A gentle introduction to Nichols algebras and their uses

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INPUT: solution of the braid equation $\longrightarrow$ OUTPUT: Nichols algebra

Plan.

(i) The quantum Yang-Baxter equation and the braid equation.

(ii) Nichols algebras: definition. The braid group. Examples. Questions and answers.

(i) The quantum Yang-Baxter equation.

Let V be a vector space, $R \in \mathsf{Aut}(V \otimes V)$.

Notation. If $N \in \mathbb{N}$, $1 \leq i < j \leq N$, $R_{ij} \in \mathsf{Aut}(\underbrace{V \otimes V \otimes \cdots \otimes V}_{N \text{ times}})$ is "R acting on the i and j tensorands"

Ex. $R_{12} = R \otimes \mathrm{id}$, $R_{23} = \mathrm{id}_V \otimes R \otimes \mathrm{id}_{V^{\otimes(N-3)}}$

Definition. R satisfies the **quantum Yang-Baxter equation** if

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \text{ in } \mathrm{Aut}(V \otimes V \otimes V).$$

Definition. Let $c \in \mathrm{Aut}(V \otimes V)$. c satisfies the **braid equation** if

$$c_{12}c_{23}c_{12} = c_{23}c_{12}c_{23}.$$

Example. The usual flip $\tau : V \otimes V \to V \otimes V$, $\tau(x \otimes y) = y \otimes x$ satisfies the braid equation: $x \otimes y \otimes z \mapsto z \otimes y \otimes x$.

Remark. R satisfies the **quantum Yang-Baxter equation** if and only if $c = \tau \circ R$ satisfies the **braid equation**.

Examples. • Let $V = V_0 \oplus V_1$ be a super vector space. If $x \in V_i$, $|x| = i$. The super flip $s : V \otimes V \to V \otimes V$, $s(x \otimes y) = (-1)^{|x||y|} y \otimes x$ satisfies the braid equation.

• Let G be a group, $\mathcal{C} \subseteq G$ a conjugacy class, $V = \oplus_{g \in \mathcal{C}} \mathbb{C} g$. Then $c : V \otimes V \to V \otimes V$, $c(g \otimes h) = ghg^{-1} \otimes g$ satisfies the braid equation.

- Let V be a vector space with a basis $(v_i)_{i \in I}$. Let $(q_{ij})_{i,j \in I}$ a family in $\mathbb{C}^\times$. Then $c : V \otimes V \to V \otimes V$, $c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i$ satisfies the braid equation.

- Let $V = \mathbb{C}^2$, $q \in \mathbb{C} - 0$. Consider the basis $e_1 \otimes e_1$, $e_1 \otimes e_2$, $e_2 \otimes e_1$, $e_2 \otimes e_2$. Then

$$\begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & q - q^{-1} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}$$

satisfies the QYBE.

How to construct systematically solutions of the braid equation (or equivalently the QYBE)?

Definition. (H, m, Δ), **Hopf algebra:**

- (H, m) alg. with unit 1,
- $\Delta : H \to H \otimes H$ algebra map (coproduct),
- Δ coassociative with counit ε,
- $\exists \mathcal{S} : H \to H$ (antipode) s. t. $m(\mathcal{S} \otimes \mathrm{id})\Delta = \varepsilon 1_H = m(\mathrm{id} \otimes \mathcal{S})\Delta$.

Definition. (Drinfeld, 1986). A quasitriangular Hopf algebra is a pair $(H, \mathcal{R})$ where H is a Hopf algebra and $\mathcal{R} \in H \otimes H$ invertible satisfies

(QT1) $(\Delta \otimes \mathrm{id})(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{23}$.

(QT2) $(\mathrm{id} \otimes \Delta)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{12}$.

(QT3) $\Delta^{\mathrm{cop}}(h) = \mathcal{R}\Delta(h)\mathcal{R}^{-1}$, $\forall h \in H$.

Moral. $(H, \mathcal{R})$ quasitriangular, V representation of H
$R =$ image of $\mathcal{R}$ in $\mathrm{Aut}(V \otimes V)$ is a solution of the QYBE.

For instance,

$$\begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & q - q^{-1} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}$$

appears in this way from a representation of $U_q(s\ell(2))$ on $\mathbb{C}^2$.

How to construct systematically examples of quasitriangular Hopf algebras?

(Drinfeld, 1986). The **double** of a (finite-dimensional) Hopf algebra H is a Hopf algebra $D(H) = H^* \otimes H$ (as vector spaces) that is quasitriangular.

Does any solution of the QYBE arise from a quasitriangular Hopf algebra?

Essentially **yes**, up to some technical hypothesis and reformulation. (Fadeev-Reshetikihin-Takhtajan).

(ii) Nichols algebras. The braid group.

Consider the transpositions $\tau_i = (i, i+1)$ in $\mathbb{S}_n$:

- $\tau_1 = (1,2), \tau_2, \dots, \tau_{n-1}$ generate $\mathbb{S}_n$
- $\tau_i^2 = \mathrm{id}$, for all i
- $\tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1}$, for all i
- $\tau_i \tau_j = \tau_j \tau_i$, if $|i - j| \geq 2$
- These are defining relations of $\mathbb{S}_n$

Definition. (Artin, 1948). The braid group in n trends is the quotient $\mathbb{B}_n$ of the free group in $T_1, T_2, \ldots, T_{n-1}$ by the defining relations

(B1) $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$, for all i.

(B2) $T_i T_j = T_j T_i$, if $|i - j| \geq 2$.

Related to: topology (braids), mathematical physics (configuration space of n particles), operator algebras (subfactors).

By definition there is a surjective map of groups $\pi : \mathbb{B}_n \to \mathbb{S}_n$.

Lemma. (Matsumoto). There is a section of sets $S : \mathbb{S}_n \to \mathbb{B}_n$ preserving the length.

Remark. Let V be a vector space and let c be a solution of the braid equation.

Define $c_i \in \mathrm{Aut}(V^{\otimes n})$ by

$$c_i = c_{i,i+1} = \mathrm{id}_{V^{\otimes(i-1)}} \otimes c \otimes \mathrm{id}_{V^{\otimes(n-i-1)}}.$$

Then $c_1, \dots, c_{n-1}$ satisfy the relations (B1), (B2).

Thus we have a representation $\rho_n : \mathbb{B}_n \to \mathrm{Aut}(V^{\otimes n})$.

Example. If $\tau : V \otimes V \to V \otimes V$ is the usual flip, $\tau(x \otimes y) = y \otimes x$, we get the representation $\rho_n : \mathbb{S}_n \to \mathrm{Aut}(V^{\otimes n})$,

$$\rho_n(\sigma)(v_1 \otimes \cdots\cdots \otimes v_n) = v_{\sigma(1)} \otimes \cdots\cdots \otimes v_{\sigma(n)}, \quad \sigma \in \mathbb{S}_n.$$

This is the starting point for the usual tensor calculus:

- The space of **symmetric** n**-tensors** $S^n(V)$ is the space of $\mathbb{S}_n$-invariants in $T^n(V) = V^{\otimes n}$

$\simeq$ the quotient of $T^n(V)$ by $\ker(\mathrm{Symmetr}_n)$,

$\mathrm{Symmetr}_n(v_1 \otimes \cdots \otimes v_n) = \sum_{\sigma \in \mathbb{S}_n} v_{\sigma(1)} \otimes \cdots\cdots \otimes v_{\sigma(n)}$

- The **symmetric algebra** $S(V) = \oplus_{n \in \mathbb{N}_0} S^n(V)$

- The space of **anti-symmetric n-tensors** $\Lambda^n(V) \simeq$ the quotient of $T^n(V) = V^{\otimes n}$ by ker(Anti-symm$_n$),

Anti-symm$(v_1 \otimes \cdots \otimes v_n) = \sum_{\sigma \in \mathbb{S}_n} (-1)^\sigma v_{\sigma(1)} \otimes \cdots\cdots \otimes v_{\sigma(n)}$

- The **exterior algebra** $\Lambda(V) = \oplus_{n \in \mathbb{N}_0} \Lambda^n(V)$

Definition. (Nichols, 1978; Woronowicz, 1988). Let c be solution of the braid equation on a vector space V. Then the Nichols algebra is

$$\mathfrak{B}(V) = \oplus_{n \in \mathbb{N}_0} \mathfrak{B}^n(V),$$

where

$$\mathfrak{B}^n(V) = T^n(V) / \ker \sum_{\sigma \in \mathbb{S}_n} \rho_n(S(\sigma)).$$

Here:

ρ_n is the representation of $\mathbb{B}_n$ induced by c
S is the Matsumoto section.

Note: c solution of the braid equation $\Longrightarrow$ $-c$ too.

Motivation.

Nichols, 1978: construction of examples of Hopf algebras.

Woronowicz, 1988: to build a "quantum differential calculus".

Lusztig, 1993, Rosso, 1994, Schauenburg 1996: abstract definition of quantized enveloping algebras.

A.-Schneider, 1998: essential tool in the classification of pointed Hopf algebras.

Examples. (V, c) = a vector space with a solution of the braid equation = **"braided vector space"**

- $(V, c) = (V, \tau) \implies \mathfrak{B}(V) = S(V)$ (symmetric algebra)

- $(V, c) = (V, -\tau) \implies \mathfrak{B}(V) = \Lambda(V)$ (exterior algebra)

- $(V, c) = (V_0 \oplus V_1, s) \implies \mathfrak{B}(V) = S(V_0) \otimes \Lambda(V_1)$ (super symmetric algebra)

In all these examples, $\mathfrak{B}(V)$ is **quadratic**: ideal of relations generated in degree 2

Definition. Let $c \in \mathrm{Aut}(V \otimes V)$ be a solution of the braid equation. c is a **symmetry** if $c^2 = \mathrm{id}$.

This means: the representations $\rho_n : \mathbb{B}_n \to \mathrm{Aut}(V^{\otimes n})$ really factorize as representations $\rho_n : \mathbb{S}_n \to \mathrm{Aut}(V^{\otimes n})$.

Lemma. If c is symmetric, then $\mathfrak{B}(V)$ is quadratic.

(More generally, for solutions satisfying the Hecke condition).

More examples.

- Let $\mathfrak{g}$ be a simple Lie algebra, $(a_{ij})_{1\leq i,j\leq\theta}$ its Cartan matrix.

$q \in \mathbb{C} - 0$, q NOT a root of 1.

$q_{ij} = q^{d_i a_{ij}}$, $1 \leq i, j \leq \theta$.

V, a vector space with a basis $(v_i)_{1\leq i,j\leq\theta}$.

$c : V \otimes V \to V \otimes V$, $c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i \implies$

$\mathfrak{B}(V) = U_q(\mathfrak{g})^+ = \mathbb{C}\langle x_1, \cdots, x_\theta | \, \mathrm{ad}_c(x_i)^{1-a_{ij}}(x_j), i \neq j\rangle$

Here

$$\mathrm{ad}_c(x_i)^{1-a_{ij}}(x_j) = \begin{cases} x_i x_j - q_{ij} x_j x_i & a_{ij} = 0; \\ x_i^2 x_j - (2)_q x_i x_j x_i + x_j x_i^2 & a_{ij} = -1; \\ etc. \end{cases}$$

In these examples, $\mathfrak{B}(V)$ is **not** quadratic.

- All the same, except $q \neq 1$ a root of 1 of order N.

$$\mathfrak{B}(V) = u_q(\mathfrak{g})^+ = \mathbb{C}\langle x_1, \cdots, x_\theta | \, \mathrm{ad}_c(x_i)^{1-a_{ij}}(x_j), i \neq j; x_\alpha^N \rangle$$

Again, $\mathfrak{B}(V)$ is **not** quadratic.

$$\dim \mathfrak{B}(V) \begin{cases} \nearrow \infty: \ S(V),\ U_q(\mathfrak{g})^+ \\ \searrow < \infty: \ \Lambda(V),\ u_q(\mathfrak{g})^+ \end{cases}$$

• (A-S). Let H be a pointed Hopf algebra (all irreducible comodules have dim. 1).

$H \rightsquigarrow (V, c)$ a braided vector space

(arises from the Drinfeld double of $\mathbb{C}G(H)$).

$\dim H < \infty \implies \dim \mathfrak{B}(V) < \infty$

Questions and answers.

(V, c) a braided vector space:

- When $\dim \mathfrak{B}(V) < \infty$? If so, compute this dimension.

- Compute the defining relations of $\mathfrak{B}(V)$.

Braided vector space of diagonal type.

$\exists$ basis $v_1, \dots, v_\theta$, $(q_{ij})_{1 \leq i,j \leq \theta}$ in $\mathbb{C}^\times$:

$$c(v_i \otimes v_j) = q_{ij} v_j \otimes v_i, \quad \forall i, j$$

Theorem. $1 \neq q_{ii}$ roots of 1. $\Rightarrow \dim \mathfrak{B}(V) < \infty$ classified.

I. Heckenberger, *Classification of arithmetic root systems*, `http://arxiv.org/abs/math.QA/0605795`.

Braided vector space of Cartan type.

$\exists\ (a_{ij})_{1\leq i,j\leq\theta}$ generalized Cartan matrix

$$q_{ij}q_{ji} = q_{ii}^{a_{ij}}.$$

Theorem. (V,c) Cartan type, $1 \neq q_{ii}$ root of 1.
$\dim \mathfrak{B}(V) < \infty \iff (a_{ij})$ of finite type.

N. A. & H.-J. Schneider, *Finite quantum groups and Cartan matrices*, Adv. Math. **154** (2000), 1-45.

I. Heckenberger, *The Weyl groupoid of a Nichols algebra of diagonal type*, Invent. Math. **164**, 175–188 (2006).