{G}$ is an automorphism having a power equal to a Frobenius map. The subgroup $\mathbb{G}^F$ of fixed points by F is a *finite group of Lie type*.

$\diamond$ $G := \mathbb{G}^F / Z(\mathbb{G}^F)$ is a simple finite group except for 8 examples that appear in low rank and char. These G are called *finite simple groups of Lie type*.

$\diamond$ Example: $\mathbf{PSL}_n(q) := \mathbf{SL}_n(q)/Z(\mathbf{SL}_n(q))$ are simple (except $\mathbf{PSL}_2(2) \simeq \mathbb{S}_3$ and $\mathbf{PSL}_2(3) \simeq \mathbb{A}_4$).

Unipotent classes in $\mathrm{PSL}_n(q)$ not collapsing.

n	type	q	Remark
2	(2)	even or not a square	sober
3	(3)	2	sober
	$(2,1)$	2	cthulhu
4	$(2,1,1)$	2	cthulhu

Non-semisimple classes in $\mathbf{PSL}_n(q)$.

$\mathbf{SL}_n(q) \ni \mathtt{x} \to x \in G = \mathbf{PSL}_n(q)$; $\mathtt{x} = \mathtt{x}_s \mathtt{x}_u$ the Jordan decomp. of $\mathtt{x}$, so $x = \pi(\mathtt{x}_s)\pi(\mathtt{x}_u)$ the Chevalley-Jordan decomp. of x. Now $\mathtt{x}_u \in \mathcal{K} := C_{\mathbf{SL}_n(q)}(\mathtt{x}_s) \implies x_u \in K := \pi(\mathcal{K})$. Then

$$\mathcal{O}^{\mathcal{K}}_{\mathtt{x}_u} \simeq \mathcal{O}^{K}_{x_u} \hookrightarrow \mathcal{O}^{G}_{x} \rightsquigarrow \text{enough to deal with } \mathcal{O}^{\mathcal{K}}_{\mathtt{x}_u}.$$

Let $\mathtt{x}_s = \begin{pmatrix} s_1 & 0 & \dots & 0 \\ 0 & s_2 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & \dots & s_k \end{pmatrix}$, with $s_i \in \mathbf{SL}_{\lambda_i}(q)$ irreducible; $h_1, \dots, h_\ell$ with h_1 is the number of blocks s_i that are isomorphic to s_1, etc.

Let $\mu_t \in \mathbb{N}$ be such that $C_{s_i} \simeq \mathbb{F}_{q^{\mu_t}}$, $i \in J_t$, $t \in I_\ell$. Then $C_{\mathbf{GL}_n(q)}(\mathtt{x}_s) \simeq \mathbf{GL}_{h_1}(q^{\mu_1}) \times \dots \times \mathbf{GL}_{h_\ell}(q^{\mu_\ell})$.

Non-semisimple classes in $\mathrm{PSL}_n(q)$ not known to collapse.

$n = h_1 + \cdots + h_\ell$	$\mathrm{x}_u = (u_1, \ldots, u_\ell)$	$\mathbf{q} = (q^{\mu_1}, \ldots, q^{\mu_\ell})$
$h_1 = 2$, $h_i \geq 2$ for $2 \leq i \leq \ell$	$(u_1, \mathrm{id}, \ldots, \mathrm{id})$	q^{μ_1} odd and not a square
$h_1 = 3$	$(u_1, \mathrm{id}, \ldots, \mathrm{id})$	$q^{\mu_1} = 2$
$h_1 = 4$	$(u_1, \mathrm{id}, \ldots, \mathrm{id})$ u_1 of type $(2, 1, 1)$	$q^{\mu_1} = 2$
$h_1 = 2$,	$(u_1, \mathrm{id}, \ldots, \mathrm{id})$	$q^{\mu_1} > 2$ even

Semisimple classes in $\mathrm{PSL}_2(q)$ not collapsing.

q	class	Remark
7	involutions	cthulhu
even or odd not a square	non-split, order 3	cthulhu
	non-split, order > 3	sober

Unipotent classes in G simple not collapsing.

G	q	type or representative	Remark
$\mathbf{PSp}_{2n}(q)$, $n \geq 2$	even	all	open
	odd, not a square	$(1^{r_1}, 2)$	cthulhu
	9	$(1^{r_1}, 2)$	open
	3	$(1^{r_1}, 2^{r_2}, 3^{r_3})$, $r_2 r_3 > 0$	open
$\mathbf{P\Omega}_{2n+1}(q)$, $n \geq 3$	3	$(1^{r_1}, 2^{r_2}, 3^{r_3})$, $r_2 r_3 > 0$	open
$\mathbf{P\Omega}^{+}_{2n}(q)$, $n \geq 4$	even	all	open
	3	$(1^{r_1}, 2^{r_2}, 3^{r_3})$, $r_2 r_3 > 0$	open

G	q	type or representative	Remark
$E_6(q)$	2,4	all except $x_{\alpha_1}(1)$	open
$E_7(q)$	2	all except y_{119}	open
	4	all except $y_{113}, y_{115}, y_{117}, y_{118}, y_{119}$	open
$E_8(q)$	2	all except z_{195}	open
	4	all except $z_{189}, z_{193}, z_{194}, z_{195}$	open
	p=2,3,5	$\subset D_8(a_7)$	open
$F_4(q)$	3		open
	2,4	all except x_4	open
$G_2(q)$	3		open