Co-Frobenius Hopf algebras

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IV Congreso Latinoamericano de Matemáticos

Sesión de Álgebra

Córdoba, 9 de agosto de 2012

I. Motivation.

Let G be a locally compact group. Then G bears a (left) Haar measure μ, which is left-invariant in the sense that

$$\int_G f(yx)d_\mu x = \int_G f(x)d_\mu x, \qquad \forall f \text{ measurable.}$$

Also μ is unique up to a scalar, but it is not always a right Haar measure. Indeed, the modular function $\Delta : G \to \mathbb{R}^+$ satisfies

$$\int_G f(xy)d_\mu x = \Delta^{-1}(y) \int_G f(x)d_\mu x, \qquad \forall f \text{ measurable;}$$

so that μ is right and left Haar $\iff$ $\Delta \equiv 1$ $\iff$ G is unimodular.

Now let K be a compact Lie group; K is the set of real points of an algebraic complex group G. Then the algebra $\mathcal{O}(G)$ of rational functions on G is contained in $L^1(K)$ so we may consider the restriction

$$\int : \mathcal{O}(G) \to \mathbb{C}.$$

The algebra decomposes as $\mathcal{O}(G) = \oplus_{\rho \in \mathrm{Irrep}\, K} V_\rho \otimes V_\rho^*$; then $\int$ coincides with the projection onto the summand corresponding to the trivial representation; hence K is unimodular. Recall

• Every finite-dimensional continuous representation K is completely reducible (semisimple).
• $\int(1) \neq 0$.

Also $\mathcal{O}(G)$ is a **Hopf algebra**.

Hopf algebra over an arbitrary field $\Bbbk$: $(H, \mu, u, \Delta, \varepsilon)$

- (H, μ, u) algebra

- (H, Δ, ε) coalgebra

- Δ, ε algebra maps

- There exists $\mathcal{S} : H \to H$ (the antipode) such that

$$H \xrightarrow{\Delta} H \otimes H \xrightarrow[\mathrm{id} \otimes \mathcal{S}]{\mathcal{S} \otimes \mathrm{id}} H \otimes H \xrightarrow{\mu} H$$

$$H \xrightarrow{\varepsilon} \Bbbk \xrightarrow{u} H$$

Example:

- Γ finite group, $H = \mathcal{O}(\Gamma) =$ algebra of functions $\Gamma \to \mathbb{k}$

- $\Delta : H \to H \otimes H \simeq \mathcal{O}(\Gamma \times \Gamma)$, $\Delta(f)(x,y) = f(x.y)$.

- $\varepsilon : H \to \mathbb{k}$, $\varepsilon(f) = f(e)$. $\mathcal{S} : H \to H$, $\mathcal{S}(f)(x) = f(x^{-1})$.

- $H^* = \mathbb{k}\Gamma =$ group alg. of $\Gamma = \langle$evaluations $e_g, g \in \Gamma\rangle$

- $\int : H \to \mathbb{k}$, $\int = \sum_{g\in\Gamma} e_g \implies \langle \int, f\rangle = \langle \int, L_h f\rangle, \forall h \in G, f \in H$.

- $\operatorname{Rep}_{\mathbb{k}} \Gamma$ is semisimple $\iff$ $\operatorname{Rep} \mathbb{k}\Gamma$ is semisimple $\iff$ $\langle \int, 1\rangle \neq 0 \iff$ $\operatorname{char} \mathbb{k} \nmid |\Gamma|$

Definition. (Hochschild, 1965; G. I. Kac, 1961; Larson-Sweedler, 1969). Let H be a Hopf algebra over $\mathbb{k}$. Then a (left) integral on H is a linear map $\int : H \to \mathbb{k}$ such that $(\mathrm{id} \otimes \int)\Delta(f) = \int(f)1$ for all $f \in H$.

Let $I_\ell(H)$, resp. $I_r(H)$, be the vector space of left, resp. right, integrals on H.

Theorem. (Sullivan, 1971). $\dim I_\ell(H) = \dim I_r(H) \leq 1$.

Definition. H is co-Frobenius when $\dim I_\ell(H) = 1$.

But $I_\ell(H) \neq I_r(H)$ in general; $\exists a \in G(H)$ (the *modular* group-like) that permutes them. So, $I_\ell(H) = I_r(H) \iff a = 1 \iff H$ is *unimodular*.

Theorem. (Larson-Sweedler, 1969; G. I. Kac-Palyutkin, 1961). A finite-dimensional Hopf algebra is co-Frobenius. Thus there are co-Frobenius but not cosemisimple Hopf algebras.

It follows that every finite-dimensional Hopf algebra is Frobenius.

Let H be a Hopf algebra (not necessarily finite-dimensional).

We denote the category of left (resp. right) H-comodules by ${}^{H}\mathcal{M}$ (resp. $\mathcal{M}^{H}$). For instance, if $H = \mathcal{O}(G)$ for G an algebraic group, then $\mathcal{M}^{H} \simeq \operatorname{Rep} G$ (the category of rational G-modules).

Definition. H is right, resp. left, cosemisimple when ${}^{H}\mathcal{M}$, resp. $\mathcal{M}^{H}$, is semisimple, i. e. any object is direct sum of simple ones.

Theorem. (Sweedler, 1969). Let H be a Hopf algebra. TFAE:

1. H is left cosemisimple.
2. H is co-Frobenius and for $\mu \in I_\ell(H) - 0, \mu(1) \neq 0$.
3. As a left comodule via Δ, H is semisimple.
4. The trivial left comodule is injective.
5. All the above with right instead of left.
6. $\mathcal{O}(H) = \oplus_{\rho \in \widehat{^H\mathcal{M}}} V_\rho \otimes V_\rho^*$

If this happens, then the projection to the trivial part in (6) is a left and right integral, so H is unimodular.

Examples:

• $\mathfrak{g}$ Lie algebra, then $H = U(\mathfrak{g})$ is not co-Frobenius.

• G algebraic connected affine group, $H = \mathcal{O}(G)$. Then $H = \mathcal{O}(G)$ is co-Frobenius $\iff$ G is linearly reductive $\iff$ H is cosemisimple. But:

∘ $\operatorname{char} \Bbbk = 0$: G is linearly reductive $\iff$ G is reductive, i.e. trivial unipotent radical.

∘ $\operatorname{char} \Bbbk \neq 0$: G is linearly reductive $\iff$ G is a torus.

• G algebraic simple affine group, $1 \neq \in \Bbbk^{\times} = \mathbb{C}^{\times}$; then $H = \mathcal{O}_q(G)$ (quantum algebra of functions on G) is co-Frobenius. But:

∘ If $q \notin \mathbb{G}_{\infty}$, then H is cosemisimple.

∘ If $q \in \mathbb{G}_{\infty}$, then H is not cosemisimple.

II. Characterizations. The coradical filtration.

The category ${}^H\mathcal{M}$ of left comodules over a Hopf algebra H is abelian and the notions of injective, resp. projective, comodules make sense; and *a fortiori*, the notions of *injective hull* (denoted $E(S)$ for $S \in {}^H\mathcal{M}$) and *projective cover* are available. However:

1. ${}^H\mathcal{M}$ has enough injectives, although the injective hull of a finite-dimensional comodule might has infinite dimension.

2. Projective non-zero objects in ${}^H\mathcal{M}$ may not exist.

Theorem. (Lin, 1977; Dăscălescu-Năstăsescu, 2009; Donkin, 1996, 98; A.-Cuadra, 2011). Let H be a Hopf algebra. TFAE:

1. H is co-Frobenius.
2. $E(S)$ is finite dimensional for every $S \in {}^H\mathcal{M}$ simple.
3. $E(\Bbbk)$ is finite dimensional.
4. ${}^H\mathcal{M}$ has a nonzero finite dimensional injective object.
5. Every $0 \neq M \in {}^H\mathcal{M}$ has a finite dimensional quotient $\neq 0$.
6. ${}^H\mathcal{M}$ possesses a nonzero projective object.
7. Every $M \in {}^H\mathcal{M}$ has a projective cover.
8. Every injective in ${}^H\mathcal{M}$ is projective.

Definition. (A.-Cuadra-Etingof, 2012). A tensor category is co-Frobenius when any object has an injective hull. This turns out to be a generalization of (2) above.

Let C be a coalgebra, $D, E \subset C$. Then
$D \wedge E = \{x \in C : \Delta(x) \in D \otimes C + C \otimes E\}$,
$\wedge^0 D = D$, $\wedge^{n+1} D = (\wedge^n D) \wedge D$.

Some invariants of a Hopf algebra H:

- The coradical H_0 = sum of all simple subcoalgebras of H.

- The *coradical filtration* is $H_n = \wedge^{n+1} H_0$.

Then: $(H_n)_{n\geq 0}$ is a coalgebra filtration and $\cup_{n\geq 0} H_n = H$.

Furthermore, if H_0 Hopf subalgebra, then $(H_n)_{n\geq 0}$ is an algebra filtration and $\operatorname{gr} H$ is a (graded) Hopf algebra.

Theorem. Let H be a Hopf algebra. Consider the statements:

1. H is co-Frobenius.

2. The coradical filtration is finite, i. e. $\exists n : H_n = H$.

Then (a) (Radford, 1977). If H_0 is a Hopf subalgebra, then (1) implies (2).

(b) (A.-Dăscălescu, 2003). (2) implies (1) always.

Conjecture. (A.-Dăscălescu, 2003). (2) $\iff$ (1) always.

A new proof of (Radford, 1977) was given in (A.-D., 2003). Assume that H_0 is a Hopf subalgebra. Then $\operatorname{gr} H \simeq R\#H_0$, where $R = \oplus_{n\geq 0} R^n$ is a Hopf algebra in ${}^{H_0}_{H_0}\mathcal{YD}$.

Theorem. (A.-Dăscălescu, 2003). TFAE

1. H is co-Frobenius.
2. $\operatorname{gr} H$ is co-Frobenius.
3. $\dim R < \infty$.
4. The coradical filtration is finite.

Theorem. (A.-Cuadra-Etingof, 2012). If H is co-Frobenius, then its coradical filtration is finite.
As a consequence the above conjecture is true.

Sketch of the proof. *First step* (Cuadra, 2006).

Let $\{S_i\}_{i\in I}$ be a full set of representatives of simple right H-comodules. Then $H \simeq \oplus_{i\in I} E(S_i)^{n_i}$, with $n_i \in \mathbb{N}$ for all $i \in I$. Since H is co-Frobenius, $E(S_i)$ is finite dimensional for all $i \in I$ and hence it has finite Loewy length. Since the Loewy series commutes with direct sums, we have:

H has finite coradical filtration iff $\{\ell\ell(E(S_i))\}_{i\in I}$ is bounded.

Observe that $\ell\ell(E(S_i)) \leq \ell(E(S_i)) \leq \dim E(S_i)$, where $\ell(E(S_i)) =$ composition length of $E(S_i)$.

Second step. Let $S \in \mathcal{M}^H$ be simple and d the largest dimension of a composition factor of $E(\Bbbk)$. Then $\ell(E(S)) \leq d \dim E(\Bbbk)$.

The above result can be extended to tensor categories:

Theorem. (A.-Cuadra-Etingof, 2012). If a tensor category is co-Frobenius and has subexponential growth, then the Loewy lengths of all simple objects are bounded.

III. More examples and structure.

Extensions.

• $A \hookrightarrow B$ inclusion of Hopf algebras; B co-Frobenius $\implies A$ co-Frobenius.

Theorem. Let $\mathcal{C} : \Bbbk \to A \to B \to C \to \Bbbk$ be an exact sequence of Hopf algebras.

(Beattie–Dăscălescu–Grünenfelder–Năstăsescu, 1996). If $\mathcal{C}$ is cleft and A, C are co-Frobenius, then B is co-Frobenius. Indeed $\int^A \otimes \int^C$ is a nonzero integral for B where $\int^A \in I_\ell(A)$, $\int^C \in I_\ell(C)$.

(A.-Cuadra, 2011). Assume B faithfully coflat as a C-comodule. Then, B is co-Frobenius if and only if A and C are co-Frobenius.

(A.-Cuadra-Etingof, 2012) give a construction of a Hopf algebra $\mathcal{D} = \mathcal{D}(m, \omega, (q_i)_{i \in I}, \alpha)$ that fits into an exact sequence

$$\Bbbk \to H \to \mathcal{D} \to \Bbbk\,\mathbb{Z}^s \to \Bbbk$$

with H finite-dimensional not semisimple; hence $\mathcal{D}$ is co-Frobenius.

Particular cases of these examples are not of finite of the Hopf socle, giving a negative answer to a question in (A.-Dāscālescu, 2003).

Theorem. (A.– Cuadra, 2011).

Every Hopf algebra H with injective antipode is a deformation of the bosonization of connected graded braided Hopf algebra R by a Hopf algebra generated by a cosemisimple coalgebra $H_{[0]}$.

Actually, $H_{[0]}$ = subalgebra generated by the coradical.

Theorem. (A.– Cuadra, 2011). TFAE

1. H is co-Frobenius.
2. $H_{[0]}$ is co-Frobenius and $\dim R < \infty$.

Question I. Classify all co-Frobenius Hopf algebras L generated by a cosemisimple coalgebra C, with a prescribed antipode.

Question II. Given L as in the previous item, classify all connected graded Hopf algebras R in ${}^L_L\mathcal{Y}D$ such that $\dim R < \infty$.

Question III. Given L and R as in previous items, classify all deformations or liftings of $R\#L$.