THE BEGINNINGS OF THE THEORY OF HOPF ALGEBRAS

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ABSTRACT. We consider issues related to the origins, sources and initial motivations of the theory of Hopf algebras. We consider the two main sources of primeval development: algebraic topology and algebraic group theory. Hopf algebras are named from the work of Heinz Hopf in the 1940's. In this note we trace the infancy of the subject back to papers from the 40's, 50's and 60's in the two areas mentioned above. Many times we just describe – and/or transcribe parts of – some of the relevant original papers on the subject.

1. Introduction

In this note, we address the following questions: Who introduced the notion of Hopf algebra? Why does it bear this name?

Succinctly, the answers are:

- The first formal definition of Hopf algebra – under the name of *hyperalgebra* – was given by Pierre Cartier in 1956, inspired by the work of Jean Dieudonné on algebraic groups in positive characteristic.
- The expression *Hopf algebra*, or more precisely, *algèbre de Hopf*, was coined by Armand Borel in 1953, honoring the foundational work of Heinz Hopf.

That is, we have two different strands, one coming from the theory of algebraic groups and the other from algebraic topology, both interwoven in the early 60's. But the story is not so simple: the notion introduced by Cartier is not exactly the same as what we use today and it took a time to arrive to the present definition. Also, what did Borel mean when speaking about Hopf algebras? We discuss these topics quoting from the original sources.

This paper has three sections besides this introduction. The second is centered around those aspects of algebraic topology that played an important role in the development of Hopf algebra theory. We organized our expositions in terms of the different authors that in our opinion were the main contributors. In particular we discuss, amongst others, the work of Heinz Hopf, Armand Borel, Edward Halpern, John Milnor and John C. Moore.

The first author was supported by Mincyt (Cba), CONICET, Foncyt and Secyt (UNC). The second author was partially supported by Pedeciba (Ur) and Foncyt (Ar).

In the third section called algebraic groups we proceeded similarly and centered our attention in the input of algebraic group theory in Hopf theory. We discuss in particular the relevant work of the following mathematicians: Jean Dieudonné, Pierre Cartier, Gerhard Hochschild, George Mostow, Bertram Kostant and Georg Kac. In the fourth section we make a few general remarks concerning later developments.

We finish this introduction by saying some words about notations and prerequisites. We assume that the reader –besides a working knowledge in Hopf algebra theory– has a certain degree of familiarity with the basic vocabulary on: algebraic topology, algebraic groups, Lie theory and (co)homology of Lie algebras, etc. In Hopf algebra theory we use freely the standard modern terminology as it appears for example in Moss Sweedler's book [Sw69].

One of the goals of the present article is to trace back the formal definition of a Hopf algebra and to describe its evolution in time. The definition we know and use today is included in the subsection on the work of Bertram Kostant, see page 14. Compare also with [SGA3, vol. 1, p. 511].

Acknowledgments. We would like to thank Gerhard Hochschild, Bertram Kostant and Moss Sweedler, for commenting on an initial version of this paper. However, any inaccuracy or omission in the paper is the responsibility of the authors, who are not professional historians and had to finish the last version under the pressure of a very tight deadline.

2. Algebraic topology

2.1. **Heinz Hopf.** In [H41] the author considers what he calls a Γ–manifold – and later called a Hopf manifold by Borel in [B53] and currently simply an H–space. They are manifolds equipped with a product operation i.e. a continuous function $M \times M \to M$. Clearly this function induces by functoriality a homomorphism from the cohomology ring of M – called H – into the cohomology of $M \times M$, i.e. into $H \otimes H$. In the above mentioned paper Hopf shows that the existence of this additional map from H into $H \otimes H$ – that is compatible with the cup product – imposes strong restrictions on the structure of H, and from these restrictions he deduces some important topological results. In accordance with W. Hurewicz's review of [H41] (appearing in MathSciNet: MR0004784, 56.0X):

This [...] is not only a far reaching generalization of the well–known results of Pontrjagin about Lie groups, but in addition it throws an entirely new light on these results by emphasizing their elementary topological nature...

The methods introduced by Hopf in this article, were soon applied to other cohomology theories – see for example [L46] and [Lef42].

In a similar direction Hans Samelson, a close collaborator of Hopf, following the ideas of the latter but working with the homology instead of the cohomology, (see [S41]) considers two products (the "intersection" and the

"Pontrjagin" product), as an alternative to the consideration *à la* Hopf of a product and a coproduct on the cohomology.

It is worth mentioning another more algebraic derivation of the methods introduced by Hopf. Using de Rham theory, the study of the real (co)homology of a Lie group – a very special case of a Γ–manifold – can be reduced to the study of the *algebraic* (co)homology of the Lie algebra $\mathfrak{g}$ of G. J.L. Koszul in [K50] considers –for a *general* Lie algebra $\mathfrak{g}$– many problems related to the cohomology and homology of Lie algebras and in particular deals with the generalization of the results of [H41].

This generalization is achieved in the form of an identification –under the hypothesis of reductivity– of the cohomology as well as the homology of $\mathfrak{g}$, to the exterior algebras of its respective primitive elements. In the proof of this result, a crucial technical role is played by the map $X \mapsto (X, X) : \mathfrak{g} \to \mathfrak{g} \times \mathfrak{g}$ that is (c.f. C. Chevalley MathSciNet MR0036511) *... an algebraic counterpart to the mapping* $(s,t) \mapsto st$ *of* $G \times G$ *into* G *...*

2.2. **Armand Borel.** In the important paper by A. Borel [B53], the author studies the homology of a principal fiber bundle and the application to the homology of homogeneous spaces. Concerning this problem he mentions many mathematicians that had studied it before: H. Cartan, Chevalley, Koszul, Weil, around 1950, with predecessors in the work of E. Cartan, Ehresmann, Pontrjagin (see the references in [B53]), Hopf and Samelson (see [H41], [HS41] and [S41]).

In the introduction to the paper, when mentioning the results he needs from the work of Hopf he uses the expression 'Hopf algebra', or more precisely 'algèbre de Hopf', that appears first in [B53, p.116]:

...la structure d'une algèbre de Hopf (c'est à dire vérifiant les conditions de Hopf)...

As we mentioned, Borel means the conditions in the paper [H41] that are satisfied by the cohomology ring of a "Γ–variety" or "Hopf variety" and that concerns the existence of an additional operation of comultiplication on the cohomology ring.

In the introduction the author mentions Hopf algebras over fields of positive characteristic and in the same page remarks about a "structure theorem":

On peut exprimer le résultat obtenue en dissant qu'une algèbre de Hopf est toujours isomorphe à un produit tensoriel gauche d'algèbres de Hopf à un générateur.

Later, in the second chapter of the paper called "Le théorème de Hopf" – more precisely in [B53, p. 137] – he considers an algebra H with the following properties: it is graded by $\mathbb{N}_0$, with finite-dimensional homogeneous components; and it is anti-commutative (or super-commutative in present terminology). He denotes by D the degree. He formally defines:

H est une algèbre de Hopf s'il existe un homomorphisme d'algèbres graduées $f : H \to H \otimes H$ et deux automorphismes ρ et σ de H tels que pour tout $h \subset H$ homogène l'on ait:

$$\begin{aligned} f(h) =& \rho(h) \otimes 1 + 1 \otimes \sigma(h) + x_1 \otimes y_1 + \cdots + x_s \otimes y_s \\ & (0 < Dx_i < Dh). \end{aligned} \tag{6.1}$$

The author then shows that without loss of generality it can be assumed that $\rho = \sigma = \mathrm{id}$.

Notice that the comultiplication map f of a 'Hopf algebra' in the sense of Borel is neither coassociative nor admits an antipode; and although it is not explicitly said, it does admit a counit. This comes from the fact that the original geometric multiplication considered by Hopf on the base space, was not assumed to be associative or to have an inverse, but in general it assumed to posses an "homotopy identity".

Then the author proves the "structure theorem" mentioned before – Theorem 6.1 – that is stated informally in p. 138 as follows:

Ce théorème affirme en somme que H est l'algèbre associative engendrée par les x_i, [..] il montre aussi qu'une algèbre de Hopf peut toujours être envisagée comme produit tensoriel (gauche) d'algèbres de Hopf à un générateur.

It is worth noticing that this structure theorem, that generalizes the corresponding result due to Hopf, gives information only about the multiplicative structure of the "Hopf algebra". Information related to the coproduct is obtained in [S41], and also in work of Leray [L45]. As we shall see in Subsection 2.4, results of this type were later generalized and unified in the work of Milnor and Moore [MM65].

The author draws immediately some important consequences of the structure theorem above, e.g. in Chapter III p. 147 he proves Proposition 10.2 – in Borel's notation K_p stands for the finite field with p elements, $\mathbf{O}(n)$ for the orthogonal group of the n–space and and $\mathbf{S}_n$ for the n–sphere:

... $\mathbf{V}_{n,n-q} = \mathbf{O}(n)/\mathbf{O}(q)$ $(1 \leq q \leq n)$... Soit $\bar{n}$, (resp. $\bar{q}$), le plus grand, (resp. le plus petit), entier impair $\leq n$ (resp. $\geq q$). Alors $\mathbf{V}_{n,n-q}$ a pour les coefficients K_p $(p \neq 2)$ même algèbre de cohomologie que le produit:

$$\mathbf{S}_{2\bar{n}-3} \times \mathbf{S}_{2\bar{n}-7} \times \cdots \times \mathbf{S}_{2\bar{q}+1}$$

multiplié encore par $\mathbf{S}_{n-1}$ si n est pair, par $\mathbf{S}_q$ si q est pair.

It seems remarkable that in the seminal paper by A. Borel: *Groupes linéaires algébriques* [B56], that set up the foundations of the modern theory of algebraic linear groups for which the algebra of polynomial functions is the archetype of a commutative Hopf algebra, neither the term 'Hopf algebra', nor any consideration about algebras of coordinate functions appear explicitly.

2.3. **Edward Halpern.** In [Ha58] the author presents some results that appeared in his thesis under the direction of S. MacLane that develops further the theory started in [H41] and [B53]. See also [Ha58b] and [Ha60] of the same author.

In the introduction to [Ha58] p. 2 the author says:

Thus, in both homology and cohomology for H–spaces one is led to consider a triple consisting of a module H, a product $H \otimes H \to H$ and a "coproduct" $H \to H \otimes H$, with product and coproduct compatible. Such a structure is called a hyperalgebra – the term hyperalgebra was suggested by S. MacLane [and had previously been used by Dieudonné][1]. Both Hopf and Pontrjagin algebras are special cases of hyperalgebras.

With the names of Hopf and Pontrjagin algebras the author is referring to the following situations. If one starts with an H–space X (endowed with a continuous map $\Delta : X \times X \to X$) and a homotopy unit, one can consider the induced maps $\Delta_* : H_* \otimes H_* \to H_*$ and $\Delta^* : H^* \to H^* \otimes H^*$ on homology and cohomology respectively. Following the nomenclature of [B53], if $\cup$ denotes the cup–product, the triple $(H^*, \cup, \Delta^*)$ was being called by some authors a *Hopf algebra*. Dually the triple $(H_*, \Delta_*, \cup_*)$ where $\cup_*$ is the dual of the cup–product was called a Pontrjagin algebra in [B54]. It is a standard fact that the maps cup and Δ^* as well as the maps $\cup_*$ and Δ_* are compatible.

Hence, in the nomenclature of the author the structure maps of an hyperalgebra need not be associative or coassociative, but in general he supposes the underlying spaces to be graded and the operations to preserve the grading, and to be unital and counital. In the case of what he calls a Hopf algebra the product is associative and in the case of the so called Pontrjagin algebra the coproduct is coassociative.

Later in Chapter 1, p. 4, the hyperalgebra is said to be associative if both product and coproduct are associative (coassociative) respectively.

In the later paper [Ha58b] the author continues to apply the name of "hyperalgebra" to the objects considered above, but in [Ha60] he switches to the name of "Hopf algebra" that was becoming more and more popular after the work of Milnor and Moore.

2.4. **John W. Milnor and John C. Moore.** The paper of John W. Milnor and John C. Moore that we consider appeared in print in March 1965, but in accordance with the review MathSciNet: MR0174052 written by I. M. James:

$\cdots$ *[A] version of this paper circulated some ten years ago. There have been many improvements, although the published version is basically the same*[2].

[1]Citation from [Ha58b]

[2]The author of the review was probably referring to J. Milnor and J. C. Moore, *On the structure of Hopf algebras*, Princeton Univ., Princeton, N.J., 1959 (mimeographed) or some predecessor.

The article has eight sections, in Section 4 the authors introduce the concept of Hopf algebra, that coincides with the usual definition of *graded bialgebra* that we use today. In the case that either the product or the coproduct are not associative, the authors speak of a *quasi Hopf algebra.*

The authors work in the category of $\mathbb{N}_0$-graded modules over a fixed commutative ring K. The definition of Hopf algebra appears as Definition 4.1 in page 226:

DEFINITION. *A Hopf algebra over K is a graded K–module together with morphisms of graded K–modules*

$$\varphi : A \otimes A \to A, \quad \eta : K \to A$$
$$\Delta : A \to A \otimes A, \quad \varepsilon : A \to K$$

such that

(1) (A, φ, η) *is an algebra over K with augmentation ε,*
(2) (A, Δ, ε) *is a coalgebra over K with augmentation η, and*
(3) *the diagram*

$$\begin{tikzcd} A \otimes A \arrow[r, "\varphi"] \arrow[d, "\Delta\otimes\Delta"] & A \arrow[r, "\Delta"] & A \otimes A \\ A \otimes A \otimes A \otimes A \arrow[rr, "A\otimes T\otimes A"] & & A \otimes A \otimes A \otimes A \arrow[u, "\varphi\otimes\varphi"'] \end{tikzcd}$$

is commutative.

In the definition above – and along all this paper – the authors use the notation A to indicate at the same time an object as well as the identity map on that object, moreover the map T stands for the usual braiding in the category of $\mathbb{N}$–graded vector spaces.

All in all, what the authors call a Hopf algebra, is what today would be called a bialgebra in the symmetric category of graded vector spaces over a commutative ring K. But connected Hopf algebras in their sense, that is those with one-dimensional 0-degree space, do have an antipode as explained below in page 9.

Later, in Section 5, the authors show the structure Theorem 5.18:

Over a field of characteristic zero, the category of graded Lie algebras is isomorphic with the category of primitively generated Hopf algebras.

In Section 7, *Some classical theorems*, a proof of the so-called Milnor–Moore theorem on the structure of graded–commutative connected Hopf algebras is presented. This theorem generalizes the array of structural theorems mentioned before due to Hopf, Samelson, Leray, Borel, etc. Along the way the authors prove the following classical results: Theorem 7.5 (Leray), Theorem 7.11 (Borel), Theorem 7.20 (Samelson, Leray). The main result proved in this section reads as follows (Theorem 7.16, page 257).

If A is a connected primitively generated Hopf algebra over the perfect field K, the multiplication in A is commutative, and the underlying vector space of A is of finite type, then there is an isomorphism of Hopf algebras of A with $\bigotimes_{i\in I} A_i$ where each A_i is a Hopf algebra with a single generator x_i.

In the case that the hypothesis of primitive generation is not valid, all that can be obtained is Borel's theorem stating the isomorphism of A with $\bigotimes_{i\in I} A_i$ as algebras.

Later, in the Appendix and using a synthesis of methods from Hopf theory and algebraic topology the authors prove the following very precise result for the complete Hopf algebra structure of the cohomology ring of a Lie group – formulated more generally for certain H–spaces:

If G is a pathwise connected homotopy associative H–space with unit, and $\lambda : \pi(G;K) \to H_*(G;K)$ is the Hurewicz morphism of Lie algebras, then the induced morphism $\widetilde{\lambda} : U(\pi(G;K)) \to H_*(G;K)$ is an isomorphism of Hopf algebras.

This is proved using the fact that for an associative connected H–space the homology is commutative and connected, thus – as was proved before – it is also primitively generated. As the image of the map λ is exactly the set of primitive elements of $H_*(G;K)$ the result follows.

The article we are considering had a wide and deep influence in the development of the subject, and maybe should be thought at the same time as a culmination of the topological line of work initiated by Hopf et. al., as well as the launching platform of an independent new area in the realm of abstract algebra. Moreover it was one of the basic references on the subject for almost a decade (1959–1969) – many papers like [A61], [Br63] or [G62] used not the final but the previous mimeographed version: J. Milnor and J. C. Moore, *On the structure of Hopf algebras*, Princeton Univ., Princeton, N.J., 1959 (mimeographed)[3] .

3. Algebraic groups

3.1. **Jean Dieudonné.** Jean Dieudonné wrote a series of papers consecrated to formal Lie groups and hyperalgebras. In the first paper of the series [D54], he starts by mentioning that:

Les récents travaux sur les groupes de Lie algébriques [...] montrent clairement que, lorsque le corps de base a une charactéristique $\neq 0$, le mécanisme de la théorie classique des groupes de Lie ne s'applique plus ...

[3]One of the earlier pioneers told us the following peculiar anecdote: *At one point I got a hold of a draft copy [of Milnor–Moore paper]. A bit later I came across an earlier draft and found it more readable, more readable by me that is. The hunt was on for earlier drafts because the earlier the draft, the more readable.*

He is referring to the problems that appear when trying to extend the dictionary 'Lie groups–Lie algebras' to positive characteristic. Dieudonné was looking for the right object to spell out an analogous dictionary. Then, in [D54, p. 87], he describes the goal of the paper:

L'objet de ce travail est de définir, pour un groupe de Lie sur un corps de caractéristique $p > 0$*, une "hyperalgèbre" associative qui correspond à l'algèbre enveloppante de l'algèbre de Lie dans le cas classique, mais possède ici une structure beaucoup plus compliquée ...*

In this paper, the author considers a formal Lie group G in the sense of [B046], to which he attaches an associative algebra $\mathfrak{G}$, called the hyperalgebra. The hyperalgebra reflects well the structural properties of G.

Dieudonné does not consider an abstract object, rather some concrete algebra attached to G – essentially the algebra of distributions with support at the identity. In characteristic 0, this is just the universal enveloping algebra of the Lie algebra of G.

The second paper of the series is devoted to the question of when an hyperalgebra arises from a formal Lie group, and to the study of abelian formal Lie groups. In particular it includes the classification of the hyperalgebras of the 1-dimensional groups. The third and the fourth parts are also about abelian formal Lie groups. For the theme of the present notes, the main novelty appears in the fifth paper of this series [D56]. In the previous papers, the hyperalgebras in sight were mostly commutative (because the formal groups under consideration were commutative). Now the author wishes to pass to the general case.

Commençons par rappeler rapidement les relations entre les notions d'hyperalgèbre et de groupe formel ... comme me l'a fait observer P. Cartier, *la manière la plus suggestive dont on puisse présenter cette thèorie consiste à faire ussage de la dualité en algèbre lineaire.*

He now gives a formal definition of hyperalgebra that includes a comultiplication map; dualizing this map, he gets the multiplication of a formal group.

3.2. **Pierre Cartier.** We discuss in this section three papers by Cartier labeled by the date of publication.

1955-56. A formal abstract definition of *hyperalgebra* is given by Pierre Cartier in [C55-6, Exposé no. 2]. He assumes that K is a commutative ring with unit.

On appelle hyperalgèbre sur K *une* K*-algèbre* U *associative, unitaire, augmentée, filtrée, munie d'une application diagonale vérifiant les axiomes suivants.*

(HA_1) *L'augmentation* $\varepsilon : U \to K \cdot 1$ *est un* K*-homomorphisme unitaire; on a donc* $U = U^+ \oplus K \cdot 1$*,* U^+ *designant le noyau de* ε*.*

(HA$_2$) *L'application diagonale* $\Delta : U \to U \otimes_K U$ *est un* K*-homomorphisme unitaire de* U *dans* $U \otimes U$ *possédant les propriétés suivantes:*

a) I *désignant l'automorphisme identique de* U*, on a:*

$$(I \otimes \Delta) \circ \Delta = (\Delta \otimes I) \circ \Delta. \tag{1}$$

L'egalité des 2 applications de U *dans* $U \otimes U \otimes U$ *exprimée par* (1) *s'énoncera en disant que* Δ *est associatif.*

b) S *désignant l'involution* $a \otimes b \mapsto b \otimes a$ *de* $U \otimes U$*, on a:*

$$S \circ \Delta = \Delta. \tag{2}$$

Autrement dit, Δ *est commutative.*

c) $P : U \otimes U \to U$ *désignant l'application* $a \otimes b \mapsto ab$ *déduite du produit, on a:*

$$P \circ (I \otimes \varepsilon) \circ \Delta = P \circ (\varepsilon \otimes I) \circ \Delta = I. \tag{3}$$

Autrement dit, Δ *est compatible avec l'augmentation.*

(HA$_3$) *Il existe une filtration d'algèbre croissante de* U *par des sous-modules* $U_{(n)}$*,* $n = 0, 1, \ldots,$ *qui vérifie:*

$$U_{(0)} = K \cdot 1 \tag{4}$$

$$\Delta(U_{(n)}) \subset \sum_{0 \le p \le n} U_{(p)} \otimes U_{(n-p)}, \qquad \text{pour tout } n \ge 0. \tag{5}$$

Axioms (HA$_1$) and (HA$_2$), define a cocommutative bialgebra. Indeed, there is no explicit reference to the antipode. However, the hypothesis (HA$_3$) implies the existence of the antipode. Indeed, (5) says that $\{U_{(n)} : n = 0, 1, \cdots\}$ is a coalgebra filtration, hence $U_{(0)}$ is the coradical of U [Mo93, 5.3.4]; it follows then from (4) that U has a bijective antipode, using a Lemma of Takeuchi – see [Mo93, 5.2.10 and 5.2.11].

The dual of an hyperalgebra is recognized as carrying the dual operations, so that it is – in present terminology – a topological commutative bialgebra but in the paper no formal definition is given.

Three examples of hyperalgebras are discussed in detail: the universal enveloping algebra of a Lie algebra, the restricted enveloping algebra of a p-Lie algebra and the algebra of divided powers.

1956. The algebra of polynomial functions on a linear algebraic group is a commutative Hopf algebra. This fact would seem to be the starting point of the theory of Hopf algebras to someone with a familiarity, at least superficial, with algebraic geometry. This is indeed recognized, albeit implicitly, in the Comptes Rendues note [C56]. Here **K** is a field. The ultimate goal of this note is a Tannaka-type theorem.

Soit G *un groupe algébrique de matrices* $g = \|g_{ij}\|$*,* $\mathfrak{o}(G)$ *l'ensemble des fonctions sur* G *de la forme* $f(g) = P(g_{ij}, \det g^{-1})$*, où* P *est un polynome. On vérifie facilement les propriétés suivants:*

G1. *L'algébre $\mathfrak{o}(G)$ est une algébre de type fini et contient la fonction constante égale a 1.*

G2. *Si $g \neq g'$, il existe $f \in \mathfrak{o}(G)$ avec $f(g) \neq f(g')$; un homomorphisme χ non nul de $\mathfrak{o}(G)$ dans $\mathbf{K}$ est de la forme $\chi(f) = f(g_0)$ $(g_0 \in G)$.*

G3. *Si $f \in \mathfrak{o}(G)$ et $f'(g_1, g_2) = f(g_1^{-1}g_2)$, on a $f' \in \mathfrak{o}(G) \otimes \mathfrak{o}(G)$.*

The convolution product in the space of linear forms on $\mathfrak{o}(G)$ is defined, and the hyperalgebra $\mathbf{U}(G)$ is described as the space of those linear forms vanishing in some power of the augmentation ideal of $\mathfrak{o}(G)$. The existence of the comultiplication of $\mathbf{U}(G)$ is made explicit, in terms of the product of $\mathfrak{o}(G)$.

The subject was continued by Gerhard Hochschild and George D. Mostow in a series of papers that will be considered later.

1962. A few years after the papers already discussed, Cartier presented a much more ample set of ideas in [C62]. Although he recognizes that

Le présent article n'est qu'un resumé . . . Nous publierons par ailleurs un exposé detaillé de la théorie.

(exposition that never saw the light), the paper contains many important concepts. Let us first select some parts of the Introduction, to highlight the main motivations of his work. He begins by saying that the theory of algebraic groups in positive characteristic has complications arising from phenomena of inseparability, and mentions explicitly the problem of isogeny, to which Barsotti and himself have devoted some work. Then he recognizes that the language of schemes – freshly introduced by Grothendieck at that time – is the natural setting to treat these questions and explains that he will proceed within it with some restrictions. Then he mentions again the work of Dieudonné on formal groups and declares:

Mais jusq'à présent, la théorie des groupes algébriques et celle des groupes formels n'avaient que des rapports superficiels. C'est le but du présent exposé d'en faire la synthèse. Pour cela, nous devrons elargir un peu la notion de groupe formel . . . ceci fait, la théorie des groupes formels est équivalente à celle des hyperalgèbres.

He immediately points out in a footnote:

En topologie algébrique, on utilise sous le nom d'Algèbre de Hopf une notion très proche de celle de hyperalgèbre.

He refers next to the duality for commutative algebraic groups and observes in a footnote:

Gabriel utilise le langage des hyperalgèbres, et non celui des groupes formels, ce qui oblige à de nombreuses contorsions.

This last remark, together with the allusions to the theory of schemes, explains the choice that Cartier does for the exposition. Namely, he rather prefers a systematic use of a functorial presentation, than *le langage des hyperalgèbres*; Hopf algebras appear only marginally in the paper.

Let us now describe the main points of the paper. He begins in Section 2 by identifying the group $GL(n, A)$, A a commutative ring, with the points of A^r, $r = 1 + n^2$, annihilated by the polynomial $D = 1 - X_0 \det X_{ij}$. He then observes that an algebraic subgroup of $GL(n, \Omega)$ is determined by an ideal I of $\Omega[X_0, X_{ij}] = \Omega[X]$ satisfying the following conditions.

(a) D belongs to I.

(b) If Y_0, Y_{ij} is another set of r variables, abbreviated Y, then set $Z_0 = X_0Y_0$, $Z_{ik} = \sum_j X_{ij}Y_{jk}$. For any $P \in I$, there exist $L_\alpha, L'_\beta \in \Omega[X]$, $P_\alpha, P'_\beta \in I$ such that

$$P(Z) = \sum_\alpha L_\alpha(X)P_\alpha(Y) + \sum_\beta P'_\beta(X)L'_\beta(Y). \tag{6}$$

(c) Let X'_0 be the determinant of the matrix (X_{kl}) and X'_{ij} the product of X_0 by the cofactor of X_{ij} in the determinant X'_0. Then $P(X')$ belongs to I whenever $P \in I$.

(d) The ideal I is an intersection of prime ideals.

In other words, condition (b) says that I is a coideal of $\Omega[X]$ and (c) that it is stable under the antipode. Hence, the polynomial functions on the algebraic subgroup – the quotient $\Omega[X]/I$ – has a natural structure of Hopf algebra without nilpotents in accordance to (d).

He then proceeds to define an algebraic group as a representable functor from the category of commutative algebras to the category of groups and devotes some time to the study of their properties.

Next, he defines in Section 10 the notion of formal group also as a functor from the category of commutative algebras to the category of groups, but instead of representable he asks for other axioms. These axioms allow him to define the hyperalgebra $U(G)$ associated to a formal group G; here the coproduct is explicitly mentioned, otherwise he refers to [C55-6]. This assignment gives an equivalence between the categories of formal groups and the category of hyperalgebras.

He next states a structure result for formal groups (Theorem 2): any formal group over a perfect field can be decomposed in a unique way as a semidirect product of a "separable" group and an "infinitesimal" group. This translates to a structure theorem for hyperalgebras (that is, cocommutative Hopf algebras) that is not explicitly spelled out; the explicit formulation of this structure theorem was discovered later by Kostant, see below. However, Theorem 3 says that the hyperalgebra of an infinitesimal group over a field of characteristic 0 is a universal enveloping algebra. Contrary to Theorem 5.18 in [MM65] quoted above, it is *not* assumed that the algebra is primitively generated.

3.3. **Gerhard Hochschild and George D. Mostow.** These authors start the Introduction of the first of the series of three papers that appeared in the late 50's [HM57], [HM58] and [HM59] as follows:

We are concerned with a number of inter–related general questions concerning the representations of a Lie Group G and the algebra $R(G)$ of the (complex valued) representative functions on G ... Instead of dealing with the characters of $R(G)$, we deal with its proper automorphisms [A = group of all algebra automorphisms commuting with right translations] ... The left translations on $R(G)$ [extend] to an operation of G^+ [the universal complexification of G] on $R(G)$, so that one obtains a continuous monomorphism of G^+ into A ... Chevalley's complex formulation of Tannaka's theorem becames the statement that, if G is compact, this monomorphism sends G^+ onto A. [If $R(G)$ is finitely generated] we may identify A and hence G^+ with a algebraic complex linear group. This links up the Lie group situation with the analogous situation for algebraic groups, which is dealt with by P. Cartier [in [C56] *].*

In the second and third of the series of papers, the same platform is considered but under hypothesis for $R(G)$ more general than the finite generation.

For example, in the first paper it is proved that if $R(G)$ is finitely generated, then the image of the natural morphism from G^+ into A is A itself. This is not true in general an in [HM58] this image is explicitly calculated.

Our result is that a proper automorphism α belongs to the natural image of G^+ if and only if $\alpha(\exp(h)) = \exp(\alpha(h))$ for every $h \in \operatorname{Hom}(G, C)$.

In the third paper [HM59] the authors deal with complex groups. In their own words:

The theory developed [so far] was adapted to real Lie groups, and used complex Lie groups as auxiliaries. It is evident from the nature of the proofs and the main results that there is an underlying analogous theory for complex Lie groups, which is actually simpler than the theory for real Lie groups. Nevertheless, the systematic development of the complex analytic case brings up a number of new questions which are of independent interest.

Constantly, along these series of papers, the authors work – without explicitly spelling out the words – with the natural Hopf algebra structure on $R(G)$. The algebra structure appears in an explicit form but the coalgebra structure appears under the guise of the formulæ that are obtained – once we know that for $f \in R(G)$ the set $G \cdot f$ generates a finite dimensional subspace – when we express $x \cdot f = \sum \alpha_i(x) g_i$ in terms of a basis $\{g_1, \cdots, g_n\}$ of this subspace. In terms of the comultiplication of $R(G)$ the above formula just means that $\Delta(f) = \sum g_i \otimes \alpha_i$.

It might be important to stress at this point that this series of papers deal globally with a generalization of the Tannaka theorem and in that sense, follow the same track initiated by Cartier in [C56] and that was considered before. Here again we are observing a fact that is very relevant even today: the deep interaction existing between Tannaka reconstruction viewpoint and Hopf theory.

Moreover, the work of Hochschild since the late 50's until the late 70's contained a systematic approach to the theory of Lie and algebraic groups and its Lie algebras, with the viewpoint of Hopf theory in mind. As time passed, the Hopf structure started to appear explicitly and its use became more and more systematic. In particular in the book "The structure of Lie groups" (1965), [Ho65], the Hopf algebra structure plays in many occasions a conspicuous role. In particular in page 26 of Chapter 2, § 3 of the mentioned reference, the author considers the $\mathbb{R}$–algebra $R(G)$ of real valued representative functions on a compact group G and proves that it admits a natural structure of "Hopf algebra".

If we abstract the structure we have just described from the group G, retaining only the $\mathbb{R}$–module $\mathcal{R}(G)$, the algebra structure μ on $\mathcal{R}(G)$ with the algebra homomorphism $u : \mathbb{R} \to \mathcal{R}(G)$ as a unit and the coalgebra structure γ with the algebra homomorphism $c : \mathcal{R}(G) \to \mathbb{R}$ as a counit, and if we assume that the above identities are satisfied by these maps and that γ is an algebra homomorphism, we have what is called a Hopf algebra $(\mathcal{R}(G), \mu, u, \gamma, c)$.

The reader should notice that the antipode was not present in the above definition of Hopf algebra. But the author goes on and says in page 27 op.cit.:

In our special case, there is one further structural item. Let η denote the algebra endomorphism of $\mathcal{R}(G)$ that is defined from the inverse of G, i.e., $\eta(f) = f'$, where $f'(x) = f(x^{-1})$. Then one sees immediately that $\mu \circ (\eta \otimes i) \circ \gamma = u \circ c$. Let us agree to call an algebra endomorphism η with this last property a symmetry of our Hopf algebra.

It is worth noticing that in that same period, the Hopf algebra language in algebraic group theory was becoming more and more standard –see for example the work by Demazure, Gabriel and Grothendieck, [DG],[SGA3].

We finish by saying some words about the evolution of the concept of integral in the setting of Hopf algebras. Since Hurwitz and Weyl, invariant integration was one of the main tools in representation theory and was formalized in the context of Hopf algebra theory in the early sixties.

The fist appearance of the general concept of integral in the framework of Hopf theory seems to be in [Ho65, page 29] under the name of a "gauge".

Now suppose that G is compact. Then a Haar integral for G gives us an $\mathbb{R}$ module homomorphism $J : \mathcal{R}(G) \to \mathbb{R}$. The invariance property $J(y \cdot f) = J(f)$ for all $y \in G$ and all $f \in \mathcal{R}(G)$, is easily seen to be equivalent to the relation $(J \otimes i) \circ \gamma = u \circ J$ ... If [A] is any Hopf algebra let us call a map $J : A \to \mathbb{R}$ with [this] property a gauge of the Hopf algebra.

The relationship between the existence of a "gauge" in the Hopf algebra of functions over group and the semisimplicity of the representations of the latter, appears clearly in the above mentioned book in the guise of the following *Tannaka duality theorem* (page 30):

For a compact group G, let $\mathcal{H}(G)$ denote the Hopf algebra attached to G in the canonical fashion. Then $\mathcal{H}(G)$ is a reduced Hopf algebra with symmetry and gauge. For a reduced Hopf algebra having a symmetry and a gauge, let $\mathcal{G}(H)$ be the topological group of homomorphisms attached to H in the canonical fashion. Then $\mathcal{G}(H)$ is compact. The canonical maps $G \to \mathcal{G}(\mathcal{H}(G))$ and $H \to \mathcal{H}(\mathcal{G}(H))$ are isomorphisms.

The same concept of gauge is also considered later in the context of Lie theory – see [Ho70] – where in Theorem 6.2 it is proved that:

Let L be a finite dimensional Lie algebra over the field F of characteristic 0. Then L is semisimple if and only if there is a linear functional J on $\mathbf{H}(L)$ such that $J \circ u$ is the identity map on F, and $u \circ J = (J \otimes i) \circ \gamma$ where u is the unit of $\mathbf{H}(L)$ and γ is the comultiplication.

Later, these ideas were generalized by M. Sweedler in [Sw69b] to the context of arbitrary Hopf algebras:

For a Hopf algebra that is the coordinate ring of a compact Lie group, there is a unique one–dimensional left invariant ideal in the linear dual, this is the space spanned by a left Haar integral. Hochschild has observed that for a Hopf algebra H which is the coordinate ring of an affine algebraic group the existence of a left invariant ideal in the linear dual which complements the linear functionals orthogonal to the unit of H, is equivalent to the group being completely reducible.

In this paper ... [we] generalize Hochschild's results.

Integrals in finite-dimensional Hopf algebras were previously considered in [LS69], cf. also [KP66]. The notion of integral plays a central role in the definition of ring group by G. I. Kac discussed below.

3.4. **Bertram Kostant.** Bertram Kostant, besides being the thesis advisor to Moss Sweedler and the originator of some of the current notations and nomenclature in Hopf theory –for example of the term *group-like element*–, produced one of the first genuine applications of Hopf algebras with his contribution to the understanding of finite groups of Lie type, in his remarkable paper [Ko66].

Here is the limpid definition of a Hopf algebra given in the Preliminaries of [Ko66]. Kostant begins by introducing the convolution product $*$ in $\mathrm{Hom}(A, R)$, where A is a coalgebra (with comultiplication d and counit ε) over a commutative ring C, and R is an algebra (with multiplication m and unit ρ) still over C. Then he says:

Now assume that A is a Hopf algebra (A is an algebra and coalgebra such that d and ε are homomorphisms and $\varepsilon\rho$ is the identity on C).
By an antipode on A we mean an element (necessarily unique if it exists) $s \in \mathrm{Hom}_C(A, A)$ such that $I * s = s * I = \varepsilon$ where I is the identity on A and $*$ is as above with A taken for R. From now on Hopf algebra means Hopf algebra with antipode.

He continues by observing that a Hopf algebra gives rise to a representable functor from the category of algebras over C to the category of groups.

Still in the Preliminaries, given a Hopf algebra B over $\mathbf{Z}$, he explains how to construct from any 'admissible' family of ideals of finite type a Hopf algebra dual to B.

The bulk of the paper contains the definition of a 'divided powers' version over $\mathbb{Z}$ of the universal enveloping algebra of a finite-dimensional semisimple Lie algebra, i.e., the so called "Kostant $\mathbb{Z}$–form". In accordance to a private communication by the author:

The ultimate purpose of this [...] was to exhibit the Chevalley (Tohoku) finite groups as the group-like elements in the $\mathbb{Z}$ dual of $U_{\mathbb{Z}}(\mathfrak{g})$. However I did not succeed in doing this. However recently George Lusztig, in a tour de force paper, did suceed in establishing what I hoped would be the case. Lusztig's paper can be found in ArXiv:0709.1286v1 [math.RT] 9 Sept. 2007.

Kostant also proved two theorems on the structure of cocommutative Hopf algebras over an algebraically closed field k. First, any cocommutative Hopf algebra H is the smash product of an irreducible Hopf algebra U with a group algebra kG. Here G turns out to be the group $G(H)$ of group-likes in H, and 'irreducible' means that $G(U)$ is trivial. Second, U is the universal enveloping algebra of the Lie algebra of primitive elements in H, provided that the characteristic of k is 0. These results were never published by Kostant, and appeared first in the book [Sw69], see Theorem 8.1.5 and Section 13.1.

Later, Kostant obtained the analogous result for cocommutative Hopf super algebras [Ko77, Th. 3.3]. In this paper he introduced the notion of super manifolds and super Lie groups with the goal to do in the super case, what Chevalley did for ordinary Lie groups. As the author told us in a private communication: *the idea is to get the super Lie group from its Lie algebra using the Hopf algebra double dual.*

3.5. **Georg Isaakovich Kac.** Inspired by work of Stinespring on a duality theory for unimodular groups (extending previous work by Tannaka and Krein), G. I. Kac introduced in 1961, see [Ka61], the notion of *group ring.* We quote from the introduction:

The basic idea of the generalization of the concept of group consists in the following. Let $\mathfrak{G}$ be a group. We denote by $\mathfrak{M}$ the commutative ring (with respect to the operation of multiplication) of bounded functions on $\mathfrak{G}$. The *mapping $f(y) \to f(xy)$ is an isomorphism of $\mathfrak{M}$ into $\mathfrak{M} \otimes \mathfrak{M}$. Thus $\mathfrak{G}$ can be considered as a commutative ring $\mathfrak{M}$ with a given isomorphism of $\mathfrak{M}$ into $\mathfrak{M} \otimes \mathfrak{M}$. We obtain the generalization under study, if we reject in this definition the requirement that $\mathfrak{M}$ is commutative.*

A group ring is essentially like a Hopf algebra in the setting of von Neumann algebras. Namely, it is defined as a collection $(\mathfrak{M}, \Phi, {}^{+}, m)$, where $\mathfrak{M}$ is a von Neumann algebra; Φ is an monomorphism of $\mathfrak{M}$ into $\mathfrak{M} \otimes \mathfrak{M}$[4], that is coassociative; $A \mapsto A^{+}$ is a sesquilinear involution of $\mathfrak{M}$ that reflects somehow the inversion in the group; m is a measure on $\mathfrak{M}$ (a 'gauge'), satisfying certain axioms, similar to what is called today an integral.

A group ring is representable as the ring of bounded functions on a unimodular group G if and only if the ring $\mathfrak{M}$ is commutative. Every group ring $\mathfrak{M}$ corresponds uniquely to a so-called dual Hilbert ring; this formalism enables to define a the dual $\mathfrak{M}\hat{}$ of $\mathfrak{M}$. There is a duality theorem stating that the double dual $\mathfrak{M}\hat{}\hat{}$ is isomorphic to $\mathfrak{M}$.

G. I. Kac developed the theory of finite group rings (group rings of finite dimension) in several papers. For instance, in [Ka62], he extends certain classical finite-group theorems to finite-ring groups, e. g. the analogue of Lagrange's theorem: The order of a subgroup of a ring group divides the order of the ring group. In [KP66, Ka68], the construction of new finite group rings by extensions is given; this was rediscovered by Takeuchi in 1981 and then again by Majid in 1988. In [KP66], the existence of an integral in a finite group ring is derived, assuming the existence of an antipode; this was later established for finite-dimensional Hopf algebras by Larson and Sweedler [LS69]. In [Ka72], among other results, he showed that a finite group ring of prime dimension is necessarily a group algebra; this fact was conjectured by Kaplansky in 1975 and proved by Y. Zhu in 1993– for Hopf algebras over $\mathbb{C}$.

4. Final considerations

With the publication of Sweedler's book [Sw69] – September 1, 1969 – the subject started to shape up as an independent part of abstract algebra and was able to walk without the help of its two honorable parents. Of the latter period we only mention that, with the appearance in 1987 of the paper by V. Drinfel'd [Dr87] and the subsequent work by him and many other mathematicians, the area experienced a very radical change in terms of methods, examples and interaction with other parts of mathematics. In this latter period, the progresses obtained in understanding the structure of Hopf algebras and its representations have been outstanding, and they have been entwined with the development of different areas of mathematics,

[4]Kac says "isomorphism. . . into" which meant at the time "monomophism"; for isomorphisms in our present meaning, it was used "isomorphism. . . onto".

most remarkably with: knot theory and topology, conformal field theory, ring theory, category theory, combinatorics, etc.

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