Quantum subgroups of a simple quantum group at a root of 1

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Let me begin by the list of finite-dimensional complex Hopf algebras I know:

- Group algebras $\mathbb{C}\Gamma$ (Γ finite)

- Their duals $\mathbb{C}^{\Gamma}$

- Their twistings $\mathbb{C}\Gamma^{J}$ and $(\mathbb{C}^{\Gamma})_{J}$, Γ finite, $J \in \mathbb{C}\Gamma \otimes \mathbb{C}\Gamma$ twist.

- $\mathfrak{g}$ simple Lie algebra, ϵ root of 1 of order N small quantum group $u_\epsilon(\mathfrak{g})$ (a.k.a. Frobenius-Lusztig kernel)

- (A.-Schneider) Pointed Hopf algebras $u(\mathcal{D}, \lambda, \mu)$, Γ finite abelian group, $\mathcal{D}$ Cartan datum, λ linking parameter, μ power root parameter

- Their duals...

- Their twistings.

Explicitly known for $\Lambda(V)\#\mathbb{C}\Gamma$ (Etingof-Gelaki), Γ non abelian.

- Bosonizations $\mathfrak{B}(V)\#\mathbb{C}\Gamma$ of the braided vector spaces of diagonal type in Heckenberger's list.

- Bosonizations $\mathfrak{B}(V)\#\mathbb{C}\Gamma$ of the known braided vector spaces of group type with finite-dimensional Nichols algebra (Milinski-Schneider, Fomin-Kirillov, Graña, . . .).

- *Ditto* replacing $\mathbb{C}\Gamma$ by a semisimple Hopf algebra (examples in paper by Dascalescu-Masuoka-Menini-...).

- Their liftings...

- Their subalgebras...

- Their duals...

- Their twistings.

- Drinfeld doubles (and generalizations)

Extensions of the preceding:

- Tensor product of any two known Hopf algebras.

- (G. I. Kac) If $\Sigma = FG$ is an exact factorization (Σ finite), then

$$\mathbb{C}^F \hookrightarrow \mathbb{C}^F \bowtie \mathbb{C}G \twoheadrightarrow \mathbb{C}G$$

- Version with cocycles $\mathbb{C}^F \hookrightarrow \mathbb{C}^F\, {}^{\tau}\!\bowtie_{\sigma} \mathbb{C}G \twoheadrightarrow \mathbb{C}G$ (control by Kac exact sequence)

- Group-theoretical Hopf algebras (Ocneanu-Ostrik)

• Non-abelian extensions, with weak actions and cocycles (very few explicit finite-dimensional examples to my knowledge; infinite-dimensional example by Majid-Soibelman, finite-dimensional version in Majid's book).

• (E. Müller) Construction of all finite-dimensional Hopf algebra quotients $\mathcal{O}_\epsilon(SL_N) \twoheadrightarrow A$, ϵ a root of 1. They fit into

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathcal{O}(SL_N) & \xrightarrow{\iota} & \mathcal{O}_\epsilon(SL_N) & \xrightarrow{\pi} & \mathbf{u}_\epsilon(\mathfrak{g})^* & \longrightarrow & 1 \\ & & {\scriptstyle t_\sigma}\downarrow & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle r} & & \\ 1 & \longrightarrow & \mathcal{O}(\Gamma) & \xrightarrow{\widehat{\iota}} & A & \xrightarrow{\widehat{\pi}} & H & \longrightarrow & 1. \end{array}$$

Here $\mathcal{O}(\Gamma) = \mathbb{C}^\Gamma$.

Questions.

- Exhaust the preceding.

- Are there more examples?

Goal. G a simple algebraic group, $\mathfrak{g} = \text{Lie } G$, ϵ a root of 1 of order ℓ (odd, prime to 3 if G is of type G_2).

Classify all (finite-dimensional or not) Hopf algebra quotients $\mathcal{O}_\epsilon(G) \twoheadrightarrow A$. They fit into

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathcal{O}(G) & \xrightarrow{\iota} & \mathcal{O}_\epsilon(G) & \xrightarrow{\pi} & \mathbf{u}_\epsilon(\mathfrak{g})^* & \longrightarrow & 1 \\ & & {\scriptstyle t_\sigma}\downarrow & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle r} & & \\ 1 & \longrightarrow & \mathcal{O}(\Gamma) & \xrightarrow{\widehat{\iota}} & A & \xrightarrow{\widehat{\pi}} & H & \longrightarrow & 1. \end{array}$$

Let $\mathbb{T} := \{K_{\alpha_1}, \ldots, K_{\alpha_n}\} = G(\mathbf{u}_\epsilon(\mathfrak{g}))$.

If $I \subset \Pi$, then $\mathbb{T}_I := \{K_{\alpha_i} : \ i \in I\}$.

Theorem. (E. Müller). The Hopf subalgebras of $\mathbf{u}_\epsilon(\mathfrak{g})$ are parameterized by triples (Σ, I_+, I_-), where

- $I_+ \subseteq \Pi$, $I_- \subseteq -\Pi$
- If $I = I_+ \cup -I_-$, then $\mathbb{T}_I < \Sigma < \mathbb{T}$.

A *subgroup datum* is a collection $\mathcal{D} = (I_+, I_-, N, \Gamma, \sigma, \delta)$ where

- $I_+ \subseteq \Pi$ and $I_- \subseteq -\Pi$. Let

$\Psi_\pm = \{\alpha \in \Phi : \operatorname{Supp} \alpha \subseteq I_\pm\}$,

$\mathfrak{l}_\pm = \sum_{\alpha \in \Psi_\pm} \mathfrak{g}_\alpha$ and $\mathfrak{l} = \mathfrak{l}_+ \oplus \mathfrak{h} \oplus \mathfrak{l}_-$;

$\mathfrak{l}$ is an algebraic Lie subalgebra of $\mathfrak{g}$.

Let L be the connected Lie subgroup of G with Lie $(L) = \mathfrak{l}$.

Let $s = n - |I_+ \cup -I_-|$.

- N is a subgroup of $(\mathbb{Z}/(\ell))^s$.
- Γ is an algebraic group.
- $\sigma : \Gamma \to L$ is an injective homomorphism of algebraic groups.
- $\delta : N \to \widehat{\Gamma}$ is a group homomorphism.

If Γ is finite, we call $\mathcal{D}$ a *finite subgroup datum*.

Let $\mathcal{D} = (I_+, I_-, N, \Gamma, \sigma, \delta)$ and $\mathcal{D}' = (I'_+, I'_-, N', \Gamma', \sigma', \delta')$ be subgroup data. We say that $\mathcal{D} \leq \mathcal{D}'$ iff

- $I'_+ \subseteq I_+$ and $I'_- \subseteq I_-$.

Hence $I' \subseteq I$, $\mathbb{T}_{I'} \subseteq \mathbb{T}_I$ and $\mathbb{T}_{I^c} \subseteq \mathbb{T}_{I'^c}$. As $\mathbb{T}_{I'^c} = \mathbb{T}_{I^c} \times \mathbb{T}_{I'^c - I^c}$, the restriction map $\widehat{\mathbb{T}_{I'^c}} \twoheadrightarrow \widehat{\mathbb{T}_{I^c}}$ admits a canonical section η and

- $\eta(N) \subseteq N'$.

- There exists a morphism of algebraic groups $\tau : \Gamma' \to \Gamma$ such that $\sigma\tau = \sigma'$.

- $\delta'\eta = {}^t\tau\delta$.

$\mathcal{D} \simeq \mathcal{D}'$ iff $\mathcal{D} \leq \mathcal{D}'$ and $\mathcal{D}' \leq \mathcal{D}$.

Theorem. *There is a bijection between*

(a) *Hopf algebra quotients* $\mathcal{O}_\epsilon(G) \to A$.

(b) *Subgroup data up to equivalence.*

N. A. & G. A. García, `http://arxiv.org/abs/0707.0070`.

Properties of $A_{\mathcal{D}}$. N. A. & G. A. García, 'Extensions of finite quantum groups by finite groups', `arXiv:math/0608647v6`.

- $\dim A_{\mathcal{D}} < \infty$ iff $|\Gamma| < \infty$
- $A_{\mathcal{D}}$ semisimple iff $|\Gamma| < \infty$, $I = \emptyset$

- If $A_{\mathcal{D}}$ is pointed, then $I_+ \cap -I_- = \emptyset$ and Γ is a subgroup of the group of upper triangular matrices of some size. In particular, if Γ is finite, then it is abelian.

- If $\dim A_{\mathcal{D}} < \infty$ and $A_{\mathcal{D}}^*$ is pointed, then $\sigma(\Gamma) \subseteq \mathcal{T}$.

- If $A_{\mathcal{D}}$ is co-Frobenius then Γ is reductive.

- Some invariants of $A_{\mathcal{D}}$ under isomorphism; complete determination if $H = \mathbf{u}_\epsilon(\mathfrak{g})^*$.

Sketch of the proof.

Let $\mathcal{D} = (I_+, I_-, N, \Gamma, \sigma, \delta)$ be a subgroup datum. We first construct a quotient $A_{\mathcal{D}}$ of $\mathcal{O}_\epsilon(G)$.

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \mathcal{O}(G) & \xrightarrow{\iota} & \mathcal{O}_\epsilon(G) & \xrightarrow{\pi} & \mathbf{u}_\epsilon(\mathfrak{g})^* & \longrightarrow & 1 \\
 & & \Big\downarrow{\scriptstyle \text{res}} & & \Big\downarrow{\scriptstyle \text{Res}} & & \Big\downarrow{\scriptstyle p} & & \\
1 & \longrightarrow & \mathcal{O}(L) & \xrightarrow{\iota_L} & \mathcal{O}_\epsilon(L) & \xrightarrow{\pi_L} & \mathbf{u}_\epsilon(\mathfrak{l})^* & \longrightarrow & 1 \\
 & & \Big\downarrow{\scriptstyle t_\sigma} & & \Big\downarrow{\scriptstyle \nu} & & \Big\| & & \\
1 & \longrightarrow & \mathcal{O}(\Gamma) & \xrightarrow{j} & A_{\mathfrak{l},\sigma} & \xrightarrow{\bar{\pi}} & \mathbf{u}_\epsilon(\mathfrak{l})^* & \longrightarrow & 1 \\
 & & \Big\| & & \Big\downarrow{\scriptstyle t} & & \Big\downarrow{\scriptstyle v} & & \\
1 & \longrightarrow & \mathcal{O}(\Gamma) & \xrightarrow{\widehat{\iota}} & A_{\mathcal{D}} & \xrightarrow{\widehat{\pi}} & H & \longrightarrow & 1.
\end{array}
$$

First step.

Let $\mathbf{u}_\epsilon(\mathfrak{l})$ be the Hopf subalgebra of $\mathbf{u}_\epsilon(\mathfrak{g})$ corresponding to the triple $(\mathbb{T}, I_+, I_-)$.

We have a commutative diagram of exact sequences of Hopf algebras

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathcal{O}(G) & \xrightarrow{\iota} & \mathcal{O}_\epsilon(G) & \xrightarrow{\pi} & \mathbf{u}_\epsilon(\mathfrak{g})^* & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \text{res}} & & \downarrow{\scriptstyle \text{Res}} & & \downarrow{\scriptstyle p} & & \\ 1 & \longrightarrow & \mathcal{O}(L) & \xrightarrow{\iota_L} & \mathcal{O}_\epsilon(L) & \xrightarrow{\pi_L} & \mathbf{u}_\epsilon(\mathfrak{l})^* & \longrightarrow & 1 \end{array}$$

Second step. Let A and K be Hopf algebras, B a central Hopf subalgebra of A such that A is left or right faithfully flat over B and $p : B \to K$ a surjective Hopf algebra map. Then $H = A/AB^+$ is a Hopf algebra and A fits into the exact sequence $1 \to B \xrightarrow{\iota} A \xrightarrow{\pi} H \to 1$. If we set $\mathcal{J} = \ker p \subseteq B$, then $(\mathcal{J}) = A\mathcal{J}$ is a Hopf ideal of A and $A/(\mathcal{J})$ is the pushout:

$$\begin{array}{ccc} B & \overset{\iota}{\hookrightarrow} & A \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ K & \underset{j}{\hookrightarrow} & A/(\mathcal{J}). \end{array}$$

K can be identified with a central Hopf subalgebra of $A/(\mathcal{J})$ and $A/(\mathcal{J})$ fits into the exact sequence

$$\begin{array}{ccccccccc} 1 & \to & B & \xrightarrow{\iota} & A & \xrightarrow{\pi} & H & \to & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \to & K & \xrightarrow{j} & A/(\mathcal{J}) & \to & H & \to & 1. \end{array}$$

We have a surjective Hopf algebra map ${}^t\sigma : \mathcal{O}(L) \to \mathcal{O}(\Gamma)$. By pushout, we construct a Hopf algebra $A_{\mathfrak{l},\sigma}$ which is part of an exact sequence of Hopf algebras and fits into the following commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathcal{O}(G) & \xrightarrow{\iota} & \mathcal{O}_\epsilon(G) & \xrightarrow{\pi} & \mathbf{u}_\epsilon(\mathfrak{g})^* & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \mathrm{res}} & & \downarrow{\scriptstyle \mathrm{Res}} & & \downarrow{\scriptstyle p} & & \\
1 & \longrightarrow & \mathcal{O}(L) & \xrightarrow{\iota_L} & \mathcal{O}_\epsilon(L) & \xrightarrow{\pi_L} & \mathbf{u}_\epsilon(\mathfrak{l})^* & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle {}^t\sigma} & & \downarrow{\scriptstyle \nu} & & \| & & \\
1 & \longrightarrow & \mathcal{O}(\Gamma) & \xrightarrow{j} & A_{\mathfrak{l},\sigma} & \xrightarrow{\bar{\pi}} & \mathbf{u}_\epsilon(\mathfrak{l})^* & \longrightarrow & 1.
\end{array}$$

Third step. Let $H^* \subseteq \mathbf{u}_\epsilon(\mathfrak{g})$ be determined by $(\mathbf{\Sigma}, I_+, I_-)$. Since $\mathbf{u}_\epsilon(\mathfrak{l})$ is determined by the triple $(\mathbb{T}, I_+, I_-)$ with $\mathbb{T} \supseteq \mathbf{\Sigma}$, we have that $H^* \subseteq \mathbf{u}_\epsilon(\mathfrak{l}) \subseteq \mathbf{u}_\epsilon(\mathfrak{g})$. Let $r : \mathbf{u}_\epsilon(\mathfrak{g})^* \to H$ and $v : \mathbf{u}_\epsilon(\mathfrak{l})^* \to H$ be the surjective Hopf algebra maps induced by the inclusions. Then

$$\begin{array}{ccc} \mathbf{u}_\epsilon(\mathfrak{g})^* & \xrightarrow{p} & \mathbf{u}_\epsilon(\mathfrak{l})^* \\ & \searrow^{r} & \downarrow v \\ & & H. \end{array}$$

Now $\mathbb{T}_I \subseteq \mathbf{\Sigma} \subseteq \mathbb{T} = \mathbb{T}_I \times \mathbb{T}_{I^c}$.

If we set $\Omega = \Sigma \cap \mathbb{T}_{I^c}$, then $\Sigma \simeq \mathbb{T}_I \times \Omega$.

TFAE:

- a subgroup $\mathbf{\Sigma}$ such that $\mathbb{T}_I \subseteq \mathbf{\Sigma} \subseteq \mathbb{T}$
- a subgroup $\Omega \subseteq \mathbb{T}_{I^c}$,
- a subgroup $N \subseteq \widehat{\mathbb{T}_{I^c}}$.

For all $1 \leq i \leq n$ such that $\alpha_i \notin I_+$ or $\alpha_i \notin I_-$ we define $D_i \in G(\mathbf{u}_\epsilon(\mathfrak{l})^*) = \mathrm{Alg}(\mathbf{u}_\epsilon(\mathfrak{l}), \mathbb{C})$ on the generators of $\mathbf{u}_\epsilon(\mathfrak{l})$ by

$$D_i(E_j) = 0 \quad \forall\, j:\ \alpha_j \in I_+, \qquad D_i(F_k) = 0 \quad \forall\, k:\ \alpha_k \in I_-,$$
$$D_i(K_{\alpha_t}) = 1 \quad \forall\, t \neq i,\ 1 \leq t \leq n,\ D_i(K_{\alpha_i}) = \epsilon_i,$$

where ϵ_i is a primitive ℓ-th root of 1. We define for all $z = (z_1, \ldots, z_s) \in \widehat{\mathbb{T}_{I^c}}$ $D^z := D_{i_1}^{z_1} \cdots D_{i_s}^{z_s} \in G(\mathbf{u}_\epsilon(\mathfrak{l})^*)$.

(a) D^z is central in $\mathbf{u}_\epsilon(\mathfrak{l})^*$, for all $z \in \widehat{\mathbb{T}_{I^c}}$.

(b) $H \simeq \mathbf{u}_\epsilon(\mathfrak{l})^*/(D^z - 1 | z \in N)$.

(c) There exists a subgroup $\mathbf{Z} := \{\partial^z |\ z \in \widehat{\mathbb{T}_{I^c}}\}$ of $G(A_{\mathfrak{l},\sigma})$ isomorphic to $\{D^z |\ z \in \widehat{\mathbb{T}_{I^c}}\}$ consisting of central elements.

Finally, $A_{\mathcal{D}}$ is given by the quotient $A_{\mathfrak{l},\sigma}/J_\delta$ where J_δ is the two-sided ideal generated by the set $\{\partial^z - \delta(z) | z \in N\}$ and the following diagram of exact sequences of Hopf algebras is commutative

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathcal{O}(G) & \xrightarrow{\iota} & \mathcal{O}_\epsilon(G) & \xrightarrow{\pi} & \mathbf{u}_\epsilon(\mathfrak{g})^* & \longrightarrow & 1 \\
 & & {\scriptstyle \text{res}}\downarrow & & \downarrow{\scriptstyle \text{Res}} & & \downarrow{\scriptstyle p} & & \\
1 & \longrightarrow & \mathcal{O}(L) & \xrightarrow{\iota_L} & \mathcal{O}_\epsilon(L) & \xrightarrow{\pi_L} & \mathbf{u}_\epsilon(\mathfrak{l})^* & \longrightarrow & 1 \\
 & & {\scriptstyle t_\sigma}\downarrow & & \downarrow{\scriptstyle \nu} & & \| & & \\
1 & \longrightarrow & \mathcal{O}(\Gamma) & \xrightarrow{j} & A_{\mathfrak{l},\sigma} & \xrightarrow{\bar{\pi}} & \mathbf{u}_\epsilon(\mathfrak{l})^* & \longrightarrow & 1 \\
 & & \| & & \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle v} & & \\
1 & \longrightarrow & \mathcal{O}(\Gamma) & \xrightarrow{\hat{\iota}} & A_{\mathcal{D}} & \xrightarrow{\hat{\pi}} & H & \longrightarrow & 1.
\end{array}$$

Fourth step. Let U be any Hopf algebra and consider the category $\mathcal{QUOT}(U)$, whose objects are surjective Hopf algebra maps $q : U \to A$. If $q : U \to A$ and $q' : U \to A'$ are such maps, then an arrow $q \xrightarrow{\alpha} q'$ in $\mathcal{QUOT}(U)$ is a Hopf algebra map $\alpha : A \to A'$ such that $\alpha q = q'$. A *quotient* of U is just an isomorphism class of objects in $\mathcal{QUOT}(U)$; let $[q]$ denote the class of the map q. There is a partial order in the set of quotients of U, given by $[q] \leq [q']$ iff there exists an arrow $q \xrightarrow{\alpha} q'$ in $\mathcal{QUOT}(U)$. Notice that $[q] \leq [q']$ and $[q'] \leq [q]$ implies $[q] = [q']$.

Lemma. *Let $\mathcal{D}$ and $\mathcal{D}'$ be subgroup data. Then*
(a) $[A_{\mathcal{D}}] \leq [A_{\mathcal{D}'}]$ iff $\mathcal{D} \leq \mathcal{D}'$.
(b) $[A_{\mathcal{D}}] = [A_{\mathcal{D}'}]$ iff $\mathcal{D} \simeq \mathcal{D}'$.

Fifth step. Let $q : \mathcal{O}_\epsilon(G) \to A$ be a surjective Hopf algebra map. We show that it is isomorphic to $q_{\mathcal{D}} : \mathcal{O}_\epsilon(G) \to A_{\mathcal{D}}$ for some subgroup datum $\mathcal{D}$.

The Hopf subalgebra $K = q(\mathcal{O}(G))$ is central in A and whence A is an H-extension of K, where $H := A/AK^+$.

There exists an algebraic group Γ and an injective map of algebraic groups $\sigma : \Gamma \to G$ such that $K \simeq \mathcal{O}(\Gamma)$.

Since $q(\mathcal{O}_\epsilon(G)\mathcal{O}(G)^+) = AK^+$, we have $\mathcal{O}_\epsilon(G)\mathcal{O}(G)^+ \subseteq \ker \widehat{\pi} q$, where $\widehat{\pi} : A \to H$ is the canonical projection. Since $\mathbf{u}_\epsilon(\mathfrak{g})^* \simeq \mathcal{O}_\epsilon(G)/[\mathcal{O}_\epsilon(G)\mathcal{O}(G)^+]$, there exists a surjective map $r : \mathbf{u}_\epsilon(\mathfrak{g})^* \to H$; H^* is determined by a triple (Σ, I_+, I_-). In particular, we have the following commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathcal{O}(G) & \xrightarrow{\iota} & \mathcal{O}_\epsilon(G) & \xrightarrow{\pi} & \mathbf{u}_\epsilon(\mathfrak{g})^* & \longrightarrow & 1 \\ & & {\scriptstyle {}^t\sigma}\downarrow & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle r} & & \\ 1 & \longrightarrow & \mathcal{O}(\Gamma) & \xrightarrow{\widehat{\iota}} & A & \xrightarrow{\widehat{\pi}} & H & \longrightarrow & 1. \end{array}$$

Lema. $\sigma(\Gamma) \subseteq L$, A is a quotient of $A_{\mathfrak{l},\sigma}$ given by pushout.

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathcal{O}(G) & \xrightarrow{\iota} & \mathcal{O}_\epsilon(G) & \xrightarrow{\pi} & \mathbf{u}_\epsilon(\mathfrak{g})^* & \longrightarrow & 1 \\
 & & \text{res}\downarrow & & \downarrow\text{Res} & & \downarrow p & & \\
1 & \longrightarrow & \mathcal{O}(L) & \xrightarrow{\iota_L} & \mathcal{O}_\epsilon(L) & \xrightarrow{\pi_L} & \mathbf{u}_\epsilon(\mathfrak{l})^* & \longrightarrow & 1 \\
 & & u\downarrow & & \downarrow\nu & & \| & & \\
1 & \longrightarrow & \mathcal{O}(\Gamma) & \xrightarrow{j} & A_{\mathfrak{l},\sigma} & \xrightarrow{\bar{\pi}} & \mathbf{u}_\epsilon(\mathfrak{l})^* & \longrightarrow & 1 \\
 & & \| & & \downarrow t & & \downarrow v & & \\
1 & \longrightarrow & \mathcal{O}(\Gamma) & \xrightarrow{\hat{\iota}} & A & \xrightarrow{\hat{\pi}} & H & \longrightarrow & 1.
\end{array}$$

Finally, there exists a group homomorphism $\delta : N \to \widehat{\Gamma}$ such that $J_\delta = (\partial^z - \delta(z)|\ z \in N)$ is a Hopf ideal of $A_{\mathfrak{l},\sigma}$ and $A \simeq A_{\mathcal{D}} = A_{\mathfrak{l},\sigma}/J_\delta$.