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HARMONIC ANALYSIS ON SEMISIMPLE HOPF ALGEBRAS

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Abstract. Let H be a semisimple Hopf algebra. The relationship is studied between the character algebra of H and that of a Hopf subalgebra. Hecke algebras are discussed, as well as their links with quantum spaces of double cosets. An explicit expression for spherical functions is given. Also, Gelfand pairs are studied, and a description of Fourier analysis on symmetric spaces via spherical functions is presented. It is shown that the pair $(D(H), H)$ is a Gelfand pair if and only if H is almost cocommutative; here $D(H)$ is the Drinfeld double of H.

§0. Introduction

A Hopf algebra is simultaneously an algebra and a coalgebra, both structures related by the multiplicativity of the coproduct and the existence of an antipode. Whereas the study of algebras or coalgebras is "Abelian" in nature, the Hopf algebras are highly non-Abelian, and the study of them has a flavor closely resembling the theory of groups; even more, various branches of the Hopf algebra theory have analogies in group theory. In particular, the theory of semisimple Hopf algebras has a striking similarity with the theory of finite groups, at least in the esthetic perspective.

In fact, this similarity was clearly mentioned many years ago by G. I. Kac and was supported by his fundamental results [Ka1, Ka2, KaP]. S. Woronowicz showed in [W] that the class of finite quantum groups precisely coincides with the class of finite dimensional Kac algebras. It should be emphasized that the existence of a compact involution in a semisimple complex Hopf algebra (that is, a structure of finite quantum group, or Kac algebra) is still an open problem; though, the uniqueness of a compact involution up to conjugation was shown in [An].

In this paper, we explore the relationships between the algebra and the coalgebra structures of semisimple Hopf algebras that make these objects so rigid. Our aim is to develop systematically the tools of harmonic analysis in this context. Harmonic analysis on quantum groups is a field of active research. The foundations of harmonic

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analysis on compact quantum groups were established by Woronowicz in [W]; see also [PW]. We also mention the paper [VS].

One of the most appealing applications of quantum groups is a new direction in the study of q-special functions, which has been explored actively after Koornwinder's paper [K]; see also [V].

In the present paper, our objective is different: we aim to develop some techniques to "induce" results from Hopf subalgebras of semisimple Hopf algebras. As an ultimate purpose, we mean applications to classification results; however, these will be the matter of further investigations.

The paper is organized as follows. In §1, we recall the definition of the convolution product in a Hopf algebra, in the setting of co-Frobenius Hopf algebras, and state some elementary properties. Some of them go back to [PW]. §2 is devoted to further properties in the case of semisimple Hopf algebras and their character algebras. In §3, we begin the study of the relationship between the character algebras of a semisimple Hopf algebra and a Hopf subalgebra. We define a map $f \mapsto \widetilde{f}$ that generalizes a well-known map in finite group theory (see [CR, Chapter 10]). In §4, we discuss Hecke algebras and their links with "quantum spaces of double cosets"; the latter are known in quantum harmonic analysis (see, e.g., [V]). In §5, we discuss spherical functions; they arise naturally in the decomposition of an induced representation. There and in the subsequent sections we invoke ideas from harmonic analysis on Lie groups [GV] (see also [Ti]) and finite groups (see [T, Te]). The topic of §6 is Gelfand pairs (see [K]) in the context of semisimple Hopf algebras; the corresponding quantum quotients are named symmetric spaces. We discuss several equivalent definitions, including some from [V], and give some criteria, inspired by similar criteria of Gelfand and Selberg. Fourier analysis on symmetric spaces via spherical functions is described in §7. The last sections are devoted to some examples: to biproducts (§8) and to Drinfeld doubles (§9).

Conventions. Except for §1, we shall work over an algebraically closed field k of characteristic zero; in §1 k is an arbitrary field. For the theory of Hopf algebras, we refer the reader to [Sw, Mo, Sch]. Given a Hopf algebra A, we shall regard A as an A^*-bimodule via the transposes of right and left multiplication, that is

$$\langle \beta, \alpha \rightharpoonup a \rangle = \langle \beta\alpha, a \rangle, \quad \langle \beta, a \leftharpoonup \alpha \rangle = \langle \alpha\beta, a \rangle, \qquad a \in A, \quad \alpha, \beta \in A^*;$$

in a similar way, A^* will be viewed as an A-bimodule. The left and right A^*-actions on A can also be written as

$$\alpha \rightharpoonup a = a_1 \langle \alpha, a_2 \rangle, \quad a \leftharpoonup \alpha = \langle \alpha, a_1 \rangle a_2; \qquad a \in A, \quad \alpha \in A^*.$$

We shall also consider the actions of A^* on A given by $a \leftharpoondown \alpha = \mathcal{S}\alpha \rightharpoonup a$, $\alpha \rightharpoondown a = a \leftharpoonup \mathcal{S}\alpha$; the same notation will be used for the corresponding actions of A on A^*. Unless explicitly stated, A (respectively, A^*) is viewed as a (left or right) module over itself via the (left or right) multiplication.

§1. Convolution product on co-Frobenius Hopf algebras

Let A be a Hopf algebra over k. A right integral $\int \in A^*$ is an element such that $\int \alpha = \langle \alpha, 1\rangle \int$, $\alpha \in A^*$. In this section we assume that A is *co-Frobenius*, i.e., that there exists a nonzero right integral $\int \in A^*$. So, the following is true:

(1.1) A^* also has a nonzero left integral (see [D]). The space of left (respectively, right) integrals in A^* is one-dimensional (Sullivan's theorem; see [A, 3.3.10], and also [DNT, Ra] for shorter proofs).
(1.2) The antipode of A is bijective (see [R1]).
(1.3) The morphism of right A^*-modules $\mathbb{F}_A\colon (A, \leftharpoonup) \to A^*$ defined by

$$\mathbb{F}_A(a) = a \rightharpoonup \textstyle\int, \quad a \in A, \tag{1.4}$$

is injective (see [D]).

The morphism $\mathbb{F}_A$ occurring in (1.4) is called the *Fourier transform* of A (associated with $\int$). Conversely, if a Hopf algebra A admits an injective morphism of left A^*-modules, $A \to A^*$, then it is co-Frobenius (see [D]). As examples of co-Frobenius Hopf algebras we mention the finite-dimensional Hopf algebras and the cosemisimple Hopf algebras (including compact quantum groups); more examples can be found in [BDG, BDGN].

We shall consider the map $\mathbb{G} = \mathbb{G}_A\colon (A, \rightharpoonup) \to A^*$ given by

$$\mathbb{G}(a) = \Lambda \leftharpoonup a, \tag{1.5}$$

where now $0 \neq \Lambda \in A^*$ is a left integral. It is a monomorphism of left A^*-modules.

The following definition appeared in [PW] in the context of compact quantum groups.

Definition 1.6. Let A be a co-Frobenius Hopf algebra, and let $\Lambda \in A^*$ be a nonzero left integral. The *convolution product* $*\colon A \otimes A \to A$ is defined by the relation

$$a * b = \langle \Lambda, a\mathcal{S}(b_1)\rangle b_2 = \langle \Lambda, a_2\mathcal{S}(b)\rangle a_1, \quad a, b \in A.$$

Here the second identity is [A, Lemma 3.3.7].

Proposition 1.7. *Convolution determines an associative product in* A, *and* $\mathbb{G}\colon (A, *) \to A^*$ *is an injective algebra homomorphism. This algebra structure possesses the following properties*:

i) *The algebra* A *has a unit element if and only if it is finite dimensional*; *in this case*, 1_* *is a right integral in* A *such that* $\langle \Lambda, 1_*\rangle = 1$.

ii) A *is an augmented algebra, with the augmentation map* $\Lambda\colon A \to k$. *The corresponding idempotent is* $1 \in A$.

iii) *If* $F\colon A \to A$ *is a Hopf algebra automorphism, then* F *preserves the convolution product.*

iv) *The antipode* $\mathcal{S}\colon A \to A$ *is an anti-algebra automorphism of* A *if and only if* $\langle \Lambda, ab\rangle = \langle \Lambda, b\mathcal{S}^2(a)\rangle$ *for all* $a, b \in A$.

Proof. We have $a * b = \mathbb{G}(a) \rightharpoondown b$ for all $a, b \in A$. Applying $\mathbb{G}$ to both sides, we see that $\mathbb{G}(a * b) = \mathbb{G}(a)\mathbb{G}(b)$, because $\mathbb{G}$ is a morphism of left A^*-modules. Hence,

$$\mathbb{G}((a * b) * c) = \mathbb{G}(a)\mathbb{G}(b)\mathbb{G}(c) = \mathbb{G}(a * (b * c))$$

for all $a, b, c \in A$. Thus, $*$ is associative since $\mathbb{G}$ is injective.

Now we prove i). An element $u \in A$ is a unit for $*$ if and only if $a * u = a = u * a$ for all $a \in A$; but this implies (by the uniqueness of the counit) that $\varepsilon(a) = \langle \Lambda, ua\rangle$, or $\varepsilon = \Lambda \leftharpoonup u$. Then, for all $a \in A$,

$$\Lambda \leftharpoonup ua = (\Lambda \leftharpoonup u) \leftharpoonup a = \varepsilon \leftharpoonup a = \varepsilon(a)\varepsilon = \Lambda \leftharpoonup \varepsilon(a)u.$$

Recalling (1.3), we see that u is a right integral in A and that $\langle \Lambda, u\rangle = 1$. In turn, this implies that A is finite-dimensional. Since the above argument can be reversed, i) follows. We leave ii) and iii) to the reader. For iv), we compute

$$\mathcal{S}(b) * \mathcal{S}(a) = \langle \Lambda, (\mathcal{S}b)_2\mathcal{S}^2(a)\rangle(\mathcal{S}b)_1 = \langle \Lambda, \mathcal{S}(b_1)\mathcal{S}^2(a)\rangle\mathcal{S}(b_2).$$

On the other hand, we have

$$\mathcal{S}(a * b) = \langle \Lambda, a\mathcal{S}(b_1)\rangle\mathcal{S}(b_2),$$

and the claim follows, because the antipode $\mathcal{S}$ is bijective. •

By [AN, (2.14)], the condition formulated in iv) is always satisfied when A is cosemisimple and $\mathcal{S}^4 = \mathrm{id}$.

In the rest of this paper, we shall consider the convolution product in the dual Hopf algebra $A = H^*$ of a finite-dimensional Hopf algebra H. In this case, $\mathbb{G} = \mathbb{G}_A\colon (H^*, *) \to H$ is an algebra isomorphism, and its inverse is the Fourier transform $\mathbb{F} = \mathbb{F}_H$. Also, occasionally, we shall use $\mathcal{F} := \mathbb{G}_H = \mathbb{F}\circ\mathcal{S}^{-1}\colon H \to H^*$ and its inverse $\mathcal{G} := \mathbb{F}_A$.

§2. Semisimple Hopf algebras

From now on, H will denote a semisimple (hence, finite-dimensional) Hopf algebra over k. Most of the results below remain valid for the semisimple cosemisimple Hopf algebras over algebraically closed fields of positive characteristic. We avoid this more general situation, because, in principle, the results of [EG] make it possible to reduce everything to the case of zero characteristic.

We fix integrals $\int \in H^*$ and $\Lambda \in H$ such that $\langle \varepsilon, \Lambda \rangle = 1$ and $\langle \int, \Lambda \rangle = 1$; then $\langle \int, 1 \rangle = \dim H$. When special emphasis is needed, we shall use the notation Λ_H for the normalized integral in H. In particular, $\mathbb{F}$ and $*$ will denote, respectively, the Fourier transform and the convolution product associated with these integrals.

Let $R(H) \subseteq H^*$ be the character algebra of H, and let $\widehat{H} := \{\chi_1, \dots, \chi_l\}$ be the set of irreducible characters of H; we have $R(H) = \bigoplus_{i=1}^{l} k\chi_i$. We recall that, for $\chi \in R(H)$, the *multiplicity* of χ_i in χ is defined as the element $m(\chi_i, \chi) \in k$ in the identity $\chi = \sum_i m(\chi_i, \chi)\chi_i$. Also by definition, for $\psi \in R(H)$ the multiplicity of χ in ψ is $m(\psi, \chi) := \sum_i m(\chi_i, \psi)m(\chi_i, \chi)$. For $\chi = \sum_i m(\chi_i, \chi)\chi_i \in R(H)$, the *degree* of χ is $\deg \chi := \sum_i m(\chi_i, \chi)\dim(V_i)$, where V_i is an irreducible module affording the character χ_i, $i = 1, \dots, l$. In particular, if χ is the character afforded by an H-module V, then $m(\chi_i, \chi) = \dim \operatorname{Hom}_H(V_i, V)$ and $\deg \chi = \dim V$. For $\chi \in R(H)$, we shall use the notation $\chi^* := \mathcal{S}(\chi)$, so that if χ is the character of an H-module V, then χ^* is the character of the dual module V^*.

It is known that the integral $\Lambda \in H$ can be written as $\Lambda = \dfrac{1}{\dim H} \displaystyle\sum_{\mu \in \widehat{H^*}} \deg \mu \mu$.

Consequently, the convolution product in H^* takes the form

$$\alpha * \beta(h) = \frac{1}{\dim H} \sum_{\mu \in \widehat{H^*}} \deg \mu \sum_{i,j=1}^{\deg \mu} \langle \alpha, \mu_{ij} \rangle \langle \beta, \mathcal{S}(\mu_{ij})h \rangle, \quad h \in H. \tag{2.1}$$

Here μ_{ij} denotes the ijth matrix entry of the corepresentation associated with μ, relative to a fixed basis of an irreducible comodule affording the character μ. This generalizes the well-known formula for the convolution product in the group algebra of a finite group.

Remark 2.2. The convolution product $*\colon H^* \otimes H^* \to H^*$ splits the comultiplication map $\Delta\colon H^* \to H^* \otimes H^*$. We have the following identities:

$$(\alpha * \beta)_1 \otimes (\alpha * \beta)_2 = \alpha * \beta_1 \otimes \beta_2,$$
$$\alpha_1 * \alpha_2 = \alpha$$

for all $\alpha, \beta \in H^*$. This is an immediate consequence of [Sch3, (1.11)].

It is known that the Fourier transform gives a linear isomorphism $Z(H) \xrightarrow{\simeq} R(H)$. Thus, $Z(H^*, *) = R(H)$. Moreover, the following statement is true.

Lemma 2.3. *The elements* $\deg \chi \chi$, $\chi \in \widehat{H}$, *form a complete set of central orthogonal idempotents in* $(H^*, *)$.

Proof. For $\chi \in \widehat{H}$, let $E_\chi \in H$ be the central primitive idempotent such that $\langle \chi, E_\chi \rangle = \deg \chi$. By [AN, (2.11)], we have $\dfrac{1}{\deg \chi} E_\chi = \Lambda \leftharpoonup \mathcal{S}(\chi)$. Now, since Λ is cocommutative, $\mathbb{G}(\chi^*) = \Lambda \leftharpoonup \mathcal{S}(\chi) = \mathcal{S}(\chi) \rightharpoonup \Lambda$, whence

$$\mathbb{G}(\chi^*) = \mathcal{S}(\chi) \rightharpoonup \Lambda = \frac{1}{\deg \chi} E_\chi. \tag{2.4}$$

Therefore, for any $\chi \in \widehat{H}$ we can write $\chi = \dfrac{1}{\deg \chi}\mathbb{F}(E_{\chi^*})$. Thus, for all $\chi' \in \widehat{H}$ we have

$$\begin{aligned}\chi * \chi' &= \frac{1}{\deg \chi \deg \chi'}\mathbb{F}(E_{\chi^*}) * \mathbb{F}(E_{\chi'^*}) = \frac{1}{\deg \chi \deg \chi'}\mathbb{F}(E_{\chi^*} E_{\chi'^*}) \\ &= \frac{1}{\deg \chi \deg \chi'}\delta_{\chi,\chi'}\mathbb{F}(E_{\chi^*}) = \frac{1}{\deg \chi}\delta_{\chi,\chi'}\chi. \quad \bullet\end{aligned}$$

Let $(\ |\ ) : H^* \times H^* \to k$ be the bilinear form defined by

$$(\alpha|\beta) := \langle \alpha\mathcal{S}(\beta), \Lambda\rangle, \quad \alpha, \beta \in H^*. \tag{2.5}$$

The basic properties of the form $(\ |\ )$ are as follows.

Proposition 2.6. i) $(\ |\ )$ *defines a nondegenerate, symmetric bilinear form on* H^*.

ii) *For all* $\alpha, \beta \in H^*$,

$$\alpha * \beta = (\alpha_2|\beta)\,\alpha_1 = (\alpha|\beta_1)\,\beta_2.$$

iii) *Associativity with respect to the convolution product. For all* $\alpha, \beta, \gamma \in H^*$,

$$(\alpha * \beta|\gamma) = (\alpha|\beta * \gamma)\,.$$

iv) *For all* $\alpha, \beta, \gamma \in H^*$, *we have*

$$(\alpha\beta|\gamma) = (\alpha|\gamma\mathcal{S}(\beta)) = (\beta|\mathcal{S}(\alpha)\gamma)$$

and

$$(\mathcal{S}(\alpha)|\mathcal{S}(\beta)) = (\alpha|\beta)\,.$$

v) *For all* $\alpha, \beta \in H^*$, $h \in H$,

$$(\alpha|h \rightharpoonup \beta) = (\alpha \leftharpoonup h|\beta) \quad \textit{and} \quad (h \rightharpoonup \alpha|\beta) = (\alpha|\beta \leftharpoonup h)\,.$$

vi) *Orthogonality relations for characters. Let* $\chi_1, \dots, \chi_l$ *be the set of irreducible characters of* H. *Then* $(\chi_i|\chi_j) = \delta_{ij}$, $1 \le i, j \le l$. *In particular, the restriction of* $(\ |\ )$ *to* $R(H)$ *is also nondegenerate, and* $\{\chi_1, \dots, \chi_l\}$ *is an orthonormal basis of* $R(H)$.

Except for ii) and iii), the above items are known.

Proof. A theorem of Larson and Sweedler (see [LS]) shows that $(\ |\ )$ is nondegenerate. Symmetry follows easily from the identities $\mathcal{S}^2 = \mathrm{id}$ and $\mathcal{S}(\Lambda) = \Lambda$. This proves i). Part ii) is an immediate consequence of the definition of the convolution product. By ii), $(\alpha|\beta) = \varepsilon(\alpha * \beta)$ for all $\alpha, \beta \in H^*$, whence iii) follows. Part iv) is straightforward and is left to the reader. Part vi) can be found, e.g., in [Sch, Theorem 4.7]. To prove part v), we use the identity

$$(\alpha|\beta) = \langle \mathcal{S}(\beta), \mathbb{G}(\alpha)\rangle = \langle \beta, \mathcal{G}(\alpha)\rangle, \quad \alpha, \beta \in H^*. \tag{2.7}$$

Indeed, since $\mathbb{G}$ is a map of left H-modules, we have

$$\begin{aligned}(\alpha \leftharpoonup h|\beta) &= (\mathcal{S}(h) \rightharpoondown \alpha|\beta) = \langle \mathcal{S}(\beta), \mathbb{G}(S(h) \rightharpoondown \alpha)\rangle \\ &= \langle \mathcal{S}(\beta), S(h)\mathbb{G}(\alpha)\rangle = \langle \mathcal{S}(\beta) \leftharpoonup \mathcal{S}(h), \mathbb{G}(\alpha)\rangle \\ &= \langle \mathcal{S}(h \rightharpoonup \beta), \mathbb{G}(\alpha)\rangle = (\alpha|h \rightharpoonup \beta)\end{aligned}$$

for all $h \in H$, $\alpha, \beta \in H^*$. By symmetry, the second identity also follows. This completes the proof of the proposition. •

Lemma 2.8. *Let* $\chi, \chi' \in R(H)$. *Then* $(\chi|\chi') = m(\chi, \chi')$. *In particular, if* χ *and* χ' *are, respectively, the characters of the* H*-modules* V_χ *and* $V_{\chi'}$, *then*

$$(\chi|\chi') = \dim \operatorname{Hom}_H(V_\chi, V_{\chi'}).$$

Proof. Both bilinear forms coincide on the basis $\{\chi_1, \dots, \chi_l\}$ of $R(H)$. •

§3. Character algebras and induced modules

In this section we discuss the relationship between the character algebras of a semisimple Hopf algebra and a Hopf subalgebra. First, we state several (mostly, well-known) versions of the Frobenius reciprocity.

Let $A \hookrightarrow B$ be an inclusion of algebras, not necessarily finite-dimensional. Let W be an A-module and V a B-module. We denote by $\operatorname{Ind}_A^B W$ (respectively, $\operatorname{Res}_A^B V$) the induced module $\operatorname{Ind}_A^B W := B \otimes_A W$ (respectively, the A-module obtained from V by restriction of scalars).

Lemma 3.1. (Frobenius reciprocity I). *There is a natural isomorphism*

$$\tau\colon \operatorname{Hom}_A(W, \operatorname{Res}_A^B V) \to \operatorname{Hom}_B(\operatorname{Ind}_A^B W, V), \quad (\tau f)(b \otimes w) = bf(w),$$

where $f \in \operatorname{Hom}_A(W, \operatorname{Res}_A^B V)$, $b \in B$, $w \in W$. *In other words, the induction functor is left adjoint to the restriction functor.*

Proof. By [CR, (2.19)], there is a natural isomorphism $\tilde{\tau}\colon \operatorname{Hom}_A(W, \operatorname{Hom}_B(B, V)) \simeq \operatorname{Hom}_B(\operatorname{Ind}_A^B W, V)$, given by $(\tilde{\tau} f)(b\otimes w) = f(w)(b)$ for all $f \in \operatorname{Hom}_A(W, \operatorname{Hom}_B(B, V))$, $b \in B$, $w \in W$. On the other hand, it is not difficult to check that $\operatorname{Hom}_B(B, V) \to \operatorname{Res}_A^B V$, $f \mapsto f(1)$, is a natural isomorphism of A-modules. •

Now, we treat the existence of a right adjoint functor to the restriction. Let $A \hookrightarrow B$ be an inclusion of algebras such that ${}_AB$ is finitely generated projective, and let $\beta\colon A \to A$ be an algebra automorphism. For a left A-module W, we denote by ${}_\beta W$ the left A-module with the same underlying vector space and with the action $a._\beta w := \beta(a).w$ for all $w \in W$, $a \in A$; similar notation will be used for right A-modules.

We recall that $A \hookrightarrow B$ is called a right *β-Frobenius extension* if there exists an isomorphism

$$ {}_BB_A \xrightarrow{F} {}_B\mathrm{Hom}_A({}_AB, {}_AA)_\beta. $$

If this is the case, let $f = F(1)$; then f is a morphism of (A,A)-bimodules $f\colon {}_AB_A \to {}_AA_\beta$, and there exist $l_i, r_i \in B$, $i = 1, \dots, n$, such that

$$ b = \sum_{i=1}^{n} f(bl_i)r_i = \sum_{i=1}^{n} l_i\beta^{-1} \circ f(r_ib) \tag{3.2} $$

for all $b \in B$ (see [FMS, 1.3]). The collections $l_i, r_i \in B$, $i = 1, \dots, n$, are called dual bases of B. Moreover, we have the following separability identity in $B \otimes_A {}_\beta B = B_{\beta^{-1}} \otimes_A B$:

$$ \sum_{i=1}^{n} l_i \otimes r_ib = \sum_{i=1}^{n} bl_i \otimes r_i \tag{3.3} $$

for all $b \in B$ (see [FMS, 1.4 (c)]).

Lemma 3.4. *The following statements are equivalent.*

i) *$A \hookrightarrow B$ is a right β-Frobenius extension.*

ii) *For every left A-module W and left B-module V there is a natural isomorphism*

$$ \zeta\colon \mathrm{Hom}_A(\mathrm{Res}^B_A V, W) \to \mathrm{Hom}_B(V, \mathrm{Ind}^B_A {}_\beta W). $$

In other words, the β-twisted induction functor is right adjoint to the restriction functor.

Proof. If ii) is true, then taking $V = B$, $W = A$ yields the desired isomorphism $F\colon {}_BB_A \to {}_B\mathrm{Hom}_A({}_AB, {}_AA)_\beta$. Conversely, assuming i), we can define ζ by

$$ \zeta(u)(v) = \sum_{i=1}^{n} l_i \otimes u(r_i.v) \in B \otimes_A {}_\beta W, \quad u \in \mathrm{Hom}_A(\mathrm{Res}^B_A V, W), \ \ v \in V. $$

Then (3.3) shows that ζ is well defined. Moreover, by using (3.2) and (3.3), it can be shown that $\sigma\colon \mathrm{Hom}_B(V, \mathrm{Ind}^B_A {}_\beta W) \to \mathrm{Hom}_A(\mathrm{Res}^B_A V, W)$,

$$ \sigma(t)(v) = (f \otimes \mathrm{id})t(v) $$

for all $t \in \operatorname{Hom}_B(V, \operatorname{Ind}_A^B {}_\beta W)$, $v \in V$, is a well-defined inverse of ζ. •

In particular, we conclude that the induction functor is right adjoint to the restriction functor if and only if $A \hookrightarrow B$ is Frobenius.

A major example of a β-Frobenius extension is provided by any inclusion of finite-dimensional Hopf algebras $A \hookrightarrow B$. This is [Sch2, 3.6 II], [FMS, Theorem 1.7]. Indeed, once again we recall that, by [NZ], B is free over A. Let $\int_B \in B^*$, respectively, $\Lambda_B \in B$, $\Lambda_A \in A$, be nonzero right integrals such that $\int_B \leftharpoonup \Lambda_B = \varepsilon$, i.e., $\langle \int_B, \Lambda_B \rangle = 1$. By [NZ], there exists $\widetilde{\Lambda} \in B$ such that

$$\Lambda_B = \widetilde{\Lambda}\Lambda_A. \tag{3.5}$$

Let $\alpha_B \in B^*$ be the right modular element, i.e., $b\Lambda_B = \langle \alpha_B, b \rangle \Lambda_B$ for all $b \in B$, and let $\eta_B \colon B \to B$ be the Nakayama automorphism $\eta_B(b) = \mathcal{S}^{-2}(b \leftharpoonup \alpha_B) = (\mathcal{S}^{-2}b) \leftharpoonup \alpha_B$, $b \in B$. Consider the algebra automorphism

$$\beta = {\eta_A}^{-1}\eta_B \colon A \to A. \tag{3.6}$$

Then the extension $A \hookrightarrow B$ is β-Frobenius with the Frobenius homomorphism $f \colon B \to A$, $f(b) = \langle \int_B, b_1 \mathcal{S}^{-1}(\Lambda_A) \rangle b_2$, $b \in B$, and with the dual basis $(\mathcal{S}^{-1}(\Lambda_2), \Lambda_1)$, where $\Lambda = \mathcal{S}(\widetilde{\Lambda}) \leftharpoonup {\alpha_B}^{-1} = {\eta_B}^{-1}(\mathcal{S}^{-1}(\widetilde{\Lambda}))$.

Lemma 3.7 (Frobenius reciprocity II). *Let $A \hookrightarrow B$ be an inclusion of finite-dimensional Hopf algebras. Then the induction functor is right adjoint to the restriction functor if and only if the relative Nakayama automorphism* (3.6) *is the identity. In turn, this is equivalent to $\alpha_A = (\alpha_B)|_A$.*

Proof. This follows from Lemma 3.4, with reference to [FMS, 1.8]. •

We recall that H denotes a semisimple Hopf algebra. Let $K \subseteq H$ be a Hopf subalgebra. The restriction and induction functors give rise, respectively, to linear maps $R(H) \to R(K)$, $\chi \mapsto \chi_K$, and $R(K) \to R(H)$, $\psi \mapsto \psi^H$.

On the other hand, if V is an H-module and W is an irreducible K-module with character χ, we denote by $V[\chi]$ the isotypic component of type W in $\operatorname{Res} V$. In particular, we denote by $V^K := V[\varepsilon_K]$ the space of K-invariants in V.

Lemma 3.8 (Frobenius reciprocity for characters). *Let $\chi \in R(H)$ and $\psi \in R(K)$; then $(\chi_K|\psi)_K = \left(\chi|\psi^H\right)_H$.* •

Let $\psi \in R(K)$ be the character of a representation of K, and let $e \in K$ be an idempotent such that Ke affords the character ψ. From [NZ] it follows that He affords the induced character ψ^H. Indeed, since H is free over K, $He = HKe \simeq H \otimes_K Ke$. In particular, $\deg \psi^H = (\deg \psi)[H{:}K]$. Here $[H{:}K] := \frac{\dim H}{\dim K}$, which is an integer by [NZ].

Now, we give a first application of the Frobenius reciprocity (Lemma 3.8).

Corollary 3.9. *Suppose K is a commutative Hopf subalgebra of H. Then for all $\chi \in \widehat{H}$ we have* $\deg \chi \leq [H{:}K]$.

Proof. Observe that $m(\chi, \psi^H) = m(\chi_K, \psi) > 0$ for some $\psi \in \widehat{K}$. Since K is commutative, we have $\deg \psi = 1$, whence $\deg \chi \leq \deg \psi^H = [H{:}K]$. •

In what follows we aim to give an explicit formula for the character of an induced representation (see Proposition 3.12). We begin with a technical definition, which will also be of use in the study of spherical functions (see §5).

For $f \in K^*$, consider the k-linear map $H^* \to k$, $\alpha \mapsto (\alpha_K|f)_K$. Since the form $(\ |\ )_H$ is nondegenerate, there exists a unique $\widetilde{f} \in H^*$ such that

$$(\alpha_K|f)_K = \left(\alpha|\widetilde{f}\right)_H \tag{3.10}$$

for all $\alpha \in H^*$. This yields a well-defined linear map $\widetilde{(\ )}\colon K^* \to H^*$, $f \mapsto \widetilde{f}$. In particular, if $\psi \in R(K)$, then, by (3.10), for all $\chi \in R(H)$ we have

$$\left(\chi|\widetilde{\psi}\right)_H = (\chi_K|\psi)_K = \left(\chi|\psi^H\right)_H.$$

Lemma 3.11. *Let $f \in K^*$. Then* $\widetilde{f} = \langle \textstyle\int_1 \mathcal{S}(f), \Lambda_K \rangle \int_2$.

Proof. Since $\int = \int_H$ is the identity element for the convolution product in H^*, we have

$$\widetilde{f} = \textstyle\int_H *_H \widetilde{f} = \left(\int_1|\widetilde{f}\right)_H \int_2 = (\int_1|f)_K \int_2 = \langle \int_1 \mathcal{S}(f), \Lambda_K\rangle \int_2. \quad \bullet$$

Example. Suppose $H = kG$, $K = kF$, where $F \subseteq G$ are finite groups. Then $\int_H = |G|\delta_1$, and $\Lambda_K = \frac{1}{|F|}\sum_{h\in F} h$. If $f \in k^F$, then $\widetilde{f} = |G| \sum_{g\in G} \langle \delta_g \mathcal{S}(f), \Lambda_K\rangle \delta_{g^{-1}}$ by Lemma 3.11. Hence, for all $x \in G$ we have

$$\begin{aligned} \widetilde{f}(x) &= |G|\langle \delta_{x^{-1}} \mathcal{S}(f), \Lambda_K\rangle = [G{:}F] \sum_{h\in F} \langle f, h\rangle \langle \delta_x, h\rangle \\ &= \begin{cases} [G{:}F] f(x), & x \in F, \\ 0, & x \notin F. \end{cases} \end{aligned}$$

The next proposition generalizes a famous formula due to Frobenius (cf. [CR, (10.3)]).

Proposition 3.12. *If* $\psi \in R(K)$, *then* $\psi^H = \operatorname{ad}\Lambda_H(\widetilde{\psi}) = \mathcal{S}(\Lambda_{H2}) \rightharpoonup \widetilde{\psi} \leftharpoonup \Lambda_{H1}$.

Proof. We put $\Lambda = \Lambda_H$. First, observe that if $\chi \in R(H)$, then $\Lambda_1 \rightharpoonup \chi \leftharpoonup \mathcal{S}(\Lambda_2) = \chi$, because $\chi \in H^*$ is a cocommutative element, $\varepsilon(\Lambda) = 1$, and

$$\langle \Lambda_1 \rightharpoonup \chi \leftharpoonup \mathcal{S}(\Lambda_2), x\rangle = \langle \chi, \mathcal{S}(\Lambda_2)x\Lambda_1\rangle = \langle \chi, \Lambda_1\mathcal{S}(\Lambda_2)x\rangle = \varepsilon(\Lambda)\langle \chi, x\rangle = \langle \chi, x\rangle$$

for all $x \in H$. So, let $\chi \in R(H)$. By Proposition 2.6v), we have

$$\Big(\chi|\operatorname{ad}\Lambda(\widetilde{\psi})\Big) = \Big(\chi|\mathcal{S}(\Lambda_2) \rightharpoonup \widetilde{\psi} \leftharpoonup \Lambda_1\Big) = \Big(\Lambda_1 \rightharpoonup \chi \leftharpoonup \mathcal{S}(\Lambda_2)|\widetilde{\psi}\Big) = \Big(\chi|\widetilde{\psi}\Big) = \left(\chi|\psi^H\right).$$

The proposition follows, because $\operatorname{ad}\Lambda_H(H^*) = R(H)$ and $(\ |\ )$ is nondegenerate on $R(H)$ by Proposition 2.6vi). •

Proposition 3.13. *Consider the convolution products in* H^* *and* K^*. *Then*:
i) $\widetilde{(\ )}\colon K^* \to H^*$, $f \mapsto \widetilde{f}$, *is an algebra map.*
ii) *The composition* $\operatorname{Res}\circ\widetilde{(\ )}\colon K^* \to K^*$ *is scalar multiplication by* $[H{:}K]$.

Proof. i) First, we claim that $\alpha_K * f = (\alpha * \widetilde{f})_K$ for all $f \in K^*$, $\alpha \in H^*$. Indeed, the definition of $\widetilde{f}$ implies that

$$\alpha_K * f = ((\alpha_2)_K|f)_K\,(\alpha_1)_K = \Big(\alpha_2|\widetilde{f}\Big)_H(\alpha_1)_K = (\alpha * \widetilde{f})_K.$$

Now, using Proposition 2.6iii), we deduce that, for all $f, g \in K^*$, $\alpha \in H^*$,

$$\begin{aligned}(\alpha_K|f * g)_K &= (\alpha_K * f|g)_K = \Big((\alpha * \widetilde{f})_K|g\Big)_K\\ &= \Big(\alpha * \widetilde{f}|\widetilde{g}\Big)_H = \Big(\alpha|\widetilde{f} * \widetilde{g}\Big)_H.\end{aligned}$$

Consequently, by the definition of $\widetilde{(\ )}$, $\widetilde{f * g} = \widetilde{f} * \widetilde{g}$. Also, from Lemma 3.11 it follows that $\widetilde{\int_K} = \int_H$.

ii) Let $f \in K^*$, $x \in K$. Then

$$\begin{aligned}\langle(\widetilde{f})_K, x\rangle &= \langle {\textstyle\int}_1\mathcal{S}(f), \Lambda_K\rangle\langle {\textstyle\int}_2, x\rangle = \langle {\textstyle\int}_1, (\Lambda_K)_1\rangle\langle f, \mathcal{S}((\Lambda_K)_2)\rangle\langle {\textstyle\int}_2, x\rangle\\ &= \langle {\textstyle\int}, (\Lambda_K)_1x\rangle\langle f, \mathcal{S}((\Lambda_K)_2)\rangle = \langle {\textstyle\int}, (\Lambda_K)_1\rangle\langle f, x\mathcal{S}((\Lambda_K)_2)\rangle\\ &= \langle {\textstyle\int}, \Lambda_K\rangle\langle f, x\rangle = [H{:}K]\langle f, x\rangle;\end{aligned}$$

here we have used the fact that, since Λ_K is an integral in K, we have $(\Lambda_K)_1x \otimes \mathcal{S}((\Lambda_K)_2) = (\Lambda_K)_1 \otimes x\mathcal{S}((\Lambda_K)_2)$ for all $x \in K$. Also, since Λ_K is an idempotent in H, we see that $\langle \int, \Lambda_K\rangle = \operatorname{Tr}_H(\Lambda_K) = \dim H\Lambda_K = [H{:}K]$. •

Remark 3.14. Observe that the relation $\widetilde{\int_K} = \int_H$ implies that $\int_H = \sum_{\psi \in \widehat{K}} \deg\psi\widetilde{\psi}$.

§4. Hecke algebras

Let $K \hookrightarrow H$ be an inclusion of Hopf algebras, with H semisimple.

Definition 4.1. Given an idempotent $e \in K$, the *Hecke algebra* associated with e is the subalgebra $\mathcal{H}(H, K, e) := eHe$ of H. The Hecke algebra $\mathcal{H}(H, K, \Lambda_K)$ will be denoted by $\mathcal{H}(H, K)$.

From [CR, (5.13), (5.18)] it follows that the Hecke algebra $\mathcal{H}(H, K, e)$ is semisimple.

Remark. The Hecke algebra $\mathcal{H}(H, K, e)$ depends only on the left module Ke, but not on the choice of an idempotent $e \in K$. Indeed, by [CR, (3.19)], the Hecke algebra $\mathcal{H}(H, K, e)$ is isomorphic to $(\operatorname{End}_H He)^{\mathrm{op}}$. If $f \in K$ is another idempotent such that $Ke \simeq Kf$, then $He \simeq H \otimes_K Ke \simeq H \otimes_K Kf \simeq Hf$. Therefore, $\mathcal{H}(H, K, e)$ and $\mathcal{H}(H, K, f)$ are isomorphic.

Lemma 4.2. *Let $e \in K$ be an idempotent such that the left K-module Ke affords the character $\psi \in R(K)$. Then for all $\chi \in \widehat{H}$, $h \in Z(H)$ we have*

$$\langle \chi, eh \rangle = \frac{m(\chi, \psi^H)}{\deg \chi} \langle \chi, h \rangle.$$

In particular,

$$m(\chi, \psi^H) = \langle \chi, e \rangle. \tag{4.3}$$

Compare (4.3) with [CR, Proposition (11.21)].

Proof. Let V denote the H-module with character ψ^H. Then the isotypic component $V[\chi]$ is a submodule of V that affords the character $\mu = m(\chi, \psi^H)\chi$ of H. Let $E \in H$ be the central primitive idempotent such that $\langle \chi, E \rangle = \deg \chi$. Then, since $V \simeq He$, we have $V[\chi] \simeq EHe = HeE$. Hence,

$$\mu(h) = \operatorname{Tr}_H(eEh) = \sum_{\tau \in \widehat{H}} \deg \tau \langle \tau, heE \rangle = \deg \chi \langle \chi, he \rangle, \quad h \in Z(H),$$

which implies the desired identity. •

Theorem 4.4. *Let $\mathcal{H} := \mathcal{H}(H, K, e)$, and let ψ be the character afforded by the K-module Ke. Then the following statements are true.*

i) *Let χ be an irreducible H-character. Then $\chi|_{\mathcal{H}} \neq 0$ if and only if $m(\chi, \psi^H) \neq 0$.*

ii) *The map $\chi \mapsto \chi|_{\mathcal{H}}$ is a bijection between the set of irreducible characters χ of H such that $\chi|_{\mathcal{H}} \neq 0$ and the set of irreducible characters of $\mathcal{H}$.*

iii) *If ϕ is an irreducible character of $\mathcal{H}$ such that $\phi = \chi|_{\mathcal{H}}$, as in* ii), *then*

$$\deg \phi = m(\chi, \psi^H).$$

iv) *If V is an H-module affording $\chi \in \widehat{H}$, then eV is an $\mathcal{H}$-module affording $\chi|_{\mathcal{H}}$.*

Proof. The proof is precisely the same as the proof of [CR, (11.25)]; we use (4.3) in place of [CR, Prop. 11.21], as well as a straightforward generalization of [CR, Prop. 11.23]. As to part ii), we note that if ϕ is an irreducible character of $\mathcal{H}$ afforded by a minimal left ideal $\mathcal{H}u$, where $u \in \mathcal{H}$ is a primitive idempotent, then the H-character χ afforded by the (minimal) left ideal Hu satisfies $\chi|_{\mathcal{H}} = \phi$. •

Precisely as in the classical case, the moral of Theorem 4.4 is that the knowledge of the representation theory of the Hecke algebra $\mathcal{H}(H, K, e)$ amounts to the knowledge of the decomposition of the induced representation He.

Corollary 4.5. $[H{:}K] \deg \psi \geq \dim \mathcal{H}(H, K, e) = \sum_{\chi \in \operatorname{Irr} H} \langle \chi, e \rangle^2 = \left(\psi^H | \psi^H\right)_H$. •

Let $E \in K$ be a central idempotent. We define

$$L^2{}_E(K \backslash H/K) := \{\alpha \in H^* : E \rightharpoonup \alpha \leftharpoonup E = \alpha\}. \tag{4.6}$$

Let $\psi \in \widehat{K}$. If $E = E_\psi \in K$ is the central primitive idempotent such that $\langle \psi, E_\psi \rangle = \deg \psi$, then $L^2{}_E(K \backslash H/K) =: L^2{}_\psi(K \backslash H/K) \subseteq H^*$ is the isotypic component of type $\psi \otimes \psi$ in the (K, K)-bimodule structure of H^* provided by $\rightharpoonup \otimes \leftharpoonup$. In the case where $E = \Lambda_K$, we shall use the notation $L^2(K \backslash H/K)$, so that $L^2(K \backslash H/K)$ coincides with the subspace of all $\alpha \in H^*$ such that $x \rightharpoonup \alpha \leftharpoonup x' = \varepsilon(x)\varepsilon(x')\alpha$ for all $x, x' \in K$.

The coalgebra $H/(K^+H + HK^+)$ can be thought of as a quantum analog of the space of double cosets; since $K^+H + HK^+ = (L^2(K \backslash H/K))^\perp$, the dual algebra $(H/K^+H + HK^+)^*$ can be identified with $L^2(K \backslash H/K)$, a subalgebra of H^*. It is also not difficult to verify that $L^2{}_E(K \backslash H/K)$ is an $L^2(K \backslash H/K)$-subbimodule of H^*; in fact, $L^2{}_E(K \backslash H/K)$ can be viewed as the space of sections of a vector bundle on the quantum space of double cosets.

We are more interested in the study of the behavior of $L^2{}_E(K \backslash H/K)$ with respect to the convolution product; for this, now we discuss a relationship between $L^2{}_E(K \backslash H/K)$ and the Hecke algebra $\mathcal{H}(H, K, E)$, which generalizes a fact known for finite groups (cf. [CR, §11D]).

Proposition 4.7. i) *$L^2{}_E(K \backslash H/K) \subseteq H^*$ is a subalgebra with the convolution product and with identity element $\langle \int_1, E \rangle \int_2$.*

ii) *The Fourier transform of H induces an algebra isomorphism $\mathbb{F}\colon \mathcal{H}(H, K, E) \to L^2{}_{\overline{E}}(K \backslash H/K)$, where $\overline{E} = \mathcal{S}(E)$.*

Proof. We show that the Fourier transform induces a bijection $\mathcal{H}(H, K, E) \to L^2{}_{\overline{E}}(K \backslash H/K)$, which implies i) and ii) immediately. Observe that the identity element in $L^2{}_{\overline{E}}(K \backslash H/K)$ will be $\mathbb{F}(E) = \langle \int_1, E \rangle \int_2$.

Let $h, u \in H$. Then

$$\begin{aligned}\langle \overline{E} \rightharpoonup \mathbb{F}(EhE) \leftharpoonup \overline{E}, u\rangle &= \langle \mathbb{F}(EhE), \overline{E}u\overline{E}\rangle = \langle \textstyle\int, \mathcal{S}(E)\mathcal{S}(h)\mathcal{S}(E)\overline{E}u\overline{E}\rangle \\ &= \langle \textstyle\int, \mathcal{S}(h)\overline{E}u\overline{E}\rangle = \langle \textstyle\int, \overline{E}\mathcal{S}(h)\overline{E}u\rangle = \langle \mathbb{F}(EhE), u\rangle.\end{aligned}$$

This proves the inclusion $\mathbb{F}(\mathcal{H}(H,K,E)) \subseteq L^2{}_{\overline{E}}(K\backslash H/K)$. The proposition will follow if we prove that

$$\dim L^2{}_{\overline{E}}(K\backslash H/K) \leq \dim \mathcal{H}(H,K,E).$$

For this, consider the evaluation map $\theta\colon L^2{}_{\overline{E}}(K\backslash H/K) \to (\mathcal{H}(H,K,E))^*$ given by $\theta(f)(EhE) = \langle \mathcal{S}(f), EhE\rangle$. Since for $f \in L^2{}_{\overline{E}}(K\backslash H/K)$ and $h \in H$ we have

$$\theta(f)(EhE) = \langle f, \overline{E}\mathcal{S}(h)\overline{E}\rangle = \langle \overline{E} \rightharpoonup f \leftharpoonup \overline{E}, \mathcal{S}(h)\rangle = \langle \mathcal{S}(f), h\rangle,$$

the relation $\theta(f) = 0$ implies $f = 0$. Therefore, θ is injective and the inequality claimed above follows. •

In the sequel, we shall only consider the algebra structure on $L^2{}_E(K\backslash H/K)$ given by convolution.

§5. Spherical functions on semisimple Hopf algebras

Let $K \hookrightarrow H$ be an inclusion of semisimple Hopf algebras. To determine the representation theory of the Hecke algebras, we need to study analogs of spherical functions.

Let χ and ψ be characters of representations of H and K, respectively. The *spherical function* associated with the pair (χ, ψ) is the element $\phi_{\chi,\psi} \in H^*$ defined by

$$\phi_{\chi,\psi} := \chi * \widetilde{\psi} = \widetilde{\psi} * \chi \tag{5.1}$$

(the second identity is true since the characters are central with respect to convolution). Compare with [T]. In the spirit of [T, 1], now we state the basic properties of spherical functions.

Lemma 5.2. *Let* $\chi \in R(H)$, $\psi \in R(K)$. *Then*

i) $\phi_{\chi,\psi}(1) = m(\psi, \chi_K) = m(\chi, \psi^H)$;

ii) $\mathcal{S}(y_2) \rightharpoonup \phi_{\chi,\psi} \leftharpoonup y_1 = \varepsilon(y)\phi_{\chi,\psi}$, $y \in K$.

Suppose $\chi, \chi' \in \widehat{H}$, $\psi, \psi' \in \widehat{K}$. *Then*

iii) $\phi_{\chi,\psi} * \chi' = \delta_{\chi,\chi'} \dfrac{1}{\deg \chi} \phi_{\chi,\psi}$;

iv) $\phi_{\chi,\psi} * \widetilde{\psi}' = \delta_{\psi,\psi'} \dfrac{1}{\deg \psi} \phi_{\chi,\psi}$.

Proof. Part i) follows from Proposition 2.6ii).

ii). Since H is semisimple, $\mathcal{S}^2 = \mathrm{id}$. This implies that ii) is equivalent to the relation $\phi(yx) = \phi(xy)$ for all $x \in H$, $y \in K$, where $\phi = \phi_{\chi,\psi}$. We shall use the fact that, for all $y \in K$,

$$y\mathcal{S}(\Lambda_{K1}) \otimes \Lambda_{K2} = \mathcal{S}(\Lambda_{K1}) \otimes \Lambda_{K2}y$$

(see [Sch, Theorem 3.2]). Let $x \in H$, $y \in K$. We compute

$$\begin{aligned}\phi(yx) &= \langle \chi_1\mathcal{S}(\psi), \Lambda_K\rangle\langle\chi_2, yx\rangle = \langle\chi, yx\Lambda_{K1}\rangle\langle\mathcal{S}(\psi), \Lambda_{K2}\rangle \\ &= \langle\chi, x(\Lambda_{K2}y)\rangle\langle\psi, \mathcal{S}(\Lambda_{K1})\rangle = \langle\chi, x\Lambda_{K2}\rangle\langle\psi, y\mathcal{S}(\Lambda_{K1})\rangle \\ &= \langle\chi, x\Lambda_{K2}\rangle\langle\psi, \mathcal{S}(\Lambda_{K1})y\rangle = \langle\chi, x\mathcal{S}(\Lambda_{K1})\rangle\langle\psi, \Lambda_{K2}y\rangle \\ &= \langle\chi, xy\mathcal{S}(\Lambda_{K1})\rangle\langle\psi, \Lambda_{K2}\rangle = \langle\chi, xy\Lambda_{K2}\rangle\langle\psi, \mathcal{S}(\Lambda_{K1})\rangle \\ &= \langle\chi, xy\Lambda_{K2}\rangle\langle\mathcal{S}(\psi), \Lambda_{K1}\rangle = \phi(xy).\end{aligned}$$

We have used the fact that the relations $\mathcal{S}(\Lambda_K) = \Lambda_K$ and $\mathcal{S}^2 = \mathrm{id}$ imply $\mathcal{S}(\Lambda_{K1}) \otimes \Lambda_{K2} = \Lambda_{K2} \otimes \mathcal{S}(\Lambda_{K1})$; indeed, this follows by applying $\mathcal{S} \otimes \mathrm{id}$ to the identity $\Lambda_{K1} \otimes \Lambda_{K2} = \mathcal{S}(\Lambda_{K2}) \otimes \mathcal{S}(\Lambda_{K1})$. Thus, ii) is proved.

Parts iii) and iv) follow from (5.1) and Lemma 2.3. •

Corollary 5.3 (Orthogonality relations for spherical functions). *If* $\chi, \chi' \in \widehat{H}$, $\psi, \psi' \in \widehat{K}$, *then*

$$(\phi_{\chi,\psi}|\phi_{\chi',\psi'}) = \delta_{\chi,\chi'}\delta_{\psi,\psi'}\frac{m(\chi, \psi^H)}{\deg\chi\deg\psi}.$$

Proof. By Lemma 5.2iii), iv), $\phi_{\chi,\psi} * \phi_{\chi',\psi'} = \delta_{\chi,\chi'}\delta_{\psi,\psi'}\dfrac{1}{\deg\chi\deg\psi}\phi_{\chi,\psi}$, and the claim follows by specialization at 1. •

Corollary 5.4. i) $\widetilde{\psi} = \sum_{\chi\in\widehat{H}} \deg\chi\phi_{\chi,\psi}$.

ii) $\chi = \sum_{\psi\in\widehat{K}} \deg\psi\phi_{\chi,\psi}$.

iii) $\int_H = \sum_{\chi\in\widehat{H},\psi\in\widehat{K}} \deg\chi\deg\psi\phi_{\chi,\psi}$.

Proof. Since $\int_H = \sum_{\chi\in\widehat{H}} \deg\chi$ is the identity element in H^* with respect to the convolution product, part i) follows. Part ii) is deduced from Remark 3.14 and Lemma 2.3 in a similar way. Part iii) follows from Remark 3.14, with the use of the relation $\int_H * \int_H = \int_H$. •

In the sequel, we give a more explicit description of the algebraic nature of the spherical functions.

Let $\chi \in \widehat{H}$, $\psi \in \widehat{K}$. Let $\rho: H \to \mathrm{End}(V)$ be the representation of H affording the character χ, and let $E_\psi \in K$ be the central primitive idempotent in K such that

$\langle \psi, E_\psi \rangle = \deg \psi$. We recall that $V[\psi]$ denotes the isotypic component of type ψ. We introduce $\Phi_{\rho,\psi} \colon H \to \operatorname{End}(V[\psi])$ by the relation

$$\Phi_{\rho,\psi}(h) = \rho(E_\psi h E_\psi),$$

i.e., $\Phi_{\rho,\psi}$ is the composition of ρ with the map $H \to \mathcal{H}(H, K, E_\psi)$ given by $h \mapsto E_\psi h E_\psi$ (cf. [GV, 1.3.1]).

Proposition 5.5. *For all $h \in H$ we have $\phi_{\chi,\psi}(h) = \dfrac{1}{\deg \psi} \operatorname{Tr}(\Phi_{\rho,\psi}(h))$.*

Proof. Let $h \in H$. Then, using (2.4), we obtain

$$\phi_{\chi,\psi}(h) = \langle \chi_1, \mathcal{S}(\psi) \rightharpoonup \Lambda_K \rangle \langle \chi_2, h \rangle = \langle \chi, (\mathcal{S}(\psi) \rightharpoonup \Lambda_K) h \rangle = \frac{1}{\deg \psi} \langle \chi, E_\psi h \rangle. \quad \bullet$$

Corollary 5.6. *$\phi_{\chi,\psi} = 0$ if and only if $m(\chi, \psi^H) = 0$.* $\bullet$

Lemma 5.7. *Let $\chi \in \widehat{H}$. Then $E_\psi \rightharpoonup \phi_{\chi,\psi} \leftharpoonup E_\psi = \phi_{\chi,\psi}$.*

Proof. This follows from Proposition 5.5. $\bullet$

Proposition 5.8. *The spherical functions $\deg \psi \phi_{\chi,\psi}|_{\mathcal{H}(H,K,E_\psi)} = \chi|_{\mathcal{H}(H,K,E_\psi)}$, with $m(\chi, \psi^H) \neq 0$, are all the irreducible characters of the Hecke algebra $\mathcal{H}(H, K, E_\psi)$.*

Proof. This is a consequence of Theorem 4.4 and Proposition 5.5. $\bullet$

For $\psi \in \widehat{K}$, $\chi \in \widehat{H}$, we shall use the notation $\varphi_\chi := \deg \chi \deg \psi \phi_{\chi,\psi}$.

Corollary 5.9. *The collection $\{ \varphi_{\chi,\psi} : m(\chi, \psi^H) \neq 0 \}$ is a complete set of central primitive idempotents in $L^2{}_\psi(K \backslash H / K)$.*

Proof. Lemma 3.8 implies that $(\psi^H)^* = (\psi^*)^H$; therefore, $m(\chi, \psi^H) > 0$ if and only if $m(\chi^*, (\psi^H)^*) > 0$. Let E_χ be the central primitive idempotent in H such that $\langle \chi, E_\chi \rangle = \deg \chi$. Then, by Theorem 4.4, $\{ E_{\chi^*} E_{\psi^*} : \chi \in \widehat{H},\ m(\chi, \psi^H) \neq 0 \}$ is a complete set of central primitive idempotents in $\mathcal{H}(H, K, E_{\psi^*})$. If we prove that

$$\mathbb{G}(\phi_{\chi,\psi}) = \phi_{\chi,\psi} \rightharpoonup \Lambda = \frac{1}{\deg \chi \deg \psi} E_{\psi^*} E_{\chi^*} \tag{5.10}$$

for all $\chi \in \widehat{H}$, then the claim will follow from Proposition 4.7. Let $\alpha \in H^*$. Then, using Proposition 5.5, we can write

$$\begin{aligned}
\langle \alpha, \mathbb{G}(\phi_{\chi,\psi}) \rangle &= \langle \alpha \phi_{\chi,\psi}, \Lambda \rangle = \langle \phi_{\chi,\psi}, \alpha \rightharpoonup \Lambda \rangle \\
&= \frac{1}{\deg \psi} \langle \chi, E_\psi (\alpha \rightharpoonup \Lambda) \rangle = \frac{1}{\deg \psi} \langle \chi, E_\psi \mathbb{G}(\alpha) \rangle \\
&= \frac{1}{\deg \psi} \langle \chi, \mathbb{G}(E_\psi \rightharpoondown \alpha) \rangle = \frac{1}{\deg \psi} \langle \alpha \leftharpoonup E_{\psi^*}, \mathbb{G}(\chi) \rangle \\
&= \frac{1}{\deg \psi \deg \chi} \langle \alpha, E_{\psi^*} E_{\chi^*} \rangle. \quad \bullet
\end{aligned}$$

Now, our aim is to give a characterization for spherical functions.

Given a vector space V, $\Omega \in \operatorname{Hom}(H, V)$, and $\alpha \in H^*$, we define $\alpha * \Omega$ (respectively, $\Omega * \alpha$) in $\operatorname{Hom}(H, V)$ by

$$\alpha * \Omega(x) = \langle \alpha, \Lambda_1 \rangle \Omega(\mathcal{S}(\Lambda_2)x) \quad \text{(respectively,} \quad \Omega * \alpha(x) = \langle \Omega(x\Lambda_1)\alpha, \mathcal{S}\Lambda_2 \rangle), \quad x \in H.$$

It is easy to check that $\alpha * (\Omega * \beta) = (\alpha * \Omega) * \beta$ whenever $\beta \in H^*$. Clearly in the above definitions we do not need the semisimplicity of H, and they also make sense in the case where $\alpha \in \operatorname{Hom}(H, \mathcal{A})$, $\mathcal{A}$ is an algebra, and V is a left (respectively, right) $\mathcal{A}$-module; of course, the associativity above requires that V be an $\mathcal{A}$-bimodule.

We recall that $\mathcal{F}(x) = \mathbb{F}\mathcal{S}(x)$, $x \in H$.

Lemma 5.11. *Let U be a finite-dimensional vector space, $E \in K$ a central idempotent, $\Phi\colon H \to \operatorname{End} U$ a linear map such that $\Phi(1) = \mathrm{id}$ and*

$$\Phi(x) = \Phi(ExE), \qquad x \in H. \tag{5.12}$$

The following statements are equivalent.

i) *$f \mapsto \Phi(f)$ yields a representation of $\mathcal{H}(H, K, E)$.*

ii) *$\Phi(xEy) = \Phi(x)\Phi(y)$ for all $x, y \in H$.*

iii) *There exists a representation $p\colon \mathcal{H}(H, K, E) \to \operatorname{End} U$ such that $\mathcal{F}(f) * \Phi = p(f)\Phi$ for all $f \in \mathcal{H}(H, K, E)$.*

iv) *There exists a representation $p'\colon \mathcal{H}(H, K, E) \to \operatorname{End} U$ such that $\Phi * \mathcal{F}(f) = \Phi p'(f)$ for all $f \in \mathcal{H}(H, K, E)$.*

If these conditions are fulfilled, then necessarily $p(f) = p'(f) = \Phi(f)$.

Proof. Observe that $\Phi(E) = \mathrm{id}$. Since

$$\Phi(xEy) = \Phi(ExEyE), \qquad \Phi(x)\Phi(y) = \Phi(ExE)\Phi(EyE)$$

by (5.12), we see that i) is equivalent to ii). We note that

$$\mathcal{F}(f) * \Phi(x) = \langle \mathcal{F}(f), \Lambda_1 \rangle \Phi(\mathcal{S}(\Lambda_2)x) = \Phi\left(\langle \mathcal{F}(f), \Lambda_1 \rangle \mathcal{S}(\Lambda_2)x\right) = \Phi(fx)$$

for all $f, x \in H$. Hence, condition iii) means that $p(f)\Phi(x) = \Phi(fx)$ for all $f \in \mathcal{H}(H, K, E)$, $x \in H$. Taking $x = 1$ we infer that $\Phi = p$ yields a representation of $\mathcal{H}(H, K, E)$. Conversely, if i) is true, then

$$\Phi(fx) = \Phi(fEx) = \Phi(f)\Phi(x), \quad f \in \mathcal{H}(H, K, E), \;\; x \in H.$$

Here, the first identity is fulfilled because $f \in \mathcal{H}(H, K, E)$ and the second because i) $\Longrightarrow$ ii). Thus, iii) is proved. In a similar way we can show that i) $\Longleftrightarrow$ iv). •

Theorem 5.13. *Let U be a finite-dimensional vector space, and let $E \in K$ be a primitive central idempotent corresponding to $\psi \in \widehat{K}$. Let $\Phi\colon H \to \operatorname{End} U$ be a linear map satisfying* (5.12) *and such that* $\Phi(1) = \mathrm{id}$. *Then $\Phi = \Phi_{\rho,\psi}$ for some representation ρ of H on V (and $U = V[\psi]$) if and only if Φ satisfies the equivalent conditions of Lemma* 5.11. *Moreover, ρ is irreducible if and only if so is $\Phi|_{\mathcal{H}(H,K,E)}$.*

Proof. It is clear that $\Phi = \Phi_{\rho,\psi}$ with $U = V[\psi]$ satisfies the relations $\Phi(1) = \mathrm{id}$ and (5.12), as well as relation ii) in Lemma 5.11. Conversely, suppose that $\Phi\colon H \to \operatorname{End} U$ satisfies $\Phi(1) = \mathrm{id}$, relation (5.12), and the conditions of Lemma 5.11. First, we assume that $\Phi|_{\mathcal{H}(H,K,E)}$ is irreducible. By Theorem 4.4, there exists an irreducible representation $\rho\colon H \to \operatorname{End} V$ such that $\rho|_{\mathcal{H}(H,K,E)} = \Phi|_{\mathcal{H}(H,K,E)}$ on $EV = V[\psi]$. By (5.12), $\Phi = \Phi_{\rho,\psi}$. In general, we decompose $U = \bigoplus_{i\in I} U_i$, where U_i is an irreducible $\mathcal{H}(H,K,E)$-submodule, $i \in I$. Then $\Phi(x)U_i \subseteq U_i$ for all $x \in H$, and we have $\Phi_i\colon H \to \operatorname{End} U_i$ satisfying the same conditions as Φ. Hence, there exist irreducible representations $\rho_i\colon H \to \operatorname{End} V_i$ such that $\Phi_i = \Phi_{\rho_i,\psi}$. Then $\Phi = \Phi_{\rho,\psi}$ with $\rho := \bigoplus_i \rho_i$. •

§6. Symmetric spaces for semisimple Hopf algebras

We keep the notation of §5.

Definition 6.1. We say that (H, K) is a *Gelfand pair* if $L^2(K\backslash H/K)$ is a commutative subalgebra of H^* with the convolution product. In this case, the quotient H-module coalgebra H/HK^+ is called a *symmetric space*.

A representation V of H is said to be *multiplicity free* if any irreducible representation of H appears with multiplicity at most one in V.

We recall that if V is an H-module, then V^K denotes the space of K-invariants in V. Part of the following result can be found in Vainerman's paper [V].

Proposition 6.2. *The following statements are equivalent.*

i) (H, K) *is a Gelfand pair.*

ii) $\mathcal{H}(H, K)$ *is a commutative subalgebra of* H.

iii) $\operatorname{Ind}_K^H \varepsilon_K$ *is multiplicity free.*

iv) *For every irreducible representation* $\rho\colon H \to \operatorname{End} V$, *we have* $\dim V^K \le 1$.

v) *For every irreducible representation* $\rho\colon H \to \operatorname{End} V$, *there exists a basis* $\{v_1, \dots, v_d\}$ *of V such that the matrix entries c_{ij} of ρ in this basis satisfy the relations*

$$\langle c_{ij}, \Lambda_K h \Lambda_K\rangle = \delta_{i1}\delta_{j1}\langle c_{11}, \Lambda_K h \Lambda_K\rangle, \quad h \in H.$$

In other words, there exists $\phi \in H^*$ such that in the basis $\{v_i\}$ the matrix of $\rho(\Lambda_K h \Lambda_K)$, $h \in H$, is of the form

$$\rho(\Lambda_K h \Lambda_K) = \begin{pmatrix} \phi(h) & 0 \\ 0 & 0 \end{pmatrix}.$$

Proof. The equivalence i) $\iff$ ii) follows from Proposition 4.7. For ii) $\iff$ iii) it suffices to put $e = \Lambda_K$ in Theorem 4.4. The fact that iii) $\iff$ iv) is a consequence of Frobenius reciprocity (Lemma 3.8). We note that v) means that $\dim \Lambda_K H \Lambda_K V \leq 1$. Since $\Lambda_K H \Lambda_K V = \Lambda_K V = V^K$ because $\Lambda_K^2 = 1$, we conclude that iv) $\iff$ v). •

Remark 6.3. Observe that $HK^+ = H(1 - \Lambda_K)$. Hence, viewed as left H-modules, $\mathrm{Ind}_K^H \varepsilon_K = H\Lambda_K \backsimeq H/H(1-\Lambda_K) = H/HK^+$. Let $H^{*\,\mathrm{co}\,K^*} \subseteq H^*$ be the right coideal subalgebra of left coinvariants of the Hopf algebra surjection $H^* \to K^*$, that is,

$$H^{*\,\mathrm{co}\,K^*} := \{\, \alpha \in H^* : \langle \alpha, xh \rangle = \varepsilon(x)\langle \alpha, h\rangle,\ \forall x \in K,\ h \in H \,\}.$$

Then $H^{*\,\mathrm{co}\,K^*}$ is an H-submodule of H^* with respect to the action $\rightharpoonup$. Clearly, $(H/K^+H)^* \backsimeq H^{*\,\mathrm{co}\,K^*}$ (we mean left H-modules).

On the other hand, it is not difficult to check that the map $\mathcal{F}\colon H\Lambda_K \to H^{*\,\mathrm{co}\,K^*}$ given by $\mathcal{F}(h\Lambda_K) = \int \leftharpoonup h\Lambda_K = h\Lambda_K \rightharpoonup \int$ is an isomorphism of left H-modules. Therefore, $\mathrm{Ind}_K^H \varepsilon_K \backsimeq H^{*\,\mathrm{co}\,K^*} \backsimeq (H/K^+H)^*$.

Now, we present sufficient conditions ensuring that (H, K) is a Gelfand pair. Our first statement is inspired by a criterion due to Selberg (cf. [Te, §2.5; GV, 1.5.2, 1.5.3]).

Proposition 6.4. *Let $\theta\colon H \to H$ be a linear isomorphism such that, for all $x \in H$,*

$$\begin{aligned}&\theta(x)\Lambda_1 \otimes \mathcal{S}(\Lambda_2)\\ &\quad = (\theta \otimes \theta)(x\Lambda_1 \otimes \mathcal{S}(\Lambda_2)) \bmod (HK^+ + K^+H) \otimes H + H \otimes (HK^+ + K^+H).\end{aligned} \tag{6.5}$$

Suppose, moreover, that $\mathcal{S}(x) - \theta(x) \in K^+H + HK^+$ for all $x \in H$. Then (H, K) is a Gelfand pair.

Proof. By Proposition 1.7iv), $\mathcal{S}\colon (H^*, *) \to (H^*, *)$ is an anti-algebra automorphism. Clearly, $\mathcal{S}(L^2(K\backslash H/K)) = L^2(K\backslash H/K)$. Let $f \in L^2(K\backslash H/K)$. If $x \in H$, then $\mathcal{S}x = \theta x + \sum_i a_i h_i + \sum_j h'_j b_j$ for some $h_i, h'_j \in H$, $a_i, b_j \in K^+$. Consequently,

$$\langle \mathcal{S}f, x\rangle = \langle f, \mathcal{S}x\rangle = \langle f, \theta x\rangle + \sum_i \langle f, a_i h_i\rangle + \sum_j \langle f, h'_j b_j\rangle = \langle f, \theta x\rangle = \langle {}^t\theta f, x\rangle.$$

This shows that ${}^t\theta$ and $\mathcal{S}$ coincide on $L^2(K\backslash H/K)$; in particular, ${}^t\theta(L^2(K\backslash H/K)) = L^2(K\backslash H/K)$.

We claim that ${}^t\theta\colon (L^2(K\backslash H/K), *) \to (L^2(K\backslash H/K), *)$ is an algebra automorphism, which implies the proposition. Let $f, g \in L^2(K\backslash H/K)$, $x \in H$. Then, by (6.5), we have

$$\begin{aligned}\langle ({}^t\theta f) * ({}^t\theta g), x\rangle &= \langle ({}^t\theta f)_2 \mathcal{S}({}^t\theta g), \Lambda\rangle\langle ({}^t\theta f)_1, x\rangle\\ &= \langle {}^t\theta f, x\Lambda_1\rangle\langle {}^t\theta g, \mathcal{S}(\Lambda_2)\rangle = \langle f, \theta(x\Lambda_1)\rangle\langle g, \theta\mathcal{S}(\Lambda_2)\rangle\\ &= \langle f, \theta(x)\Lambda_1\rangle\langle g, \mathcal{S}(\Lambda_2)\rangle = \langle f * g, \theta(x)\rangle = \langle {}^t\theta(f * g), x\rangle.\end{aligned}$$ •

Corollary 6.6 (Gelfand criterion). *Suppose there exists a Hopf algebra automorphism θ of H such that $\mathcal{S}(x) - \theta(x) \in K^+H + HK^+$ for all $x \in H$. Then (H, K) is a Gelfand pair.*

Proof. Since θ is a Hopf algebra automorphism, it is clear that θ satisfies (6.5). •

Corollary 6.7. *Suppose there exists a Hopf algebra automorphism θ of H such that*
i) $\theta(a) = a$ *for all* $a \in K$;
ii) $H = KP$, *where* $P = \{ h \in H : \theta(h) = \mathcal{S}(h) \}$.
Then (H, K) is a Gelfand pair.

We note that, necessarily, the automorphism θ is an involution of H.

Proof. If $h \in H$, then $h = \sum_i a_i p_i$ with $a_i \in K$, $p_i \in P$. Then $\theta(h) = \sum_i a_i \mathcal{S}(p_i)$ and $\mathcal{S}(h) = \sum_i \mathcal{S}(p_i)\mathcal{S}(a_i)$. Now, for all i, $a_i\mathcal{S}(p_i) = \varepsilon(a_i)\mathcal{S}(p_i)$ modulo K^+H and $\mathcal{S}(p_i)\mathcal{S}(a_i) = \varepsilon(a_i)\mathcal{S}(p_i)$ modulo HK^+, so that $\theta(h) = \mathcal{S}(h)$ modulo $HK^+ + K^+H$, and it remains to refer to Corollary 6.6. •

§7. Harmonic analysis on symmetric spaces

In this section we fix a Gelfand pair (H, K). Let $\widehat{H}^K$ be the set of all irreducible characters $\chi \in \widehat{H}$ such that $m(\chi, \varepsilon_K^H) = 1$. For all $\chi \in \widehat{H}$, we denote the spherical function $\phi_{\chi,\varepsilon_K}$ by ϕ_χ; recalling §5, we see that $\varphi_\chi := \deg \chi \phi_\chi$. By Proposition 5.5, $\phi_\chi(h) = \langle \chi, \Lambda_K h \rangle$ for all $h \in H$. Rewriting the integral of K in terms of the matrix entries of K, we obtain the following expression for the spherical functions:

$$\phi_\chi(h) = \frac{1}{\dim K} \sum_{\tau \in \widehat{K^*}} \sum_{1 \le i,j \le \deg \tau} (\deg \tau) \chi(\tau_{ij} h). \tag{7.1}$$

We note that the spherical function ϕ_χ satisfies $\phi_\chi = \phi$, where ϕ is as in Proposition 6.2v), and V is the irreducible module affording χ. In other words, ϕ_χ is the composition of a distinguished matrix entry of V with $h \mapsto \Lambda_K h \Lambda_K$.

In the symmetric case there are elegant characterizations of the spherical functions.

Proposition 7.2. *Let $\phi \colon H \to k$ be a linear map. Then the following statements are equivalent*:
i) $\phi = \phi_\chi$ *for some* $\chi \in \widehat{H}^K$;
ii) $\phi(1) = 1$, $\phi(\Lambda_K x \Lambda_K) = \phi(x)$, *and* $\phi(x)\phi(y) = \phi(x\Lambda_K y)$ *for all* $x, y \in H$.

Proof. This follows from Theorem 5.13. •

The next proposition is a consequence of Corollaries 5.3 and 5.9.

Proposition 7.3. i) *Relative to the nondegenerate bilinear form* $(\ |\ )$ *introduced in* (2.5), *the set* $\{\varphi_\chi : \chi \in \widehat{H}^K\}$ *forms an orthonormal basis of* $L^2(K\backslash H/K)$.

ii) *The collection* $\{\varphi_\chi : \chi \in \widehat{H}^K\}$ *is a complete set of orthogonal primitive idempotents in* $L^2(K\backslash H/K)$. •

Corollary 7.4 (Fourier expansion in $L^2(K\backslash H/K)$. *Let* $f \in L^2(K\backslash H/K)$. *Then*

$$f = \sum_{\chi\in\widehat{H}^K} (f|\varphi_\chi)\,\varphi_\chi = \sum_{\chi\in\widehat{H}^K} \deg\chi\,(f|\phi_\chi)\,\phi_\chi. \quad \bullet$$

We define a map $\mathfrak{S}\colon L^2(K\backslash H/K) \to R(H)^*$ as follows: $\langle\mathfrak{S}(f),\chi\rangle := (f|\phi_\chi)$, $f \in L^2(K\backslash H/K)$, $\chi \in \widehat{H}$. The map $\mathfrak{S}$ might be called the *spherical transform*; then Corollary 7.4 would yield an analog of the inversion formula in Fourier analysis. Observe that

$$\langle\mathfrak{S}(f),\chi\rangle = (f|\phi_\chi) = (f|\Lambda_K \rightharpoonup \chi \leftharpoonup \Lambda_K) = (\Lambda_K \rightharpoonup f \leftharpoonup \Lambda_K|\chi) = (f|\chi) = \langle\mathcal{G}(f),\chi\rangle$$

by Propositions 5.5 and 2.6v) and by (2.7). Thus, $\mathfrak{S}(f)$ is nothing but the restriction of $\mathcal{G}(f)$ to $R(H)$.

Proposition 7.5. *Let* $x, y \in \mathcal{H}(H,K)$. *Then*

$$\langle\textstyle\int, xy\rangle = \displaystyle\sum_{\chi\in\widehat{H}^K} \deg\chi\,\chi(x)\chi(y).$$

Proof. Using Corollary 5.4iii), Lemma 5.7, and Proposition 5.8, we obtain

$$\begin{aligned}
\langle\textstyle\int, xy\rangle &= \sum_{\chi\in\widehat{H},\psi\in\widehat{K}} \deg\chi\,\deg\psi\,\phi_{\chi,\psi}(xy)\\
&= \sum_{\chi\in\widehat{H},\psi\in\widehat{K}} \deg\chi\,\deg\psi\,\phi_{\chi,\psi}(\Lambda_K xy\Lambda_K)\\
&= \sum_{\chi\in\widehat{H},\psi\in\widehat{K}} \deg\chi\,\deg\psi\,\phi_{\chi,\psi}(E_\psi\Lambda_K xy\Lambda_K E_\psi)\\
&= \sum_{\chi\in\widehat{H}} \deg\chi\,\phi_\chi(xy) = \sum_{\chi\in\widehat{H}^K} \deg\chi\,\chi(x)\chi(y). \quad \bullet
\end{aligned}$$

§8. Smash products and biproducts

Let $K \hookrightarrow H$ be an inclusion of semisimple Hopf algebras. Assume there is an algebra isomorphism $H \simeq R\#K$, where R is a left K-module algebra, and $\#$ stands for smash product. Let $R^K \subseteq R$ be the subalgebra of all K-invariant elements in R,

$$R^K = \{\, r \in R : x.r = \varepsilon(x)r,\ x \in K\,\}.$$

By [Mo, Lemma 4.3.4], the Hecke algebra $\mathcal{H}(H,K)$ is isomorphic to R^K. Hence, (H,K) is a Gelfand pair if and only if R^K is a commutative subalgebra of R.

In particular, the above applies when R is a braided Hopf algebra in the category of Yetter–Drinfeld modules over K and $H = R\#K$ is a *biproduct* over K. In this case, we have the left H-module coalgebra isomorphisms

$$\operatorname{Ind}_K^H \varepsilon_K \simeq H/HK^+ \simeq R$$

(see [R2]), where the action $\rightharpoonup$: $H \otimes R \to R$ of H on R is given by $(r\#x) \rightharpoonup s := r(x \rightharpoonup s)$ for all $r, s \in R$, $x \in K$.

§9. Almost cocommutative semisimple Hopf algebras and the Drinfeld double

Let H be a semisimple Hopf algebra. We denote by $D(H)$ the Drinfeld double of H and identify $D(H) = H^* \otimes H$ and $D(H^*) = H \otimes H^*$ as vector spaces. For $h \in H$, $\alpha \in H^*$, the elements $\alpha \otimes h \in D(H)$ and $h \otimes \alpha \in D(H)^*$ will be denoted by $\alpha\#h$ and $h\#\alpha$, respectively. There are Hopf algebra inclusions $H \xrightarrow{\iota} D(H)$, $h \mapsto \varepsilon\#h$, and $H^{*\mathrm{cop}} \xrightarrow{\iota'} D(H)$, $\alpha \mapsto \alpha\#1$. We identify H and $H^{*\mathrm{cop}}$ with their images in $D(H)$.

We recall (see [Z]) that a left $D(H)$-module algebra structure of $D(H)$ on H can be given by

$$\alpha\#h.x = (h_1 x \mathcal{S}(h_2)) \leftharpoonup \mathcal{S}^{-1}(\alpha), \quad \alpha \in H^*,\ h, x \in H. \tag{9.1}$$

On the other hand, if we consider the left (faithful) $R(H)$-action $\rightharpoonup$ on H, $f \rightharpoonup h = \langle f, h_2\rangle h_1$, then, as was shown in [Z], this action together with the $D(H)$-action (9.1) form a commuting pair. This implies that there is a bijective correspondence between the irreducible $D(H)$-summands and the irreducible $R(H)$-summands in H. Also, if $V_1, \dots, V_s$ are representatives of the isomorphism classes of irreducible $R(H)$-modules, then, viewed as a $D(H) \otimes R(H)$-module, H is isomorphic to a direct sum,

$$H \simeq \bigoplus_{i=1}^{s} W_i \otimes V_i,$$

where W_i is the irreducible $D(H)$-module corresponding to V_i. In particular, if $e_i \in R(H)$ is the primitive idempotent affording V_i, then $\dim W_i = \dim(H^* e_i)$ for all $i = 1, \dots, s$ (see [Z]).

Lemma 9.2. *We have an isomorphism of left $D(H)$-modules $H \simeq (\operatorname{Ind}_H^{D(H)} \varepsilon_H)^*$.*

Proof. Consider the sequence of maps

$$H \xrightarrow{\iota} D(H) \xrightarrow{\pi} H^{*\mathrm{cop}},$$

where π is the coalgebra map defined by $\pi(\alpha\#h) := \varepsilon(h)\alpha$. Clearly, $H = D(H)^{\mathrm{co}\,\pi}$, and there is isomorphism of left $D(H)$-module coalgebras $H^{*\mathrm{cop}} \simeq D(H)/D(H)H^+$, where the $D(H)$-module structure on $H^{*\mathrm{cop}}$ is given by

$$\alpha\#h.f = (\mathrm{id}\otimes\varepsilon)\,(\alpha\#hf\#1) = \alpha\left(h_1 \rightharpoonup f \leftharpoonup \mathcal{S}^{-1}(h_2)\right), \quad \alpha, f \in H^*, \ \ h \in H. \qquad (*)$$

As was mentioned before, $D(H)/D(H)H^+ \simeq \operatorname{Ind}_H^{D(H)} \varepsilon_H$ (isomorphism of $D(H)$-modules). On the other hand, dualizing the left action of $D(H)$ on $H^{*\mathrm{cop}}$ given by $(*)$, results in the left $D(H)$-module algebra structure on H determined by $\langle f, \alpha\#h.x\rangle := \langle \mathcal{S}(\alpha\#h).f, x\rangle$, $\alpha, f \in H^*$, $h, x \in H$. In turn, this $D(H)$-module structure coincides with (9.1), and the claim follows. •

Lemma 9.3. *The Hecke algebra $\mathcal{H}(D(H), H)$ is isomorphic to $R(H)$.*

Proof. Let $\Lambda \in H$ be a normalized integral. Then, for all $\alpha \in H^*$, $h \in H$ we have

$$\Lambda(\alpha\#h)\Lambda = \varepsilon(h)\langle \alpha, \mathcal{S}^{-1}(\Lambda_2)\_\Lambda_1\rangle\#\Lambda = \varepsilon(h)\,\mathrm{ad}\,\Lambda(\alpha)\#\Lambda.$$

Now, $\mathrm{ad}\,\Lambda(H^*)$ is the isotypical component of trivial type in H^* with respect to the adjoint action of H, $\mathrm{ad}\colon H \otimes H^* \to H^*$, $h \otimes f \mapsto \mathrm{ad}\,h(f) = \langle f, \mathcal{S}^{-1}(h_2)\_h_1\rangle$. As is well known, this coincides with $R(H)$. Hence, $\mathcal{H}(D(H), H) = R(H)\#\Lambda$. Also, it is not difficult to see that if $\alpha, \beta \in R(H)$, then $(\alpha\#\Lambda)(\beta\#\Lambda) = \alpha\beta\#\Lambda$. This proves the claim. •

Remark 9.4. Let $e \in R(H)$ be a primitive idempotent. If χ is the irreducible $D(H)$-character afforded by $D(H)(\varepsilon\#e)$, and $\Lambda \in H$ is the normalized integral, then Theorem 4.4 shows that

$$\dim R(H)e = \langle \chi, \Lambda\rangle.$$

We recall that a Hopf algebra H is said to be *almost cocommutative* if there exists $Q \in (H \otimes H)^\times$ such that $\Delta^{\mathrm{cop}}(h) = Q\Delta(h)Q^{-1}$ for all $h \in H$. If H is semisimple, then H is almost cocommutative if and only if $R(H)$ is a commutative algebra.

Corollary 9.5. *The pair $(D(H), H)$ is a Gelfand pair if and only if H is an almost cocommutative Hopf algebra.*

Proof. This is a consequence of Proposition 6.2 and Lemma 9.3. •

Thus, whenever H is a quasitriangular Hopf algebra, $(D(H), H)$ is a Gelfand pair. In particular, if G is a finite group, then $(D(kG), kG)$ is a Gelfand pair.

References

[A] Abe E., *Hopf algebras*, Cambridge Tracts in Math., vol. 74, Cambridge Univ. Press, Cambridge–New York, 1980.

[An] Andruskiewitsch N., *Compact involutions of semisimple quantum groups*, Czechoslovak J. Phys. **44** (1994), 963–972.

[AN] Andruskiewitsch N., Natale S., *Plancherel identity for semisimple Hopf algebras*, Comm. Algebra **25** (1997), no. 10, 3239–3254.

[BDG] Beattie M., Dăscălescu S., Grünenfelder L., *Constructing pointed Hopf algebras by Ore extensions*, J. Algebra **225** (2000), 743–770.

[BDGN] Beattie M., Dăscălescu S., Grünenfelder L., Năstăsescu C., *Finiteness conditions, co-Frobenius Hopf algebras and quantum groups*, J. Algebra **200** (1998), 312–333.

[CR] Curtis C. W., Reiner I., *Methods of representation theory.* Vol. I, John Wiley and Sons, Inc., New York, 1990.

[D] Doi Y., *Homological coalgebra*, J. Math. Soc. Japan **33** (1981), no. 1, 31–50.

[DNT] Dăscălescu S., Năstăsescu C., Torrecillas B., *Co-Frobenius Hopf algebras: integrals, Doi-Koppinen modules and injective objects*, J. Algebra **220** (1999), 542–560.

[EG] Etingof P., Gelaki S., *On finite-dimensional semisimple and cosemisimple Hopf algebras in positive characteristic*, Internat. Math. Res. Notices **1998**, no. 16, 851–864.

[FMS] Fischman D., Montgomery S., Schneider H.-J., *Frobenius extensions of subalgebras of Hopf algebras*, Trans. Amer. Math. Soc. **349** (1997), 4857–4895.

[GV] Gangolli R., Varadarajan V. S., *Harmonic analysis of spherical functions on real reductive groups*, Ergeb. Math. Grenzgeb., Bd. 101, Springer-Verlag, Berlin–New York, 1988.

[Ka1] Кац Г. И., *Расширения групп, являющиеся кольцевыми группами*, Мат. сб. **76** (1968), №3, 473–496.

[Ka2] Кац Г. И., *Некоторые арифметические свойства кольцевых групп*, Функц. анал. и его прил. **6** (1972), №2, 88–90.

[KaP] Кац Г. И., Палюткин В. Г., *Конечные кольцевые группы*, Тр. Моск. мат. о-ва **15** (1966), 224–261.

[K] Koornwinder T., *Positive convolution structures associated with quantum groups*, Probability Measures on Groups, X (Oberwolfach, 1990), Plenum, New York, 1991, pp. 249–268.

[LS] Larson R., Sweedler M., *An associative orthogonal bilinear form for Hopf algebras*, Amer. J. Math. **91** (1969), 75–94.

[Mo] Montgomery S., *Hopf algebras and their actions on rings*, CBMS Regional Conf. Ser. in Math., vol. 82, Amer. Math. Soc., Providence, RI, 1993.

[NZ] Nichols W., Zoeller M., *A Hopf algebra freeness theorem*, Amer. J. Math. **111** (1989), 381–385.

[PW] Podleś P., Woronowicz S. L., *Quantum deformation of Lorentz group*, Comm. Math. Phys. **130** (1990), 381–431.

[R1] Radford D., *Finiteness conditions for a Hopf algebra with nonzero integral*, J. Algebra **46** (1977), 189–195.

[R2] Radford D., *The structure of Hopf algebras with a projection*, J. Algebra **92** (1985), 322–347.

[Ra] Raianu S., *An easy proof for the uniqueness of integrals*, Preprint, 1999.

[Sch] Schneider H.-J., *Lectures on Hopf algebras*, Trabajos de Matemática no. 31/95, Fa. M. A. F., Córdoba, 1995.

[Sch2] Schneider H.-J., *Normal basis and transitivity of crossed products for Hopf algebras*, J. Algebra **152** (1992), 289–312.

[Sch3] Schneider H.-J., *Lectures on Hopf algebras.* II, Trabajos de Matemática, Fa. M. A. F., Córdoba (to appear).

[Sw] Sweedler M., *Hopf algebras*, Benjamin, New York, 1969.

[Te] Terras A., *Fourier analysis on finite groups and applications*, London Math. Soc. Stud. Texts, vol. 43, Cambridge Univ. Press, Cambridge, 1999.

[Ti] Tirao J., *Spherical functions*, Rev. Un. Mat. Argentina **28** (1976/77), 75–98.

[T] Travis D., *Spherical functions on finite groups*, J. Algebra **29** (1974), 65–76.

[V] Vaĭnerman L., *Gel′fand pairs of quantum groups, hypergroups and q-special functions*, Applications of Hypergroups and Related Measure Algebras (Seattle, WA, 1993), Contemp. Math., vol. 183, Amer. Math. Soc., Providence, RI, 1995, pp. 373–394.

[VS] Ваксман Л. Л., Сойбельман Я. С., *Алгебра функций на квантовой группе* $\mathrm{SU}(2)$, Функц. анал. и его прил. **22** (1988), №3, 1–14.

[W] Woronowicz S. L., *Compact matrix pseudogroups*, Comm. Math. Phys. **111** (1987), 613–665.

[Z] Zhu Y., *A commuting pair in Hopf algebras*, Proc. Amer. Math. Soc. **125** (1997), 2847–2851.

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