It is proved that a complex cosemisimple Hopf algebra has at most one compact involution modulo automorphisms.

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Introduction

Let $H$ be a complex cosemisimple Hopf algebra, that is, any finite dimensional $H$-comodule is completely reducible, or equivalently $H$ is completely reducible as comodule via the comultiplication (see 1.3 (c) in [1]). We prove that two compact involutions of $H$ [2] are necessarily conjugated by a Hopf algebra automorphism. This extends a well-known theorem of Cartan to the quantum case. Using results from [3], this was proved recently for finite Hopf algebras [4]. Since then, the author noticed however the paper [5] which contains a weak form of those results from [3] and enables him to extend the theorem to the infinite case. The second part of the proof is a variation of Mostow's proof of the above mentioned Cartan's theorem — see p. 182 in [6]. In the first section of this paper, we recall some results on cosemisimple Hopf algebras (some of them go back to [7]) and give a formula (1.8) for the Killing form — an invariant bilinear form on $H$ arising from (a choice of) the integral and normalized by a further invariant condition. In the second, we prove the theorem. For this, we use an invariant sesquilinear form on $H$ also derived from the integral, first considered in [8].

1 Killing forms on cosemisimple Hopf algebras

We shall work over an arbitrary field $\mathbb{K}$ in this section. The notation for Hopf algebras is standard: $\Delta$, $S$, $\varepsilon$, denote respectively the comultiplication, the antipode, the counit; we use Sweedler [9] notation but drop the summatory.

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1.1. Let $H$ be a Hopf algebra. Recall that for a finite dimensional right comodule $c : V \rightarrow V \otimes H$, its left and right duals $d^c$ and $c^d$ are the right $H$-comodule structures on $V^*$ defined as follows. Let $(h_i)_{i \in I}$ be a basis of $H$. Then $c(v) = \sum_i T_i(v) \otimes h_i$, with $T_i \in \text{End} V$, $T_i = 0$ for all but a finite number of $i$. Define

$$d^c(\alpha) = \sum_i T_i(\alpha) \otimes S^{-1}(h_i), \quad c^d(\alpha) = \sum_i T_i(\alpha) \otimes S(h_i),$$

for $\alpha \in V^*$. $d^V$, $V^d$ denote $V^*$ considered as $H$-comodule via, respectively, $d^c$, $c^d$. In the category of finite dimensional right comodules, the functors $V \mapsto d^V$ and $V \mapsto V^d$ are inverse to each other; therefore, the following are equivalent:

(a) $V \simeq (V^d)^d$; (b) $V \simeq (c^d V)$; (c) $V^d \simeq d V$.

1.2. $H^*$ has an algebra structure provided by the transposes of the multiplication and the counit. Any (left or right) $H$-comodule is then a (right or left) $H^*$-module; such $H^*$-modules are called rational. For example, $H$ is an $H^*$-bimodule via $x \cdot h = h(1)(x, h(2))$, $h' \cdot x = (x, h(0)) h(2)$; $h \in H$, $x \in H^*$. This correspondence is in fact an isomorphism between the categories of $H$-comodules and rational $H^*$-comodules. By psychological reasons, it is often helpful to state properties in terms of $H^*$-actions. By abuse of notation, we write $S : H^* \rightarrow H^*$ for the transpose of the antipode and $\varepsilon : H^* \rightarrow \mathbb{K}$ for evaluation in 1. The representations $\rho^d$ and $d^\rho$ can be defined for any representation $\rho$ of $H^*$; for rational ones, they agree with those derived from the previous $c^d$, $d^c$.

1.3. Define $\psi^V : (\text{End} V)^* \rightarrow H$ by $\psi^V(\alpha) = \sum i \langle \alpha, T_i \rangle h_i$. Then $\psi^V$ is a morphism of coalgebras. Furthermore, it is injective if $V$ is irreducible, and the simple subcoalgebras of $H$ are exactly the $\text{Im} \psi^V$ for $V$ irreducible [1]. Thus, if $H$ is cosemisimple,

$$H = \bigoplus_{V \in \tilde{H}} \text{Im} \psi^V,$$

where $\tilde{H}$ denotes the set of isomorphism classes of irreducible $H$ comodules. (We often confuse a class with a representant). $\text{Im} \psi^V$ is the isotypic component of $H$, for the coaction given by the multiplication, of type $V$. We shall denote it alternatively as $H_\rho$ or $H_\rho^\rho$; $\rho$ will be then the representation of $H^*$ derived from the coaction $c$. We shall also identify $\tilde{H}$ with the set of isomorphism classes of irreducible rational $H^*$-modules.

Given a finite dimensional representation $\rho : H^* \rightarrow \text{End} U$, let $\varphi^U : U^* \otimes U \rightarrow H^{**}$ be the "matrix coefficient" map defined, for $v \in U$, $\alpha \in U^*$, by $(\varphi^U_{\alpha \otimes v}, x) = \langle \alpha, \rho(x)v \rangle$. Modulo the usual identifications $(\text{End} U)^* \simeq \text{End} U$ (provided by the trace) and End $U \simeq U^* \otimes U$, it coincides with the usual transpose map $^t \rho : (\text{End} U)^* \rightarrow H^{**}$:

$$^t \rho(T) = \varphi^U_{\alpha \otimes v}, \quad \text{if} \quad T \in \text{End} U, \quad T(u) = \langle \alpha, u \rangle v.$$
Note that \( \Theta (\phi^V \otimes \omega) = \phi^V \). Let \( \Theta : H \to H^{**} \) be the natural injection; then \( \Theta \phi^V = \phi^V \) (\( V \) is an \( H \)-comodule and hence a rational \( H^* \)-module). \( \Theta \) is a morphism of \( H^* \)-bimodules.

1.4. Let \( d : W \to W \otimes H \) be another finite dimensional right comodule structure; then \( V \otimes W \) also is an \( H \)-comodule whose coaction we shall denote \( c \otimes d \). Let \( S_j \in \text{End}_W \) be, similarly as above, such that \( d(w) = \sum S_j(w) \otimes h_j \). Define a comodule structure on \( \text{Hom}(V, W) \) by \( A \mapsto \sum_{i,j} S_j \circ A \circ T_i \otimes h_j S(h_i) \). The natural isomorphism between \( \text{Hom}(V, W) \) and \( W \otimes V^* \) is in fact an \( H \)-comodule isomorphism between \( \text{Hom}(V, W) \) and \( W \otimes V^d \). The isotypic component of trivial type of \( \text{Hom}(V, W) \) with respect to the adjoint action is exactly the space of \( H \)-comodule maps. Therefore, if \( W \) and \( V \) are irreducible, the multiplicity of the trivial representation in \( W \otimes V^d \) is 1 (resp., 0) if \( W \) and \( V \) are (resp., are not) isomorphic. In other words, \( W \otimes V \) contains the trivial representation if and only if \( W \cong dV \).

1.5. Recall that a linear functional \( f : H \to \mathbb{K} \) is a right integral if
\[
(f, h)_1 = (f, h(1)) h(2), \quad \text{for all } h \in H. \tag{1.1}
\]
It is equivalent to provide \([10]\)

(a) A right integral \( f \).

(b) A bilinear form \( (\cdot | \cdot) : H \times H \to \mathbb{K} \) satisfying
\[
((uv|w)) = ((u|vw)), \tag{1.2}
\]
\[
((x - v|w)) = ((v|Sx - w)), \tag{1.3}
\]
for all \( u, v, w \in H, x \in H^* \).

Explicitly, \( (f, v) = ((v|1)), ((v|u)) = (f, uv) \). In general, if \( (\cdot | \cdot) \) is a bilinear form which satisfies (1.3), then \( A \in H^* \) given by \( (A, v) = (v|1) \) is a right integral; (1.2) is a "normalization" condition which ensures the bijectivity of the correspondence. Indeed, if \( (\cdot | \cdot) \) satisfies (1.3) then \( (uv|v) = (uv|1) \) also does, and in addition satisfies (1.2).

Now let \( M, N \subseteq H \) be submodules for \( \to \) and let \( \theta : M \to N^d \) be given by \( \langle \theta(m), n \rangle = ((m|n)) \); \( \theta \) is a morphism of \( A \)-modules by (1.3). Therefore if \( M \) and \( N \) are both irreducible, \( \theta \) is either 0 or an isomorphism. Taking \( M = \mathbb{K}1 = H_e \), the trivial submodule of \( H \), we conclude that \( (f, v) = 0 \) for all \( v \in N \), for all irreducible, non-trivial, \( N \).

Now assume that \( H \) is cosemisimple. For \( a \in H \), write \( a = \sum_{\rho \in \hat{H}} a_{\rho} \), with \( a_{\rho} \in H_{\rho} \). By abuse of notation, we shall write \( a_{\cdot 1} \) instead of \( a_{\cdot} \) with \( a_{\cdot} \in \mathbb{K} \). Then
\[
(f, h) = a_{\cdot} (f, 1). \tag{1.4}
\]
Conversely, the linear map defined by (1.4) and an arbitrary value of \( (f, 1) \) is a right integral, because \( H_{\rho} \) is a subcoalgebra of \( H \). It follows that, for \( H \) cosemisimple,
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the space of right integrals is one-dimensional. Interchanging right by left and viceversa, one sees that any left integral also is expressed by (1.4); hence $H$ is unimodular. In particular, by the “dual hand” version of the equivalence above, $((\,))$ also satisfies

$$((v \leftarrow Sx|w)) = ((v|w \leftarrow x)). \quad (1.5)$$

Finally, if $H$ is an arbitrary Hopf algebra admitting a right integral such that $(f, 1) \neq 0$ then $H$ is cosemisimple. See [7], where the formula (1.4) appears for the first time.

**Lemma 1.6.** Let $H, H'$ be Hopf algebras, let $T : H' \to H$ be an isomorphism of coalgebras such that $T(1) = 1$ and let $f$ be a right integral for $H$. Then $f \circ T$ is a right integral for $H'$. In particular, $f \circ S$ is a left integral for $H$. If $H$ is cosemisimple, $T$ is an automorphism of Hopf algebras of $H$ and $f$ is normalized by $(f, 1) = 1$, then $((Tu|Tv)) = ((u|v))$, for all $u, v \in H$.

*Proof.* Straightforward. \(\square\)

**1.7.** Let $H$ be a cosemisimple Hopf algebra as above.

**Theorem (Thm. 3.3 in [5]).** For each simple subcoalgebra $C$ of $H$, $S^2C = C$.

**Corollary.** For any irreducible $H$-comodule $c$, $c^{\ddagger}$ is isomorphic to $c$.

*Proof.* Let $V$ be the space of $c$. Then $S^2(\phi^V) = \phi^{V^{\ddagger}} \in H_c \cap H_{c^{\ddagger}}$ (modulo identification by $\Theta$). Thus $H_c = H_{c^{\ddagger}}$ and hence $c \simeq c^{\ddagger}$.

As observed in [5], the proof of this theorem implies that $((\,))$ is non-degenerate. This fact will also follow from formula (1.8) below.

**1.8.** We still assume that $H$ is cosemisimple and normalize $f$ by $(f, 1) = 1$. The corresponding $((\,))$ will be named the Killing form of $H$. We shall give a formula for it in the spirit of [3]. Let $a = \sum_{c \in \widehat{H}} a_c$, $b = \sum_{c \in \widehat{H}} b_c \in H$. Then

$$((a|b)) = \sum_{c \in \widehat{H}} ((a_c|b_c)).$$

So we need only to precise $((\,)) : H_c \otimes H_c \to \mathbb{K}$, for $c : V \to V \otimes H$ irreducible. Recall that we have identified $H_c \simeq (\text{End} V)^*$ with $\text{End} V$ via the trace map. Fix $\mathcal{M} \in \text{Aut} V$ such that

$$\sum_i T_i \mathcal{M} \otimes h_i = \sum_i \mathcal{M} T_i \otimes S^2(h_i). \quad (1.6)$$

Let $\rho : H^* \to \text{End} V$ be the representation corresponding to $c$. Then (1.6) means that $\mathcal{M}\rho(S^2x) = \rho(x)\mathcal{M}$, for all $x \in H^*$. Let $S \in \text{End}(V^d)$, $T \in \text{End} V$ and define

$$B_c(S, T) = \text{Tr}(^{t}STM). \quad (1.7)$$

Then

$$B_c(x \leftarrow S, T) = \text{Tr}(^{t}(\rho^d(x)S)TM) = \text{Tr}(^{t}S^t(\rho^d(x))TM) = \text{Tr}(^{t}S\rho(Sx)TM) = B_c(S, Sx \leftarrow T).$$
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On the other hand,

\[ B_c(S - Sx, T) = \text{Tr} \left( \iota(S \rho^d(Sx)) \iota T M \right) = \text{Tr} \left( \iota(S \rho^d(Sx)) \iota T M \rho(S^2 x) \right) = \text{Tr} \left( \iota ST \rho(x) M \right) = B_c(S, T - x). \]

As \( \text{End} V \) is irreducible as \( H^* \)-bimodule, there is only one bilinear form satisfying (1.3) and (1.5), up to scalars. Therefore,

\[ ((a_c \rho | b_c)) = C_c B_c(S, T) = C_c \text{Tr} \left( \iota ST M \right), \]

for some scalar \( C_c \), where \( S \in \text{End} (V^d) \) corresponds to \( a_c \), and \( T \) to \( b_c \). Next we compute \( C_c \). The preceding \( B_c(\, \, \, ) \) depends on \( M \) and hence is also defined up to a scalar; what we need, therefore, is to take \( C_c = 1 \) and adjust \( M \).

So let \( a_c \rho, b \rho, S \) and \( T \) be as above. We wish to compute \( ((a_c \rho | b \rho)) = ((a_c \rho b \rho | 1)) = d_c \), if \( a_c \rho b \rho = \sum_{\tau \in \mathbb{H}} d_{\tau} \), with \( d_{\tau} \in H_{\tau} \) and \( d_c, 1, d \in \mathbb{K} \), instead of \( d_c \). We compute \( a_c \rho b \rho \) (compare with [11]). \( V^d \otimes V \) decomposes as direct sum of irreducible \( A \)-submodules: \( V^d \otimes V = \bigoplus_{\tau \in \mathbb{H}} U_{\tau} \). Let \( \iota_{\tau} : U_{\tau} \to V^d \otimes V \) be the inclusion and \( \pi_{\tau} : V^d \otimes V \to U_{\tau} \), the projection with respect to this direct sum. Let \( R_{\tau \mu} = \pi_{\mu}(S \otimes T) \iota_{\tau} \in \text{Hom}(U_{\tau}, U_{\mu}) \). Then \( S \otimes T = \sum_{\tau, \mu} \iota_{\tau} R_{\tau \mu} \pi_{\tau} \); that is, \( (R_{\tau \mu}) \) is the “partition” of \( S \otimes T \) in blocks with respect to the decomposition above, and \( d_c \) corresponds to \( R_{\tau \tau} \). We already know that \( (V^d \otimes V)_{\epsilon} \) is one dimensional. A generator is \( Z = \sum_{1 \leq h \leq n} \alpha_h \otimes M v_h \), where \( (v_h) \) is a basis of \( V \) and \( (\alpha_h) \) is the dual basis. Indeed,

\[ (c^d \otimes c)(Z) = \sum_{1 \leq h \leq n, i, j \in I} t_j(\alpha_h) \otimes t_i(M v_h) \otimes S(h_j) h_i \]

\[ = \sum_{1 \leq k, h \leq n, i, j \in I} (v_k, \iota t_j(\alpha_h)) \alpha_k \otimes t_i(M v_h) \otimes S(h_j) h_i \]

\[ = \sum_{1 \leq k, h \leq n, i, j \in I} \alpha_k \otimes t_i(M t_j v_k) \otimes S(h_j) h_i \]

\[ = \sum_{1 \leq k, h \leq n, i, j \in I} \alpha_k \otimes t_i t_j M (v_k) \otimes S^{-1}(h_j) h_i = Z \otimes 1. \]

Now the projector \( \pi_{\epsilon} : V^d \otimes V \to \mathbb{K} Z \) must be of the form \( \pi_{\epsilon}(P) = \langle \Omega, P \rangle Z \), for \( P \in V^d \otimes V \), with \( \Omega \in (V^d \otimes V)^* \). Let \( \Omega = \sum_{1 \leq i \leq n} v_i \otimes \alpha_i \) (with the usual vector space identification of \( (V^d \otimes V)^* \) with \( V \otimes V^d \)) and write tentatively \( \pi \) for \( P \mapsto \langle \Omega, P \rangle Z \). Then \( c_{\text{Hom}(V^d \otimes V, \mathbb{K} z)}(\pi) = \sum_{i, j \in I} \text{id} \circ \pi \circ (t_i \otimes t_j) \otimes S(S(h_i) h_j) \). Evaluating in \( \beta \otimes w \) the first factor, we get

\[ \sum_{i, j \in I} \langle \Omega, t_i(\beta) \otimes t_j(w) \rangle Z \otimes S(S(h_i) h_j) \]

\[ = \sum_{1 \leq k \leq n} (v_k, \iota t_i(\beta))(\alpha_k, t_j(w)) Z \otimes S(S(h_i) h_j) = \]

that is, \( \pi \) is invariant, and nonzero. As some multiple of it is a projector, \( \pi(Z) = (\Omega, Z)Z = \text{Tr} Z \neq 0 \). Therefore, we can normalize \( \mathcal{M} \), as promised, by \( \text{Tr} \mathcal{M} = 1 \).

We can now write \( \pi_\epsilon \) instead of \( \pi \). But \( d_\epsilon Z = \pi_\epsilon((S \otimes T)Z) = (\Omega, (S \otimes T)Z)Z \) and hence

\[
d_\epsilon = (\Omega, (S \otimes T)Z) = \left( \sum_{i} \nu_i \otimes \alpha_i, \sum_{j} S\alpha_j \otimes T\mathcal{M}v_j \right)
\]

\[
= \sum_{i,j} \langle \alpha_i, T\mathcal{M}v_j \rangle (\alpha_j, ^1Sv_i) = \text{Tr} \left( ^1ST\mathcal{M} \right).
\]

We have proved

\[
((a_\rho|b_\rho)) = \text{Tr} \left( ^1ST\mathcal{M} \right),
\]

where \( a_\rho \) corresponds to \( S \in \text{End } (V^d) \), \( b_\rho \) to \( T \) and \( \mathcal{M} \in \text{End } V \) satisfies (1.6) and \( \text{Tr} \mathcal{M} = 1 \).

1.10. Is the Killing form symmetric? We compute \( ((b_\rho|a_\rho)) = ((b_\tau|a_\tau)) \), for \( \tau = \rho^d \). Note that (1.6) is equivalent to

\[
(^1\mathcal{M})^{-1} \rho^d(S^2x) = \rho^d(x)(^1\mathcal{M})^{-1}, \quad \text{for all } x \in H^*.
\]

Also, if \( b_\rho \) corresponds to \( T \in \text{End } V \) then it corresponds to \( \mathcal{M}^{-1}T\mathcal{M} \in \text{End } V^{dd} \).

Let \( \mu = (\text{Tr} \mathcal{M}^{-1} \mathcal{M})^{-1} \). Applying (1.8) to \( \rho^d \) we get

\[
((b_\rho|a_\rho)) = \mu \text{Tr} \left( ^1(M^{-1}T\mathcal{M})S(^1\mathcal{M})^{-1} \right) = \mu \text{Tr} \left( ^1T(^1\mathcal{M})^{-1}S \right) = \mu \text{Tr} \left( ^1SM^{-1}T \right).
\]

Thus the Killing form is symmetric if and only if \( \mathcal{M} = (\dim V)^{-1} \text{id}_V \) for all irreducible \( V \), if and only if \( S^2 = \text{id} \). Indeed, \( S^2b_\rho \) corresponds to \( \mathcal{M}T\mathcal{M}^{-1} \in \text{End } V \).

2 Killing forms and *-Hopf algebras

We assume in this section that \( K = \mathbb{C} \). We suppose further that \( H \) is a *-Hopf algebra, i.e., it is a *-algebra and the comultiplication is a morphism of *-algebras; \( H^* \) is then considered as *-algebra by \( (x^*, v) = (x, S(v)^*) \). It is known that \( (Sx)^* = S^{-1}(x^*) \). For convenience, we shall denote \( T(x) = (Sx)^* = S^{-1}(x^*) \).

Lemma 2.1. (i) The following data are equivalent:

(a) A right integral \( f : H \to \mathbb{C} \).
(b) A bilinear form \((\langle \| \rangle)\) satisfying (1.2), (1.3).

(c) A sesquilinear form \((\langle \| \rangle)\) satisfying
\[
\begin{align*}
(\langle uv|w \rangle)_\ell &= (\langle v|u^*w \rangle)_\ell, \\
(\langle x - v|w \rangle)_\ell &= (\langle v|x^* - w \rangle)_\ell.
\end{align*}
\] (2.1)

(ii) Also, the following are equivalent:

(d) A left integral \(f : H \to \mathbb{C}\).

(e) A bilinear form \((\langle \| \rangle)_r\) satisfying (1.2), (1.6).

(f) A sesquilinear form \((\langle \| \rangle)_r\) satisfying
\[
\begin{align*}
(\langle uv|w \rangle)_r &= (\langle u|v^*w \rangle)_r, \\
(\langle v - x|w \rangle)_r &= (\langle v|w - x^* \rangle)_r.
\end{align*}
\] (2.3)

Proof. We have already discussed the equivalence between (a) and (b), resp. (d) and (e). The correspondence between (b) and (c), resp. (e) and (f), is given by
\[
(\langle vlw \rangle)_\ell = \langle (w'Iv) \rangle, \quad \text{resp.} \quad (\langle vlw \rangle)\_r = \langle (v*l.:) \rangle\_r, (2.5)
\]
and correspondingly, \((\langle v|w \rangle) = (\langle w|v^* \rangle)_\ell, (\langle v|w \rangle)\_r = (\langle v^*|w \rangle)_r\). For the proof, we need the formulas
\[
(\langle x - v \rangle\_r = (\langle Sx \rangle^* - v^*), \quad (\langle v - x \rangle\_r = v^* - (\langle Sx \rangle^*).
\]
Thus \((\langle v|x^* - w \rangle)_\ell = ((\langle x^* - w \rangle^*|v)) = ((\langle S^{-1}x - w^*|v)) = (\langle w^*|x - v \rangle) = (x - v|w\rangle), \text{and the rest is similar.}
\]

2.2. Let \(\int\) be a right integral and let \(\Lambda\) be defined by \(\langle \Lambda, h \rangle = (\int, h^*)\). Then \(\Lambda\) is also a right integral:
\[
\langle \Lambda, h(1) \rangle h(2) = \langle \int, h(1)^* \rangle h(2) = ((\langle \int, h(1)^* \rangle h(2))^* = (\langle \int, h^* \rangle 1)^* = \langle \Lambda, h \rangle 1.
\]

Assume now that \(H\) is cosemisimple. We shall normalize, in what follows, \(\int\) by \(\langle \int, 1 \rangle = 1\). Then, by the uniqueness of the right integral, \(\int = \Lambda\). It follows that the corresponding sesquilinear form \((\langle \| \rangle)_\ell\) is Hermitian:
\[
(\langle v|w \rangle)_\ell = \langle \int, w^*v \rangle = \langle \Lambda, w^*v \rangle = \langle \int, (w^*v)^* \rangle = \langle w|v \rangle_\ell.
\]

Remark. These facts were essentially first observed by Majid [8].

2.3. A \(\ast\)-representation of \(H^*\) is a representation \(\rho : H^* \to \text{End} \, V\) together with a non-degenerate sesquilinear form \((\langle \| \rangle)\) such that \((\rho(x)v|w) = (v|\rho(x^*)w), \text{for all} \ x \in H^*, v, w \in V\). Such form shall be called invariant. We consider in the following only finite dimensional rational representations. A representation is a
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\textit{*-representation if and only if there exists a sesquilinear isomorphism }J : V \to V^d\text{ such that }J(\rho(x)w) = \rho^d(J(x))J(w).\text{ Explicitly, }\langle Jw, v \rangle = \langle v|w \rangle.\text{ If }T \in \text{End} V,\text{ define as usual }T^* \in \text{End} V \text{ by }\langle Tv|w \rangle = \langle v|T^*w \rangle,\text{ or equivalently by }T^* = J^{-1}TJ.

\text{Let }V \text{ be a right } H\text{-comodule and let }T_i \text{ as in 1.1. Let }\mathcal{G} = \sum_i T_i \otimes h_i; \text{ it follows easily from the comodule axioms that }\mathcal{G} \text{ is invertible and }\mathcal{G}^{-1} = \sum_i T_i \otimes S(h_i), \text{ in the algebra }\text{End} V \otimes H. \text{ The last is a }*\text{-algebra once a non-degenerate sesquilinear form is chosen. It can be shown that the corresponding rational representation of } H^* \text{ is a }*\text{-representation if and only if }\mathcal{G}^{-1} = \mathcal{G}^*; \text{ hence the present definition agrees with that of [2].}

\text{Let }V \text{ be a }*\text{-representation. Let }\langle J^{-1}\rangle^\dagger : V^* \to V \text{ be given by }\langle \mu, (J^{-1})^\dagger \alpha \rangle = \langle \alpha, J^{-1} \mu \rangle. \text{ Then the }* \text{ in } H \text{ of the matrix coefficients is given (modulo }\Theta\text{) by [11], p. 306}

\begin{align*}
\phi^V_{\alpha \otimes v}^* &= \phi^{V^d}_{(J^{-1})^\dagger \alpha \otimes Jv}. \quad (2.6)
\end{align*}

\text{Equivalently, if }T \in \text{End} V \text{ corresponds to }w \in H, \text{ then w}^* \text{ corresponds to}

\begin{align*}
JTJ^{-1} \in \text{End } V^d. \quad (2.7)
\end{align*}

\text{Here one uses that }\text{Tr}(JAJ^{-1}) = \overline{\text{Tr}A}, \text{ for }A \in \text{End} V.

\text{If }\langle \rangle \text{ is an invariant form, then }\langle \rangle_{\text{opp}}, \text{ given by }\langle v|w \rangle_{\text{opp}} = \overline{\langle w|v \rangle}, \text{ also is. Assume that }V \text{ is irreducible. Then invariant forms are unique up to multiplication of a scalar; in particular }\langle \rangle_{\text{opp}} = \lambda\langle \rangle \text{ for some scalar }\lambda. \text{ Applying this twice, we see that }\lambda^2 = 1. \text{ Multiplying }\langle \rangle \text{ by a suitable scalar, we can assume that }\lambda = 1, \text{ i.e., that }\langle \rangle \text{ is Hermitian.}

\text{Let }V \text{ be a }*\text{-representation, with invariant form }\langle \rangle, \text{ and let }\mathcal{M} \in \text{Aut } V \text{ satisfying (1.6). Let }\langle \rangle_d \text{ be the form on } V^d \text{ defined by }\langle \mu|\eta \rangle_d = \langle \mathcal{M}^{-1}J^{-1}\eta|J^{-1}\mu \rangle; \text{ it is also invariant. If }V \text{ is irreducible, then }V^d \text{ also is; assuming this, we shall normalize first }\langle \rangle \text{ to get an Hermitian form, and second }\mathcal{M}, \text{ to get an Hermitian form on } V^d. \text{ In such case, }\mathcal{M} = \mathcal{M}^*, \text{ i.e., }\mathcal{M} \text{ is self-adjoint. Now assume in addition that }\langle \rangle \text{ is an inner product. Then }\langle \rangle_d \text{ also is, if and only if }\mathcal{M} \text{ is positive definite; in such case, }\text{Tr}\mathcal{M} > 0. \text{ Conversely, if }V^d \text{ admits an invariant inner product, then some multiple of }\mathcal{M} \text{ is positive definite.}

\text{A representation is not always a }*\text{-representation. For example, let }H^* \text{ be the group algebra of an abelian finite group with the involution }\langle \sum_{g \in G} \lambda_g e_g \rangle^* = \sum_{g \in G} \overline{\lambda_g} e_g. \text{ Let }\chi \text{ be a one-dimensional representation of } G \text{ which is not real; this admits no sesquilinear invariant form.}

\text{2.4. Now we are ready to state the key point of the proof of the main result. We first recall a definition [2].}

\textbf{Definition.} \text{We shall say that } H \text{ is a }\textit{compact quantum group} \text{ if any rational, finite dimensional, representation of } H^* \text{ carries an invariant inner product.}

\text{By a standard argument, if } H \text{ is compact, then is cosemisimple. It is known (see e.g. [12], [13]) that completions of compact quantum groups as in the preceding definition with respect to a suitable norm give rise to compact quantum groups as}
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in [2]; the preceding notion corresponds to that of "algebras of regular functions" in Woronowicz definition [2].

**Proposition.** $H$ is a compact quantum group if and only if the hermitian form $(\langle \cdot | \cdot \rangle)_I$ is positive defined.

**Proof.** If $(\langle \cdot | \cdot \rangle)_I$ is positive defined then any $H^*$-submodule of $H$ (for $\rightarrow$) carries an invariant inner product and $H$ is a compact quantum group. Conversely, assume that $H$ is a compact quantum group. Let $v \in H_\rho, w \in H_\tau$; then $w^* \in H_{\rho^4}$ by (2.6), and $(v|w)_I = 0$ if $\rho$ and $\tau$ are not isomorphic, by (2.5). So assume that $\rho = \tau$ and let $S, T \in \text{End } V$ correspond to $v, w$, respectively. By (1.7) and (2.7), we have

$$
\langle v|w\rangle_I = (\langle w^*|v\rangle) = \text{Tr} \left( (JTJ^{-1})SM \right) = \text{Tr} \left( T^*SJTJ^{-1} \right)
$$

This formula also implies that $(\langle \cdot | \cdot \rangle)_I$ is Hermitian. Thus $\langle v|v\rangle_I = \text{Tr}(S^*SM) > 0$ if $S \neq 0$, because $\mathcal{M}$, normalized by $\text{Tr}\mathcal{M} = 1$, is positive definite. 

**2.5.** The preceding Proposition enables us to adapt Mostow's proof of Cartan's theorem of the uniqueness of compact involutions (see Ch. II, Thm. 7.1 in [6]) to our setting. See also Proposition 2 in [4].

**Proposition.** Let $H$ be a compact quantum group with respect to $*$ and let $x \rightarrow x^#$ be another structure of $^*\text{-Hopf algebra}$ on $H$. Then there exists a Hopf algebra automorphism $T$ of $H$ such that $#$ and $T^*T^{-1}$ commute.

**Proof.** Let $N$ be given by $N(u) = (u^*)#$; this is a Hopf algebra automorphism and any finite dimensional submodule of $H$ is contained in some finite dimensional submodule $W$ such that $N(W) = W$. By Proposition 2.4, the Hermitian form $(\langle \cdot | \cdot \rangle)_I$ (defined with respect to $*$) is positive definite. From Lemma 1.7, we deduce that $N$ is self-adjoint with respect to $(\langle \cdot | \cdot \rangle)_I$. Then the Hopf algebra automorphism $P = N^2$ is diagonalizable with positive eigenvalues; let $(X_i)_{i \in I}$ be a basis of $H$ such that $PX_i = \lambda_i X_i$. For each $s \in \mathbb{R}$, one has a well-defined linear automorphism $P^s$ of $H$. We claim that $P^s$ is also a Hopf algebra automorphism. Let $c_{ij}^k$ be constants such that $\Delta(X_k) = \sum_{i,j} c_{ij}^k X_i \otimes X_j$, for all $k$. Hence

$$
\lambda_i \lambda_j c_{ij}^k = \lambda_k c_{ij}^k
$$

for all $i, j, k$ and a fortiori $\lambda_i^* \lambda_j^* c_{ij}^k = \lambda_k^* c_{ij}^k$, that is, $P^s$ preserves the comultiplication. With similar arguments, one shows that $P^s$ is a morphism of Hopf algebras. Now $T = P^{1/4}$ does the job, cf. p. 183 in [6].

**Theorem 2.6.** Let $H$ be a compact quantum group with respect to $*$ and also with respect to #. Then there exists a Hopf algebra automorphism $T$ such such that $*T = T^*$. 

**Proof.** Taking into account that $H_\rho$ is $^*$- and $#-$stable, the proof in [6], p. 184, (see also [4]) can be adapted here.

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Nicolas Andruskiewitsch: Compact involutions . . .

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