On the classification of finite-dimensional Hopf algebras

Nicolás Andruskiewitsch

Universidad de Córdoba, Argentina

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Plan of the talk.

I. Introduction.

II. Semisimple Hopf algebras.

III. The lifting method.

IV. Pointed Hopf algebras with abelian group.

V. Pointed Hopf algebras with non-abelian group.

VI. A generalized lifting method.

I. Introduction. $\mathbb{C}$ algebraically closed field.

A algebra: product $\mu : A \otimes A \to A$, unit $u : \mathbb{C} \to A$

Associative:

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{\text{id} \otimes \mu} & A \otimes A \\ {\scriptstyle \mu \otimes \text{id}}\Big\downarrow & & \Big\downarrow{\scriptstyle \mu} \\ A \otimes A & \xrightarrow{\quad \mu \quad} & A \end{array}$$

Unitary:

$$\begin{array}{ccccc} \mathbb{C} \otimes A & \xrightarrow{u \otimes \text{id}} & A \otimes A & \xleftarrow{\text{id} \otimes u} & A \otimes \mathbb{C} \\ & {\scriptstyle \sim}\searrow & \Big\downarrow{\scriptstyle \mu} & \swarrow{\scriptstyle \sim} & \\ & & A & & \end{array}$$

C coalgebra: coproduct $\Delta : C \to C \otimes C$, counit $\varepsilon : C \to \mathbb{C}$

Co-associative:

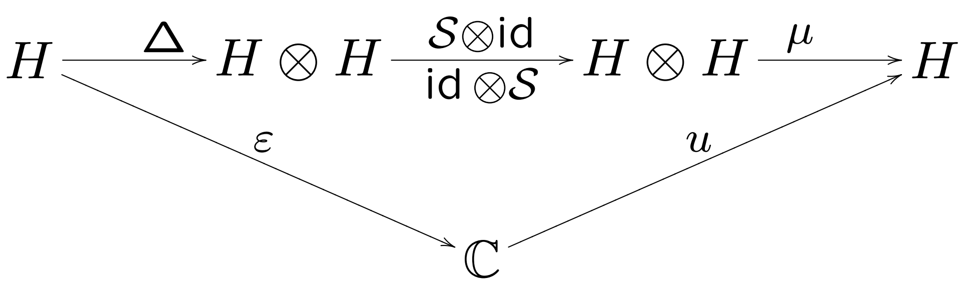
$$\begin{array}{ccc} C \otimes C \otimes C & \xleftarrow{\mathrm{id} \otimes \Delta} & C \otimes C \\ \uparrow{\scriptstyle \Delta \otimes \mathrm{id}} & & \uparrow{\scriptstyle \Delta} \\ C \otimes C & \xleftarrow{\Delta} & C \end{array}$$

Co-unitary:

$$\begin{array}{ccccc} \mathbb{C} \otimes C & \xleftarrow{\varepsilon \otimes \mathrm{id}} & C \otimes C & \xrightarrow{\mathrm{id} \otimes \varepsilon} & C \otimes \mathbb{C} \\ & {\scriptstyle \sim}\searrow & \uparrow{\scriptstyle \Delta} & \swarrow{\scriptstyle \sim} & \\ & & C & & \end{array}$$

Hopf algebra: $(H, \mu, u, \Delta, \varepsilon)$

- (H, μ, u) algebra
- (H, Δ, ε) coalgebra
- Δ, ε algebra maps
- There exists $\mathcal{S} : H \to H$ (the antipode) such that



Example:

- G finite group
- $H = \mathcal{O}(G) =$ algebra of functions $G \to \mathbb{C}$
- $\Delta : H \to H \otimes H \simeq \mathcal{O}(G \times G)$, $\Delta(f)(x, y) = f(x.y)$.
- $\varepsilon : H \to \mathbb{C}$, $\varepsilon(f) = f(e)$.
- $\mathcal{S} : H \to H$, $\mathcal{S}(f)(x) = f(x^{-1})$.

Remark: $(H, \mu, u, \Delta, \varepsilon)$ finite-dimensional Hopf algebra
$\Longrightarrow$ $(H^*, \Delta^t, \varepsilon^t, \mu^t, u^t)$ Hopf algebra

Example: $H = \mathcal{O}(G)$; for $x \in G$, $E_x \in H^*$, $E_x(f) = f(x)$. Then

$$E_x E_y = E_{xy}, \qquad \mathcal{S}(E_x) = E_{x^{-1}}.$$

Hence $H^* = \mathbb{C}G$, group algebra of G.

Remark: $(H, \mu, u, \Delta, \varepsilon)$ Hopf algebra with $\dim H = \infty$,
H^* NOT a Hopf algebra,
but contains a largest Hopf algebra with operations transpose to those of H.

Example:

- G affine algebraic group

- $H = \mathcal{O}(G) =$ algebra of regular (polynomial) functions $G \to \mathbb{C}$ is a Hopf algebra with analogous operations.

- $H^* \supset \mathbb{C}G$
- $H^* \supset \mathcal{U} :=$ algebra of distributions with support at e; this is a Hopf algebra

- If $\operatorname{char} \mathbb{C} = 0$, then $\mathcal{U} \simeq U(\mathfrak{g})$, $\mathfrak{g} =$ Lie algebra of G

- If $\mathfrak{g}$ is any Lie algebra, then the enveloping algebra $U(\mathfrak{g})$ is a Hopf algebra with $\Delta(x) = x \otimes 1 + 1 \otimes x$, $x \in \mathfrak{g}$.

Short history:

- Since the dictionary *Lie groups* $\leftrightsquigarrow$ *Lie algebras* fails when char > 0, Dieudonné studied in the early 50's the hyperalgebra $\mathcal{U}$. Pierre Cartier introduced the abstract notion of hyperalgebra (cocommutative Hopf algebra) in 1955.

- Armand Borel considered algebras with a coproduct in 1952, extending previous work of Hopf. He coined the expression *Hopf algebra*.

- George I. Kac introduced an analogous notion in the context of von Neumann algebras.

- The first appearance of the definition (that I am aware of) as it is known today is in a paper by Kostant (1965).

First invariants of a Hopf algebra H:
$G(H) = \{x \in H - 0 : \Delta(x) = x \otimes x\}$, group of grouplikes.

$\mathrm{Prim}(H) = \{x \in H : \Delta(x) = x \otimes 1 + 1 \otimes x\}$, Lie algebra of primitive elements.

$\tau : V \otimes W \to W \otimes W$, $\tau(v \otimes w) = w \otimes v$ the *flip*.
H is commutative if $\mu\tau = \mu$. H is cocommutative if $\tau\Delta = \Delta$.

Group algebras, enveloping algebras, hyperalgebras are cocommutative.

Theorem. (Cartier-Kostant, early 60's). $\mathrm{char}\,\mathbb{C} = 0$.
Any cocommutative Hopf algebra is of the form $U(\mathfrak{g})\#\mathbb{C}G$.

$H = \mathbb{C}[X]$, $\Delta(X) = X \otimes 1 + 1 \otimes X$. Then

$$\Delta(X^n) = \sum_{0 \leq j \leq n} \binom{n}{j} X^j \otimes X^{n-j}.$$

If $\operatorname{char} \mathbb{C} = p > 0$, then $\Delta(X^p) = X^p \otimes 1 + 1 \otimes X^p$.

Thus $\mathbb{C}[X]/\langle X^p \rangle$, $\Delta(X) = X \otimes 1 + 1 \otimes X$ is a Hopf algebra, commutative and cocommutative, dim p.

(Kulish, Reshetikhin and Sklyanin, 1981). Quantum $SL(2)$: if $q \in \mathbb{C}$, $q \neq 0, \pm 1$, set

$$\begin{aligned} U_q(\mathfrak{sl}(2)) = \mathbb{C}\langle E, F, K, K^{-1} | KK^{-1} &= 1 = K^{-1}K \\ KE &= q^2 EK, \\ KF &= q^{-2} FK, \\ EF - FE &= \frac{K - K^{-1}}{q - q^{-1}} \rangle \end{aligned}$$

$$\begin{aligned} \Delta(K) &= K \otimes K, \\ \Delta(E) &= E \otimes 1 + K \otimes E, \\ \Delta(F) &= F \otimes K^{-1} + 1 \otimes F. \end{aligned}$$

These Hopf algebras, neither commutative nor cocommutative, are analogues of the enveloping algebra of $\mathfrak{sl}(2)$.

(Lusztig, 1989). If q is a root of 1 of order N odd, then

$$\mathfrak{u}_q(\mathfrak{sl}(2)) = \mathbb{C}\langle E, F, K, K^{-1} | \text{same relations plus}$$
$$K^N = 1, \quad E^N = F^N = 0\rangle.$$

These Hopf algebras, neither commutative nor cocommutative, are analogues of the Frobenius kernel of $\mathfrak{sl}(2)$.

In 1983, Drinfeld and Jimbo introduced quantized enveloping algebras $U_q(\mathfrak{g})$, for q as above and $\mathfrak{g}$ any simple Lie algebra.

- Quantum function algebras $\mathcal{O}_q(G)$: Faddeev-Reshetikhin and Takhtajan (for $SL(N)$) and Lusztig (any simple G).

- Finite-dimensional versions when q is a root of 1.

II. Semisimple Hopf algebras. = the underlying Hopf algebra is semisimple (finite-dimensional).

- Group algebras $\mathbb{C}G$ and their duals $\mathcal{O}(G)$ are semisimple.

Methods of construction of Hopf algebras:

- Duals. *The dual of a semisimple Hopf algebra is semisimple.*

- Extensions.

- Twisting.

- Bosonization.

- Extensions:

- Two Hopf algebras A, B;
- Extra data: action, coaction, cocycle, dual cocycle, $\rightsquigarrow$ Hopf algebra structure in $C := A \otimes B$ such that

$$1 \longrightarrow A \longrightarrow C \longrightarrow B \longrightarrow 1$$

is an extension of Hopf algebras.

- If A and B are semisimple, then so is C.

- C is a *simple* Hopf algebra if it can not be presented as an extension; $\mathbb{C}G$ is simple iff G is simple.

In practice, very hard to produce explicit examples.

- Abelian extensions (G. I. Kac, Takeuchi):

- Two finite groups F, G; compatible actions $F \overset{\triangleleft}{\leftarrow} G \times F \overset{\triangleright}{\rightarrow} G$ $\leftrightsquigarrow$ exact factorization $\Gamma := F \cdot G \rightsquigarrow$ Hopf algebra structure in $\mathcal{O}(G) \bowtie \mathbb{C}F := \mathcal{O}(G) \otimes \mathbb{C}F$ such that

$$1 \longrightarrow \mathcal{O}(G) \longrightarrow \mathcal{O}(G) \bowtie \mathbb{C}F \longrightarrow \mathbb{C}F \longrightarrow 1$$

is an extension of Hopf algebras. Even with cocycles:

$$1 \longrightarrow \mathcal{O}(G) \longrightarrow \mathcal{O}(G) \bowtie_{\sigma}^{\tau} \mathbb{C}F \longrightarrow \mathbb{C}F \longrightarrow 1$$

Note $(\mathcal{O}(G) \bowtie \mathbb{C}F)^* \simeq \mathcal{O}(F) \bowtie \mathbb{C}G$, w.r.t. $\Gamma := G \cdot F$.

• Twisting (Drinfeld).

H a Hopf algebra, $J \in H \otimes H$ invertible. Assume:

$$(1 \otimes J)(\mathrm{id} \otimes \Delta)(J) = (J \otimes 1)(\Delta \otimes \mathrm{id})(J),$$
$$(\mathrm{id} \otimes \varepsilon)(J) = 1 = (\varepsilon \otimes \mathrm{id})(J).$$

Let $\Delta_J := J\Delta J^{-1} : H \to H \otimes H \rightsquigarrow H_J = (H, \Delta_J)$ is again a Hopf algebra, *the twisting* H *by* J. If H is semisimple, then so is H_J.

• Twistings of group algebras classify *triangular* semisimple Hopf algebras (Etingof-Gelaki).

• G simple group $\Longrightarrow$ twistings $(\mathbb{C}G)_J$ are simple (Nikshych).

• $\exists G$ solvable group and a *simple* twisting $(\mathbb{C}G)_J$ (Galindo-Natale).

All the known semisimple Hopf algebras (that I know) are:

- Group algebras and their duals.
- Twistings of group algebras and their duals.
- Abelian extensions, their twistings and duals.
- Two-step extensions $1 \longrightarrow \mathcal{O}(G) \bowtie \mathbb{C}F \longrightarrow H \longrightarrow \mathbb{C}L \longrightarrow 1$ (Nikshych)

Question: Is any semisimple Hopf algebra constructed from group algebras using twistings, duals and extensions?

• Bosonization (Radford, Majid).

A braided vector space is a pair (V, c), where V is a vector space and $c : V \otimes V \to V \otimes V$ is a linear isomorphism that satisfies

$$(c \otimes \mathrm{id})(\mathrm{id} \otimes c)(c \otimes \mathrm{id}) = (\mathrm{id} \otimes c)(c \otimes \mathrm{id})(\mathrm{id} \otimes c).$$

This is called the braid equation (closely related to the quantum Yang-Baxter equation).

• Any Hopf algebra (with bijective antipode) gives a machine of solutions of the braid equation.

• The solutions associated to $U_q(\mathfrak{g})$ are very important in low dimensional topology and some areas of theoretical physics.

Braided Hopf algebra: $(R, c, \mu, u, \Delta, \varepsilon)$

- (R, c) braided vector space
- (R, μ, u) algebra, (R, Δ, ε) coalgebra
- Δ, ε algebra maps, with the multiplication μ_2 in $R \otimes R$

$$\begin{array}{ccc} R\otimes R\otimes R\otimes R & \xrightarrow{\mathrm{id}\otimes c\otimes \mathrm{id}} & R\otimes R\otimes R\otimes R \\ & {\scriptstyle \mu_2}\searrow \qquad \swarrow{\scriptstyle \mu\otimes\mu} & \\ & R\otimes R & \end{array}$$

- There exists $\mathcal{S} : R \to R$, the antipode.

Braided Hopf algebras appear in nature: Let $\pi : H \to K$ be a surjective morphism of Hopf algebras that admits a section $\iota : K \to H$, also a morphism of Hopf algebras. Then

$$R = \{x \in H : (\mathrm{id} \otimes \pi)\Delta(x) = x \otimes 1\}$$

is a braided Hopf algebra; it bears an action and a coaction of K. Also

$$H \simeq R \# K.$$

We say that H is the bosonization of R by K.

H is semisimple iff R and K are so, but no new example appears in this way.

III. The lifting method.

We describe a method, joint with H.-J. Schneider.

Let C be a coalgebra, $D, E \subset C$. Then

$D \wedge E = \{x \in C : \Delta(x) \in D \otimes C + C \otimes E\}$.

$\wedge^0 D = D$, $\wedge^{n+1} D = (\wedge^n D) \wedge D$.

Invariants of a Hopf algebra H:

- The coradical $H_0 =$ sum of all simple subcoalgebras of H.
- The *coradical filtration* is $H_n = \wedge^{n+1} H_0$.

Hypothesis: The coradical H_0 is a Hopf subalgebra of H.

Example: H is pointed when $H_0 = \mathbb{C}G(H)$.

In this case the coradical filtration is a Hopf algebra filtration $\rightsquigarrow$ the *associated graded Hopf algebra* $\operatorname{gr} H = \oplus_{n\in\mathbb{N}} H_n/H_{n-1}$.

It turns out that $\operatorname{gr} H \simeq R\#H_0$, where

- $R = \oplus_{n\in\mathbb{N}} R^n$ is a graded connected algebra and it is a braided Hopf algebra.

- The subalgebra of R generated by R^1 is a braided Hopf subalgebra of a special sort– a Nichols algebra.

Hypothesis: The coradical H_0 is a Hopf subalgebra of H.

$H_0 \subset H_1 \cdots \subset H_n \subset \cdots \subset \bigcup_{n\in\mathbb{N}} H_n = H$

$$H \rightsquigarrow \mathrm{gr}\, H = \oplus_{n\in\mathbb{N}} H_n/H_{n-1} = R\# H_0 \rightsquigarrow R \supset \mathfrak{B}(V)$$

Here $V = R^1$ is a braided vector space of a special sort.

Question I. Classify all $V \in {}^{H_0}_{H_0}\mathcal{YD}$ s. t. $\dim \mathfrak{B}(V) < \infty$; for them find a presentation by generators and relations.
Question II. $\mathfrak{B}(V) = R$?
Question III. Given V, classify all H s. t. $\mathrm{gr}\, H \simeq \mathfrak{B}(V)\# H_0$.

Nichols algebras:

given a braided vector space (V, c), its Nichols algebra is a braided Hopf algebra

$$\mathfrak{B}(V) = \oplus_{n \in \mathbb{N}_0} \mathfrak{B}^n(V),$$

- $\mathfrak{B}^0(V) = \mathbb{C}$, $\mathfrak{B}^1(V) = V$,
- $\mathrm{Prim}(\mathfrak{B}(V)) = V$,
- V generates $\mathfrak{B}(V)$ as an algebra.

IV. Pointed Hopf algebras with abelian group.

G a finite abelian group; $H_0 \simeq \mathbb{C}G$

Question I. Classify all $V \in {}^{H_0}_{H_0}\mathcal{YD}$ s. t. $\dim \mathfrak{B}(V) < \infty$; for them find a presentation by generators and relations.

In this case, all $V \in {}^{H_0}_{H_0}\mathcal{YD}$ are of diagonal type:
$\exists$ basis $v_1, \dots, v_\theta$, $(q_{ij})_{1 \le i,j \le \theta}$ in $\mathbb{C}^\times$:

$$c(v_i \otimes v_j) = q_{ij} v_j \otimes v_i, \quad \forall i, j$$

Theorem 1 (Heckenberger). V of diagonal type, $\dim \mathfrak{B}(V) < \infty$ classified.

I. Heckenberger, *Classification of arithmetic root systems*, Adv. Math. (2009).

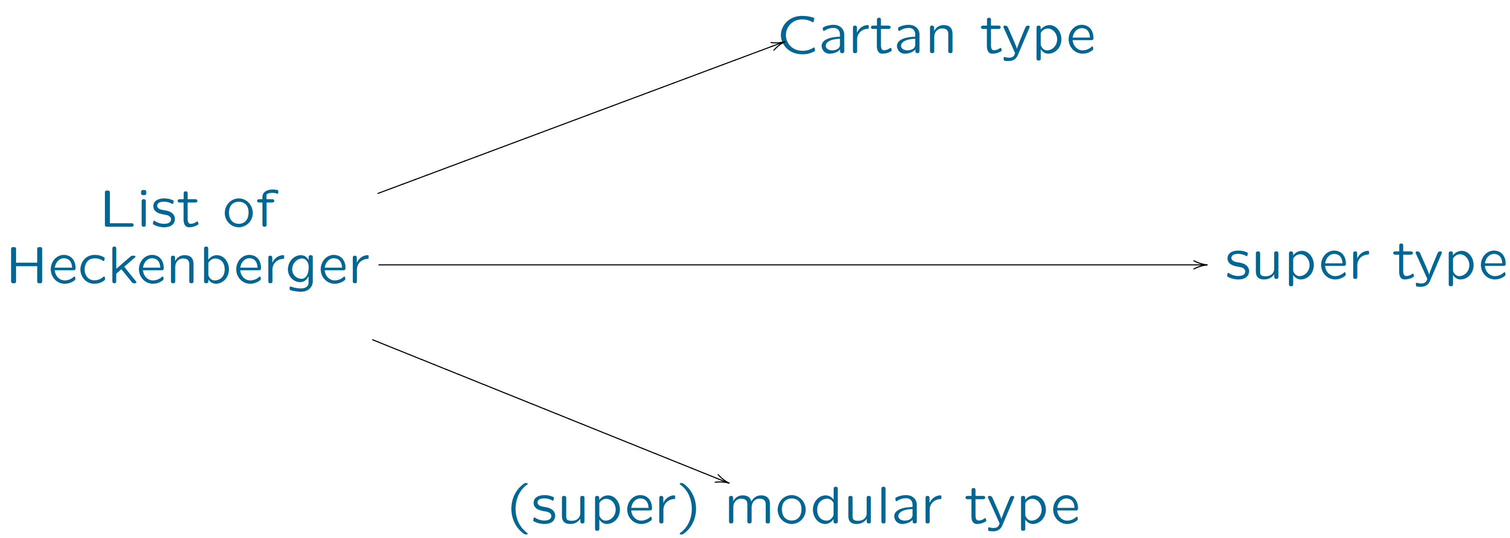
Theorem 2 (Angiono). (V, c) of diagonal type with $\dim \mathfrak{B}(V) < \infty$. Then a minimal presentation by generators and relations is known.

I. Angiono, *A presentation by generators and relations of Nichols algebras of diagonal type and convex orders on root systems*, J. Eur. Math. Soc., to appear.

I. Angiono, *On Nichols algebras of diagonal type*, Crelle, to appear.

Question II. $\mathfrak{B}(V) = R$? Solved in this context:

Theorem 3 (Angiono). H a finite-dimensional pointed Hopf algebra with $G(H)$ abelian. Then H is generated by group-like and primitive elements.



If the prime divisors of $|G|$ are > 7, then the possible V are of Cartan type. In this context, the classification of all pointed Hopf algebras with group G is known.

N. A. and H.-J. Schneider, *On the classification of finite-dimensional pointed Hopf algebras*, Ann. Math. Vol. 171 (2010), No. 1, 375-417.

The outcome is that all are variations of the Lusztig's small quantum groups.

V. Pointed Hopf algebras with non-abelian group.

- There is a class of braided vector spaces playing the role of diagonal type: related to racks (and 2-cocycles).

- Very difficult to compute the corresponding Nichols algebras; only known in a few examples.

- Some criteria to decide that a Nichols algebras has infinite dimension.

Application. If $G = \mathbb{A}_n$, $n \geq 5$, or sporadic (except Monster, Baby Monster and Fi_{22}), then the only finite-dimensional pointed Hopf algebra with group G is the group algebra $\mathbb{C}G$.

VI. A generalized lifting method.

- The *Hopf coradical* $H_{[0]}$ is the subalgebra generated by H_0.
- The *standard filtration* is $H_{[n]} = \wedge^{n+1} H_{[0]}$.
- The *associated graded Hopf algebra* $\operatorname{gr} H = \oplus_{n\in\mathbb{N}} H_{[n]}/H_{[n-1]}$.

It turns out that $\operatorname{gr} H \simeq R\# H_{[0]}$, where

- $R = \oplus_{n\in\mathbb{N}} R^n$ is a graded connected algebra and it is a braided Hopf algebra.

Theorem. (A.– Cuadra).

Any Hopf algebra with injective antipode is a deformation of the bosonization of connected graded braided Hopf algebra by a Hopf algebra generated by a cosemisimple coalgebra.

To provide significance to this result, we should address some fundamental questions.

Question I. Let C be a finite-dimensional cosemisimple coalgebra and $T : C \to C$ a bijective morphism of coalgebras. Classify all finite-dimensional Hopf algebras L generated by C such that $\mathcal{S}_{|C} = T$.

Question II. Given L as in the previous item, classify all finite-dimensional connected graded Hopf algebra in ${}^{L}_{L}\mathcal{Y}D$.

Question III. Given L and R as in the previous items, classify all deformations, or liftings, H, that is, such that $\operatorname{gr} H \simeq R\#L$.

About Question I:

Theorem. (Stefan).

Let H be a Hopf algebra and C an $\mathcal{S}$-invariant 4-dimensional simple subcoalgebra. If $1 < \operatorname{ord} \mathcal{S}^2_{|C} = n < \infty$, then there are a root of unity ω and a Hopf algebra morphism $\mathcal{O}_{\sqrt{-\omega}}(SL_2(\mathbb{C})) \to H$.

- Classification of finite-dimensional quotients of $\mathcal{O}_q(SL_N(\mathbb{C}))$: E. Müller.

- Classification of quotients of $\mathcal{O}_q(G)$: A.–G. A. García.