ON THE QUIVER-THEORETICAL QUANTUM YANG-BAXTER EQUATION

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Abstract. Quivers over a fixed base set form a monoidal category with tensor product given by pullback. The quantum Yang-Baxter equation, or more properly the braid equation, is investigated in this setting. A solution of the braid equation in this category is called a "solution" for short. Results of Etingof-Schedler-Soloviev, Lu-Yan-Zhu and Takeuchi on the set-theoretical quantum Yang-Baxter equation are generalized to the context of quivers, with groupoids playing the role of groups. The notion of "braided groupoid" is introduced. Braided groupoids are solutions and are characterized in terms of bijective 1-cocycles. The structure groupoid of a non-degenerate solution is defined; it is shown that it is braided groupoid. The reduced structure groupoid of a non-degenerate solution is also defined. Non-degenerate solutions are classified in terms of representations of matched pairs of groupoids. By linearization we construct star-triangular face models and realize them as modules over quasitriangular quantum groupoids introduced in papers by M. Aguiar, S. Natale and the author.

Introduction

The problem of classification of set-theoretical solutions to the quantum Yang-Baxter equation (QYBE) was raised by Drinfeld [D], who also mentioned the example of a subset of a group stable under conjugation. The question was considered by Etingof-Schedler-Soloviev [ESS, S] and Lu-Yan-Zhu [LYZ1]. In these works, an abstract characterization of solutions to the QYBE in group-theoretical terms was given. Indecomposable solutions with an underlying set with a prime number of elements were classified in [EGS]. Later, Takeuchi simplified some arguments and provided a unified presentation in terms of braided groups [T].

It is natural to extend Drinfeld's question and ask for solutions to the QYBE in an arbitrary monoidal category. In the linear or in the set-theoretical settings, or more generally in a symmetric monoidal category, the QYBE is equivalent to the braid equation. Actually, the main goal of the above mentioned articles is the study of the braid equation; information on the QYBE is obtained as a bonus. Such an equivalence does not exist in arbitrary, not symmetric, monoidal categories.

In this paper, the question of solutions to the braid equation is addressed in the special case of the category of quivers over a fixed set $\mathcal{P}$ of vertices; we refer to this as the quiver-theoretical braid equation. Results of Etingof-Schedler-Soloviev [ESS, S] and Lu-Yan-Zhu [LYZ1] on the set-theoretical quantum Yang-Baxter equation are generalized to the context of quivers, with groupoids playing the role of groups. Our approach, inspired by the presentation of Takeuchi [T], is different from the original papers [ESS, LYZ1, S]; it is systematically based on the notion of braided groupoid, whereas in those papers much emphasis was put on the derived structure groupoid.

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Any solution of the set-theoretical quantum Yang-Baxter equation gives rise to a solution of the quantum Yang-Baxter equation in the category of vector spaces, by linearization. In a similar way, any solution of the braid equation in the category of quivers over $\mathcal{P}$ gives rise by linearization to a solution of the braid equation in the category of bimodules over the function algebra $\Bbbk^{\mathcal{P}}$. It seems that solutions of the braid equation in categories of bimodules have not been considered systematically in the literature. There is however a relation with the dynamical QYBE, not explicit in the literature to the best of our knowledge.

Solutions of the quiver-theoretical quantum Yang-Baxter equation do appear in nature. Indeed, let $(\mathcal{V}, \mathcal{H})$ be any matched pair of groupoids; see subsection 1.5. The double $D(\mathcal{V}, \mathcal{H})$ is another matched pair of groupoids, introduced in [AA], playing an analogous role to the Drinfeld double of a Hopf algebra. Any representation of $D(\mathcal{V}, \mathcal{H})$ carries a natural braiding– in the same way as any representation of the Drinfeld double of a Hopf algebra; see [AA, Example 4.6]. Thus, any matched pair of groupoids gives rise to solutions of the quiver-theoretical quantum Yang-Baxter equation. Now, matched pairs of groupoids appear in nature; in fact a description of matched pairs of groupoids in terms of groups is given in [AN, Th. 2.16], and explicit examples are given in [AM, Section 3]. In particular, there is a matched pair of groupoids attached to a triple (D, V, H), where D is a finite group, and V, H are subgroups of D such that V intersects trivially any conjugate of H [AM, Subsection 3.3].

A natural question raised by the results of the present paper is the explicit description of all finite braided groupoids. This problem is addressed in [MM]. Essentially, one is reduced to consider triples (D, V, H) as in the preceding paragraph, with the groups V and H isomorphic, plus some extra data.

In Section 1, basic notions on quivers and groupoids are recalled. The apparently new construction of the free groupoid generated by a quiver is given. This is needed in Section 2 to define the structure groupoid of a non-degenerate solution. In Section 2, braided groupoids are investigated. The main results on characterization of non-degenerate solutions are stated and proved in Section 3. Our main result, Theorem 3.10, gives a classification of non-degenerate solutions via representations of suitable matched pairs of groupoids. This result generalizes [S, Th. 2.7] but the formulation is new even in the set-theoretical case. The concept of representations of matched pairs of groupoids was introduced in [AA]. In this paper, "positive" universal R-matrices for the quantum groupoids constructed in [AN] are presented. In Section 4, devoted to linearization, we construct star-triangular face models and realize some of them over quasitriangular quantum groupoids from [AN, AA]. Section 5 is an Appendix due to M. Takeuchi, devoted to the FRT-construction for matched pairs of groupoids.

We close this introduction with three remarks.

The category of quivers over $\mathcal{P}$, with fiber product as tensor product, does not appear to be symmetric in a reasonable way. However there are some substitutes of the symmetry and a formulation of the quiver-theoretical quantum Yang-Baxter equation is still possible (2.2), and there is still an equivalence between solutions to the braid equation and solutions to (2.2).

The analogy of some of the results in the set-theoretical and quiver-theoretical settings suggests that these might be particular cases of a general description of solutions to the braid equation in arbitrary monoidal categories with extra hypothesis (e. g. existence of equalizers and direct products).

After release of the first version of this paper, it was pointed out to the author that a solution of the set-theoretical Yang-Baxter equation is essentially the same thing as a birack; see [CES, FJK, FRS, SW, W] and references therein.

Acknowledgements. This work is a continuation of [AN, AA]. Aprovecho esta ocasión para agradecer a Sonia y Marcelo por compartir conmigo su entusiasmo e intuición. I also thank Professor M. Takeuchi for sending me a copy of [T]; the strategy in the present paper owes a lot to it.

1. Quivers and groupoids

1.1. Quivers.

Let $(\mathcal{A}, \mathcal{P}, \mathfrak{s}, \mathfrak{e})$ be a quiver; thus $\mathcal{A}$ and $\mathcal{P}$ are sets, with $\mathcal{P}$ non-empty, and $\mathfrak{s}, \mathfrak{e} : \mathcal{A} \rightrightarrows \mathcal{P}$ are functions. An element a of $\mathcal{A}$ is an "arrow" from its source $\mathfrak{s}(a)$ to its end $\mathfrak{e}(a)$. We shall fix $\mathcal{P}$ and say that '$\mathcal{A}$ is a quiver or $\mathcal{A}$ is a quiver over $\mathcal{P}$. Quivers are also called oriented graphs. We denote by $\mathcal{A}(P, Q)$ the set of arrows from P to Q, if $P, Q \in \mathcal{P}$; and $\mathcal{A}(P) = \mathcal{A}(P, P)$. Morphisms of quivers over $\mathcal{P}$ are defined in the usual way; they should be the identity on $\mathcal{P}$.

A quiver $\mathcal{B}$ differs from $\mathcal{A}$ in the orientation if it has the same arrows as $\mathcal{A}$ but with different $\mathfrak{s}$, $\mathfrak{e}$ for some arrows. For instance, the *opposite quiver* is $\mathcal{A}^{\rm op} = \mathcal{A} \times \{-1\}$; if $x \in \mathcal{A}$, then $x^{-1} := (x, -1)$ has $\mathfrak{s}(x^{-1}) = \mathfrak{e}(x)$, $\mathfrak{e}(x^{-1}) = \mathfrak{s}(x)$. Also, by abuse of notation, we set $(\mathcal{A}^{\rm op})^{\rm op} = \mathcal{A}$, and $(x^{-1})^{-1} = x$ for $x \in \mathcal{A}$.

If $\mathcal{A}$ and $\mathcal{B}$ are quivers over $\mathcal{P}$, then we can form the disjoint union $\mathcal{A} \coprod \mathcal{B}$ that is again a quiver over $\mathcal{P}$. The *double* of $\mathcal{A}$ is $\mathcal{D}\mathcal{A} := \mathcal{A} \coprod \mathcal{A}^{\rm op}$; it does not depend on the orientation of $\mathcal{A}$. The quiver $\mathcal{D}\mathcal{A}$ is occasionally denoted by $\overline{\mathcal{A}}$ in the literature.

Let n be a positive integer. A *path of length* n in $\mathcal{A}$ is a sequence $w = (x_1, \dots, x_n)$ of elements in $\mathcal{A}$ such that $\mathfrak{s}(x_{i+1}) = \mathfrak{e}(x_i)$, $1 \le i < n$; we shall denote it by $w = x_1 x_2 \dots x_n$. A *path of length* 0 is a symbol $\operatorname{id} P$, $P \in \mathcal{P}$. The set of all paths of length n in $\mathcal{A}$ is a quiver $\operatorname{Path}_n(\mathcal{A})$ with $\mathfrak{s}(w) = \mathfrak{s}(x_1)$, $\mathfrak{e}(w) = \mathfrak{e}(x_n)$ if $w = x_1 x_2 \dots x_n$ (if $n > 0$), and $\mathfrak{s}(\operatorname{id} P) = \mathfrak{e}(\operatorname{id} P) = P$ (if $n = 0$). The quiver of all paths in $\mathcal{A}$ is $\operatorname{Path}(\mathcal{A}) = \coprod_{n \ge 0} \operatorname{Path}_n(\mathcal{A})$.

The quiver $\mathcal{A}$ induces an equivalence relation on $\mathcal{P}$: $P \approx Q$ if and only if there exists $w \in \operatorname{Path}(\mathcal{D}\mathcal{A})$ with $\mathfrak{s}(w) = P$, $\mathfrak{e}(w) = Q$, $P, Q \in \mathcal{P}$. Then $\mathcal{A}$ is *connected* if $P \approx Q$ for all $P, Q \in \mathcal{P}$.

Any map $p : \mathcal{L} \to \mathcal{P}$ can be considered as a quiver with $\mathfrak{s} = \mathfrak{e} = p$; such a quiver shall be called a *loop bundle*. In this context we shall sometimes use the fiber notation $\mathcal{L}_P := \mathcal{L}(P) = p^{-1}(P)$.

A quiver $\mathcal{A}$ gives rise to two loop bundles: $\mathfrak{s} : \mathcal{A} \to \mathcal{P}$ and $\mathfrak{e} : \mathcal{A} \to \mathcal{P}$; we shall denote them by $\mathcal{A}^{\mathfrak{s}}$ and $\mathcal{A}^{\mathfrak{e}}$, respectively. It might be useful to visualize them as follows:

$$\mathcal{A}^{\mathfrak{s}} = \coprod_{P \in \mathcal{P}} \mathcal{A}^{\mathfrak{s}}(P), \qquad \mathcal{A}^{\mathfrak{s}}(P) = \{(y, y^{-1}) : y \in \mathcal{A}(P, Q),\, Q \in \mathcal{P}\}, \tag{1.1}$$

$$\mathcal{A}^{\mathfrak{e}} = \coprod_{Q \in \mathcal{P}} \mathcal{A}^{\mathfrak{e}}(Q), \qquad \mathcal{A}^{\mathfrak{e}}(Q) = \{(x^{-1}, x) : x \in \mathcal{A}(P, Q),\, P \in \mathcal{P}\}. \tag{1.2}$$

Also, we denote by $x \mapsto \overline{x} = (x^{-1}, x)$ the canonical map $\mathcal{A} \to \mathcal{A}^{\mathfrak{e}}$.

If $T : \mathcal{A} \to \mathcal{B}$ is a morphism of quivers, then $\overline{T} : \mathcal{A}^{\mathfrak{e}} \to \mathcal{B}^{\mathfrak{e}}$ is the morphism of loop bundles given by $\overline{T}(\overline{x}) = \overline{T(x)}$.

If $\mathcal{A}$, $\mathcal{B}$ are quivers over $\mathcal{P}$, then $\mathcal{A}\, {}_{\mathfrak{e}}{\times}_{\mathfrak{s}}\, \mathcal{B} = \{(a, b) \in \mathcal{A} \times \mathcal{B} : \mathfrak{e}(a) = \mathfrak{s}(b)\}$ is a quiver over $\mathcal{P}$ with $\mathfrak{s}(a, b) = \mathfrak{s}(a)$, $\mathfrak{e}(a, b) = \mathfrak{e}(b)$. Thus, the category $\operatorname{Quiv}(\mathcal{P})$ of quivers over $\mathcal{P}$ is monoidal, with $\otimes = {}_{\mathfrak{e}}{\times}_{\mathfrak{s}}$ and with unit object $(\mathcal{P}, \mathcal{P}, \operatorname{id}, \operatorname{id})$.

This monoidal category does not seem to be symmetric in any reasonable way. We have nevertheless two natural isomorphisms playing the role of a "weak" symmetry. The first is the natural isomorphism $\vartheta : \mathcal{B}^{\rm op}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}}\mathcal{A}^{\rm op} \to (\mathcal{A}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}}\mathcal{B})^{\rm op}$ given by

$$\vartheta(y^{-1}, x^{-1}) = (x, y), \qquad (x, y) \in \mathcal{A}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}}\mathcal{B}. \tag{1.3}$$

The second possibility is very similar. Let $\mathcal{B}\,{}_{\mathfrak{s}}\!\times_{\mathfrak{e}}\mathcal{A} = \{(b, a) \in \mathcal{B} \times \mathcal{A} : \mathfrak{s}(b) = \mathfrak{e}(a)\}$, a quiver over $\mathcal{P}$ with $\mathfrak{s}(b, a) = \mathfrak{s}(a)$, $\mathfrak{e}(b, a) = \mathfrak{e}(b)$. Then we define $\tau : \mathcal{A}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}}\mathcal{B} \to \mathcal{B}\,{}_{\mathfrak{s}}\!\times_{\mathfrak{e}}\mathcal{A}$ by

$$\tau(x, y) = (y, x), \qquad (x, y) \in \mathcal{A}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}}\mathcal{B}. \tag{1.4}$$

These are related as follows. Let $\mu : \mathcal{B}\,{}_{\mathfrak{s}}\!\times_{\mathfrak{e}}\mathcal{A} \to (\mathcal{B}^{\rm op}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}}\mathcal{A}^{\rm op})^{\rm op}$ be given by $\mu(y, x) = (y^{-1}, x^{-1})$. Then the following diagram commutes:

$$\begin{array}{ccc} \mathcal{A}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}}\mathcal{B} & \xrightarrow{\quad\tau\quad} & \mathcal{B}\,{}_{\mathfrak{s}}\!\times_{\mathfrak{e}}\mathcal{A} \\ & {\scriptstyle\vartheta}\nwarrow \qquad \swarrow{\scriptstyle\mu} & \\ & (\mathcal{B}^{\rm op}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}}\mathcal{A}^{\rm op})^{\rm op}\,. & \end{array}$$

1.2. **Groupoids.**

Let $\mathcal{G}$ be a groupoid with base $\mathcal{P}$ and source and end maps $\mathfrak{s}, \mathfrak{e} : \mathcal{G} \rightrightarrows \mathcal{P}$. We identify $\mathcal{P}$ with a subset of $\mathcal{G}$ via the identity. We indicate the composition $m(f, g)$ of two elements in a groupoid by juxtaposition: $m(f, g) = fg$, and not gf. A group bundle is a groupoid $\mathcal{N}$ with source = end; thus $\mathcal{N} = \coprod_{P\in\mathcal{P}} \mathcal{N}(P)$.

A morphism of groupoids $T : \mathcal{G} \to \mathcal{K}$ is a map preserving the product; thus, it preserves also source and end, and induces a map between the bases. If $\mathcal{G}$ and $\mathcal{K}$ have the same base $\mathcal{P}$, we shall say that $T : \mathcal{G} \to \mathcal{K}$ is a morphism of groupoids *over* $\mathcal{P}$ if the restriction $\mathcal{P} \to \mathcal{P}$ is the identity. Groupoids over $\mathcal{P}$ form a category $\mathrm{Gpd}(\mathcal{P})$.

A wide subgroupoid is a subgroupoid such that the inclusion is a morphism of groupoids over $\mathcal{P}$. A subgroup bundle of a groupoid is a subgroupoid that is a group bundle. There is a largest wide subgroup bundle of a groupoid $\mathcal{G}$, namely $\mathcal{G}^{\rm bundle} = \coprod_{P\in\mathcal{P}} \mathcal{G}(P)$; that is, we forget the arrows between distinct points.

If $(\mathcal{N}_i)_{i\in I}$ is a family of (wide) subgroupoids of a groupoid $\mathcal{G}$ then $\bigcap_{i\in I} \mathcal{N}_i$, defined by $\bigcap_{i\in I} \mathcal{N}_i(P;Q) = \bigcap_{i\in I} (\mathcal{N}_i(P;Q))$, is a (wide) subgroupoid of $\mathcal{G}$.

A groupoid over $\mathcal{P}$ is a group object in the monoidal category $\mathrm{Quiv}(\mathcal{P})$. Any groupoid being a quiver, we shall use all the terminology above also for groupoids. Clearly, $\mathcal{G}(P)$ is a group and it acts freely and transitively on the left on $\mathcal{G}(P, Q)$ and on the right on $\mathcal{G}(Q, P)$, for any $P, Q \in \mathcal{P}$.

Basic examples of groupoids are:

- A group G, considered as the set of arrows of a category with a single object.
- An equivalence relation R on $\mathcal{P}$; $\mathfrak{s}$ and $\mathfrak{e}$ are respectively the first and the second projection, and the composition is given by $(x, y)(y, v) = (x, v)$.

The equivalence relation where all the elements of $\mathcal{P}$ are related is denoted $\mathcal{P}^2$ and called the *coarse* groupoid on $\mathcal{P}$.

If $\mathcal{G} \rightrightarrows \mathcal{P}$ is any groupoid, then $\mathcal{G} \simeq \coprod_{X\in\mathcal{P}/\approx} \mathcal{G}_X$. Here $\mathcal{G}_X$ is the subgroupoid on the base X defined by $\mathcal{G}_X(x,y) = \mathcal{G}(x,y)$, for all $x,y \in X$. Furthermore, $\mathcal{G}_X = \mathcal{G}(x) \times X^2$ for any $x \in X$, $X \in \mathcal{P}/\approx$. This description can be viewed as a structure theorem for groupoids.

Let $\mathcal{N}$ be a wide subgroup bundle of a groupoid $\mathcal{G}$ over $\mathcal{P}$. Then there are quivers $\mathcal{N}\backslash\mathcal{G}$, $\mathcal{G}/\mathcal{N}$, equipped with surjective morphisms of quivers $\mathcal{G} \to \mathcal{N}\backslash\mathcal{G}$ and $\mathcal{G} \to \mathcal{G}/\mathcal{N}$, defined by

$$(\mathcal{N}\backslash\mathcal{G})(P,Q) = \mathcal{N}(P)\backslash\mathcal{G}(P,Q), \qquad (\mathcal{G}/\mathcal{N})(P,Q) = \mathcal{G}(P,Q)/\mathcal{N}(Q), \qquad P,Q \in \mathcal{P}.$$

We shall say that $\mathcal{N}$ is *normal* if the following equivalent conditions hold for any $P,Q \in \mathcal{P}$:

- For any $x \in \mathcal{G}(P,Q)$ and $n \in \mathcal{N}(Q)$, $xnx^{-1} \in \mathcal{N}(P)$.
- For any $x \in \mathcal{G}(P,Q)$ and $n \in \mathcal{N}(Q)$, there exists $m \in \mathcal{N}(P)$ such that $xn = mx$.

If $(\mathcal{N}_i)_{i\in I}$ is a family of normal subgroup bundles of $\mathcal{G}$, then $\bigcap_{i\in I} \mathcal{N}_i$ is a normal subgroup bundle.

If $\mathcal{N}$ is a normal wide subgroup bundle of $\mathcal{G}$, then $\mathcal{G}/\mathcal{N}$ has groupoid multiplication, with the canonical map $\pi : \mathcal{G} \to \mathcal{G}/\mathcal{N}$ being a morphism of groupoids.

Let $T : \mathcal{G} \to \mathcal{K}$ be a morphism of groupoids over a $\mathcal{P}$. The *kernel* of T is the (wide and normal) subgroup bundle

$$\operatorname{Ker} T = \{g \in \mathcal{G} : T(g) \in \mathcal{P}\} = \coprod_{P\in\mathcal{P}} \operatorname{Ker} T(P), \qquad \text{where } \operatorname{Ker} T(P) = \ker(T : \mathcal{G}(P) \to \mathcal{K}(P)).$$

Let $f,g \in \mathcal{G}$. Then $T(f) = T(g)$ iff there exists $n \in \operatorname{Ker} T$ with $f = ng$ iff there exists $m \in \operatorname{Ker} T$ with $f = gm$. That is, $\operatorname{Im} T \simeq \mathcal{G}/\operatorname{Ker} T$.

The largest subgroup bundle $\mathcal{G}^{\text{bundle}}$ is clearly normal; the quotient $\mathcal{G}/\mathcal{G}^{\text{bundle}}$ is the groupoid associated to the equivalence relation $\approx$.

1.3. The free groupoid generated by a quiver.

It is natural to look for the left adjoint of the obvious forgetful functor from $\operatorname{Gpd}(\mathcal{P})$ to $\operatorname{Quiv}(\mathcal{P})$. This leads us to the construction of the free groupoid generated by a quiver.

Theorem 1.1. *Let $\mathcal{A}$ be a quiver over $\mathcal{P}$. Then there exists a groupoid $F(\mathcal{A})$ over $\mathcal{P}$ provided with a morphism of quivers $\iota : \mathcal{A} \to F(\mathcal{A})$ satisfying the usual universal property:*

If $\mathcal{G}$ is a groupoid over $\mathcal{P}$ provided with a morphism of quivers $\nu : \mathcal{A} \to \mathcal{G}$, then there is a unique map of groupoids $\widehat{\nu} : F(\mathcal{A}) \to \mathcal{G}$ such that $\nu = \widehat{\nu}\,\iota$.

The groupoid $F(\mathcal{A})$ is unique up to isomorphisms with respect to this property.

Proof. We first consider the quiver $\operatorname{Path}(\mathcal{D}\mathcal{A})$; its elements will be called "words in the alphabet $\mathcal{A} \cup \mathcal{A}^{\text{op}}$". A word $w = x_1x_2\dots x_n$, $x_i \in \mathcal{D}\mathcal{A}$, is *reduced* if either $n = 0$, or there is no i such that $x_i = x_{i+1}^{-1}$.

Let $w = x_1x_2\dots x_n$, $x_i \in \mathcal{D}\mathcal{A}$, be a word of length $n > 0$ and assume there exists i, $1 \le i < n$, such that $x_i = x_{i+1}^{-1}$. Then set $w' := x_1x_2\dots x_{i-1}x_{i+2}\dots x_n$, if $n > 2$, or $w' := \operatorname{id} \mathfrak{s}(x_1)$ if $n = 2$. The word w' is called an *elementary reduction* of w. Furthermore, a word $\widetilde{w}$ is called a *reduction* of w if it can be attained from w by a sequence of elementary reductions.

Reduction generates an equivalence relation in the usual way. Two words u and v are *equivalent*, denoted $u \sim v$, if there is a sequence of words $w_1, w_2, \dots, w_N$ with $N \ge 1$, $u = w_1$, $v = w_N$ and either

w_i a reduction of w_{i+1}, or w_{i+1} a reduction of w_i, for all i, $1 \le i < N$. This is clearly an equivalence relation and the class of a word u is denoted $[u]$. Furthermore,

Step 1. *In any class there is one and only one reduced word.*

Let w be a word. By a standard recurrence argument, there is at least one reduced word in $[w]$, which is a reduction of w. To prove the uniqueness, we consider the "W-process" for a word. We first set $\mathcal{W}_0 = \operatorname{id} \mathfrak{s}(w)$. If the length of w is $n > 0$, say $w = x_1 x_2 \dots x_n$, then we define recursively

$$\mathcal{W}_1 = x_1,$$

$$\mathcal{W}_{i+1} = \begin{cases} X, & \text{if } \mathcal{W}_i \text{ is of the reduced form } X x_{i+1}^{-1}, \\ \mathcal{W}_i x_{i+1}, & \text{if } \mathcal{W}_i \text{ is } \textit{not} \text{ of the reduced form } X x_{i+1}^{-1}. \end{cases}$$

Then $\mathcal{W}_0, \dots, \mathcal{W}_n$ are all reduced (by induction) and $\mathcal{W}_n = w$ if w is reduced. We have then a map $\operatorname{Path}(\mathcal{DA}) \to \{p \in \operatorname{Path}(\mathcal{DA}) : p \text{ is reduced}\}$, $w \mapsto \mathcal{W}_n$, which is a retraction of the inclusion. We will now check that $w \sim u$ implies $\mathcal{W}_n = \mathcal{U}_m$, where m is the length of u, and $\mathcal{U}_0, \dots, \mathcal{U}_m$ is the W-process for u. In particular if both w and u are reduced and equivalent, then necessarily $w = u$.

So, assume that w is an elementary reduction of $u = x_1 \dots x_r y y^{-1} x_{r+1} \dots x_n$, with $y \in \mathcal{DA}$. Clearly, $\mathcal{U}_0 = \mathcal{W}_0, \dots, \mathcal{W}_r = \mathcal{U}_r$. Now two cases can happen:

a) $\mathcal{W}_r = \mathcal{U}_r$ is of the reduced form Xy^{-1}. Then X is *not* of the reduced form Yy. Thus $\mathcal{U}_{r+1} = X$ and $\mathcal{U}_{r+2} = Xy^{-1} = \mathcal{W}_r$. Hence $\mathcal{U}_{r+2+i} = \mathcal{W}_{r+i}$, $i \ge 0$.

b) $\mathcal{W}_r = \mathcal{U}_r$ is *not* of the reduced form Xy^{-1}. Then $\mathcal{U}_{r+1} = \mathcal{U}_r y$ and $\mathcal{U}_{r+2} = \mathcal{U}_r = \mathcal{W}_r$. Hence, again $\mathcal{U}_{r+2+i} = \mathcal{W}_{r+i}$, $i \ge 0$.

This finishes the proof of the step.

The map $\operatorname{Path}(\mathcal{DA}) \, {}_{\mathfrak{e}}\!\times_{\mathfrak{s}} \operatorname{Path}(\mathcal{DA}) \to \operatorname{Path}(\mathcal{DA})$, $(x_1 \dots x_n, y_1 \dots y_m) \mapsto x_1 \dots x_n y_1 \dots y_m$, if $n > 0$, $m > 0$; $(\operatorname{id} \mathfrak{s}(y_1), y_1 \dots y_m) \mapsto y_1 \dots y_m$, etc., is called the *juxtaposition*. Let $F(\mathcal{A}) := \operatorname{Path}(\mathcal{DA}) / \sim$.

Step 2. *Juxtaposition induces a groupoid structure on $F(\mathcal{A})$.*

We first claim that juxtaposition descends to a map $\cdot : F(\mathcal{A}) \, {}_{\mathfrak{e}}\!\times_{\mathfrak{s}} F(\mathcal{A}) \to F(\mathcal{A})$. We omit the straightforward verification of the claim: "$w \sim \tilde{w}$, $u \sim \tilde{u}$ and $\mathfrak{e}(w) = \mathfrak{s}(u)$ implies $wu \sim \tilde{w}\tilde{u}$." Since $([u][v])[w] = [uvw] = [u]([v][w])$, $\cdot$ is associative. The elements $[\operatorname{id} P]$, $P \in \mathcal{P}$, are partial identities of the product $\cdot$. If $w = x_1 x_2 \dots x_n$, then set $w^{-1} = x_n^{-1} x_{n-1}^{-1} \dots x_1^{-1}$. Then $[w]^{-1} := [w^{-1}]$ is the inverse of $[w]$. Thus $F(\mathcal{A})$ is a groupoid.

Step 3. *The map $\iota : \mathcal{A} \to F(\mathcal{A})$, $x \mapsto [x]$, is an injective morphism of quivers and satisfies the required universal property.*

Injectivity of ι follows from Step 1. If $\mathcal{G}$ is a groupoid and $\nu : \mathcal{A} \to \mathcal{G}$ is a morphism of quivers, then ν can be extended to $\mathcal{A}^{\mathrm{op}}$, then to $\operatorname{Path}(\mathcal{DA})$, and finally to a morphism of groupoids $\hat{\nu} : F(\mathcal{A}) \to \mathcal{G}$, which is easily seen to be unique. □

Remark 1.2. This proof is an adaptation of the construction of the free group generated by a set.

Examples 1.3.
- The groupoid $F(\mathcal{A})$ does not depend on the orientation of $\mathcal{A}$.
- If $\mathcal{P}$ has one element, then the groupoid $F(\mathcal{A})$ is just the free group generated by $\mathcal{A}$.
- Suppose that $\mathcal{P}$ has exactly two elements P and Q. If $\mathcal{A}$ consists of only one arrow from P to Q, then $F(\mathcal{A}) \simeq \mathcal{P}^2$. If $\mathcal{A}$ consists only of arrows from P to Q, say $\operatorname{card} \mathcal{A}(P,Q) = n$, then $F(\mathcal{A}) \simeq F^{n-1} \times \mathcal{P}^2$, where F^{n-1} is the free group in $n-1$ variables.

Let $\mathcal{A}$ be any quiver and let $\mathcal{R}$ be any subset of $\coprod_{P \in \mathcal{P}} F(\mathcal{A})(P)$. Then the *groupoid presented by $\mathcal{A}$ with relations $\mathcal{R}$* is the quotient of the free groupoid $F(\mathcal{A})$ by the smallest normal wide subgroupoid containing $\mathcal{R}$.

Let $\mathcal{G}$ be a groupoid and $\mathcal{A}$, a sub-quiver of $\mathcal{G}$. The (wide) *subgroupoid generated by $\mathcal{A}$* is $< \mathcal{A} >:=$ the image of the induced map $F(\mathcal{A}) \to \mathcal{G}$. In words, the elements of $< \mathcal{A} >$ are compositions of elements of $\mathcal{A}$ or their inverses. We say that $\mathcal{A}$ *generates* $\mathcal{G}$ if $< \mathcal{A} >= \mathcal{G}$.

1.4. **Actions of groupoids.**

Let $\mathcal{G}$ be a groupoid with base $\mathcal{P}$ and let $p : \mathcal{E} \to \mathcal{P}$ be a map. A *left action* of $\mathcal{G}$ on p is a map $\rightharpoonup : \mathcal{G} \,{}_{\mathfrak{e}}\!\times_p \mathcal{E} \to \mathcal{E}$ such that

$$p(g \rightharpoonup e) = \mathfrak{s}(g), \qquad g \rightharpoonup (h \rightharpoonup e) = gh \rightharpoonup e, \qquad \operatorname{id} p(e) \rightharpoonup e = e, \tag{1.5}$$

for all $g, h \in \mathcal{G}$, $e \in \mathcal{E}$ composable appropriately. Intertwiners of actions of $\mathcal{G}$ on $p : \mathcal{E} \to \mathcal{P}$ and $p' : \mathcal{E}' \to \mathcal{P}$ are defined in the usual way.

Given a set X there is an action of $\mathcal{G}$ on $p : \mathcal{P} \times X \to \mathcal{P}$, where p is the first projection, given by $g \rightharpoonup (\mathfrak{e}(g), x) = (\mathfrak{s}(g), x)$ for all $x \in X$, $g \in \mathcal{G}$. An action on $p : \mathcal{E} \to \mathcal{P}$ is *trivial* if there exists a set X and a bijective intertwiner of actions $\nu : \mathcal{E} \to \mathcal{P} \times X$.

Definition 1.4. Let $p : \mathcal{E} \to \mathcal{P}$ be a map. The groupoid $\mathfrak{aut}\, p$, or indistinctly $\mathfrak{aut}\, \mathcal{E}$, is defined by

$$\mathfrak{aut}\, p = \{(P, x, Q) : P, Q \in \mathcal{P}, \text{ and } x : \mathcal{E}_Q \to \mathcal{E}_P \text{ is a bijection}\};$$

with source and end $\mathfrak{s}, \mathfrak{e} : \mathfrak{aut}\, p \to \mathcal{P}$ given by $\mathfrak{s}(P, x, Q) = P$, $\mathfrak{e}(P, x, Q) = Q$, $(P, x, Q) \in \mathfrak{aut}\, p$; with composition $(P, x, Q)(Q, y, R) = (P, xy, R)$, $(P, x, Q), (Q, y, R) \in \mathfrak{aut}\, p$; and with identities $\operatorname{id} P = (P, \operatorname{id}, P)$, $P \in \mathcal{P}$.

Then there is an equivalence between left actions of $\mathcal{G}$ on p, and morphisms of groupoids $\mathcal{G} \to \mathfrak{aut}\, p$. Namely, if $\rightharpoonup$ is a left action, then the corresponding morphism $\rho : \mathcal{G} \to \mathfrak{aut}\, p$ is $\rho(g) = (\mathfrak{s}(g), g \rightharpoonup \_, \mathfrak{e}(g))$, $g \in \mathcal{G}$.

Similarly, a *right action* of $\mathcal{G}$ on p is a map $\leftharpoonup : \mathcal{E} \,{}_p\!\times_{\mathfrak{s}} \mathcal{G} \to \mathcal{E}$ such that

$$p(e \leftharpoonup g) = \mathfrak{e}(g), \qquad (e \leftharpoonup g) \leftharpoonup h = e \leftharpoonup gh, \qquad e \leftharpoonup \operatorname{id} p(e) = e, \tag{1.6}$$

for all $g, h \in \mathcal{G}$, $e \in \mathcal{E}$ composable appropriately. Left and right actions are equivalent, by the rule $e \leftharpoonup g = g^{-1} \rightharpoonup e$.

1.5. **Matched pairs of groupoids.**

It is convenient to introduce some alternative notation for quivers and groupoids. We shall say that a groupoid $\mathcal{V}$ is denoted *vertically* if the source and end are named respectively $t, b : \mathcal{V} \rightrightarrows \mathcal{P}$, where t means top and b means bottom. The elements of $\mathcal{V}$ will be consequently depicted as vertical arrows going down. Similarly, a groupoid $\mathcal{H}$ is denoted *horizontally* if the source and end are named respectively $l, r : \mathcal{H} \rightrightarrows \mathcal{P}$, where l means left and r means right; the elements of $\mathcal{H}$ will be depicted as horizontal arrows going right.

Definition 1.5. [M, Definition 2.14]. A *matched pair of groupoids* is a pair of groupoids $(\mathcal{V}, \mathcal{H})$ over $\mathcal{P}$ with $\mathcal{V}$ denoted vertically and $\mathcal{H}$ horizontally, endowed with a left action $\rightharpoonup \, : \mathcal{H}\,{}_r\!\times_t \mathcal{V} \to \mathcal{V}$ of $\mathcal{H}$ on $t : \mathcal{V} \to \mathcal{P}$, and a right action $\leftharpoonup \, : \mathcal{H}\,{}_r\!\times_t \mathcal{V} \to \mathcal{H}$ of $\mathcal{V}$ on $r : \mathcal{H} \to \mathcal{P}$, satisfying

(1.7) $b(x \rightharpoonup g) = l(x \leftharpoonup g)$,

(1.8) $x \rightharpoonup fg = (x \rightharpoonup f)((x \leftharpoonup f) \rightharpoonup g)$,

(1.9) $xy \leftharpoonup g = (x \leftharpoonup (y \rightharpoonup g))(y \leftharpoonup g)$,

for all $f, g \in \mathcal{V}$, $x, y \in \mathcal{H}$ such that the compositions are possible.

Given $P \in \mathcal{P}$, there are two identities: $\mathrm{id}_{\mathcal{H}} P \in \mathcal{H}$ and $\mathrm{id}_{\mathcal{V}} P \in \mathcal{V}$. We omit the subscript unless some emphasis is needed.

Definition 1.6. [M]. Let $\mathcal{D} \rightrightarrows \mathcal{P}$ be a groupoid. An *exact factorization* of $\mathcal{D}$ is a pair of wide subgroupoids $\mathcal{V}$ AND $\mathcal{H}$, such that the multiplication map $\mathcal{V}\,{}_b \times_l \mathcal{H} \to \mathcal{D}$ is a bijection.

We shall not give the complete definition of a vacant double groupoid (due to Ehresmann), see [AN] for a full discussion and some historical references. Informally, a double groupoid is a set of boxes with two partial compositions, a horizontal one and a vertical one. Each box has horizontal and vertical sides; the set of all horizontal sides form a groupoid on their own, and the same holds for the vertical sides. These two groupoids share the set of points, which are the corners of the boxes. In short, a double groupoid is a collection of sets and maps

$$\begin{array}{ccccc} & \mathfrak{B} & \overset{t,b}{\rightrightarrows} & \mathcal{H} & \\ l,r \downdownarrows & & & & \downdownarrows l,r \\ & \mathcal{V} & \underset{t,b}{\rightrightarrows} & \mathcal{P} & \end{array}$$

such that all four sides in this diagram are groupoids, satisfying some compatibility conditions.

A double groupoid is vacant if any pair of a horizontal and a vertical side with a common point determines a unique box.

We recall some notation needed later. If X is a box, then X^{-1} is the box obtained by inverting the horizontal and the vertical arrows. If $g \in \mathcal{V}$ with $t(g) = P$ and $b(g) = Q$, then the horizontal identity of g is the box

$$\mathbf{id}\, g = g \begin{array}{|c|} \hline \overset{\mathrm{id}_{\mathcal{H}} P}{} \\ \underset{\mathrm{id}_{\mathcal{H}} Q}{} \\ \hline \end{array} g.$$

The vertical identity of a horizontal arrow is defined similarly.

Proposition 1.7. [M, Theorems 2.10 and 2.15] *The following notions are equivalent.*

1. *Matched pairs of groupoids.*
2. *Groupoids with an exact factorization.*
3. *Vacant double groupoids.*

We sketch the main parts of the correspondence needed later. See [M, Theorems 2.10 and 2.15] and also [AN, Prop 2.9] for more details.

Proof. If $(\mathcal{V}, \mathcal{H})$ is a matched pair of groupoids, then the *diagonal* groupoid $\mathcal{V} \bowtie \mathcal{H} \rightrightarrows \mathcal{P}$ is defined as follows: $\mathcal{V} \bowtie \mathcal{H} := \mathcal{V}\,{}_b\!\times_l \mathcal{H}$, with composition $(f, y)(h, z) = (f(y \rightharpoonup h), (y \leftharpoonup h)z)$, with source $\mathfrak{s} : \mathcal{V} \bowtie \mathcal{H} \to \mathcal{P}$, $\mathfrak{s}(f, y) = t(f)$, and with end $\mathfrak{e} : \mathcal{V} \bowtie \mathcal{H} \to \mathcal{P}$, $\mathfrak{e}(f, y) = r(y)$. Clearly, $\mathcal{V}$ and $\mathcal{H}$ can be identified with subgroupoids of $\mathcal{V} \bowtie \mathcal{H}$ forming an exact factorization.

Conversely, let $\mathcal{V}$ and $\mathcal{H}$ be an exact factorization of a groupoid $\mathcal{D}$; that is, for any $\alpha \in \mathcal{D}$, there exist unique $f \in \mathcal{V}$, $y \in \mathcal{H}$, such that $\alpha = fy$. Let $\mathcal{H} \xleftarrow{\leftharpoonup} \mathcal{H}\,{}_r\!\times_t \mathcal{V} \xrightarrow{\rightharpoonup} \mathcal{V}$ be given by $xg = (x \rightharpoonup g)(x \leftharpoonup g)$, $(x, g) \in \mathcal{H}\,{}_r\!\times_t \mathcal{V}$. Then $\mathcal{V}$, $\mathcal{H}$, together with these actions, form a matched pair.

Similarly, if $(\mathcal{V}, \mathcal{H})$ is a matched pair of groupoids then we define $\mathfrak{B} := \mathcal{H}\,{}_r \times_t \mathcal{V}$. We represent $X = (x, g) \in \mathcal{H}\,{}_r \times_t \mathcal{V}$ by

$$X = \begin{array}{ccc} & x & \\ x \rightharpoonup g & \boxed{} & g. \\ & x \leftharpoonup g & \end{array}$$

Then $\mathcal{T} = (\mathfrak{B}, \mathcal{V}, \mathcal{H}, \mathcal{P})$ is a vacant double groupoid. For later use, we record the description of the groupoid $\mathfrak{B} \rightrightarrows \mathcal{V}$: $l(x, g) = x \rightharpoonup g$, $r(x, g) = g$, $(x, g)(y, h) = (xy, h)$ if $g = y \rightharpoonup h$. The construction is reversible and gives the opposite implication. □

1.6. **Semidirect products.**

Let $\mathcal{V}$ be a groupoid denoted vertically. Assume that $\mathcal{H} = \mathcal{N}$ is a group bundle, with $p := l = r$. The *trivial* action of $\mathcal{N}$ on $\mathcal{V}$ is $x \rightharpoonup g = g$, $(x, g) \in \mathcal{N}\,{}_p\!\times_t \mathcal{V}$. Also, a right action of $\mathcal{V}$ on $\mathcal{N}$ is *by group bundle automorphisms* if $xy \leftharpoonup g = (x \leftharpoonup g)(y \leftharpoonup g)$, $x, y \in \mathcal{N}$, $g \in \mathcal{V}$, composable.

It is easy to see that a right action and the trivial action form a matched pair of groupoids in the sense of Definition 1.5 iff it the right action is by group bundle automorphisms. We shall denote the corresponding diagonal groupoid by $\mathcal{V} \ltimes \mathcal{N}$ and call it a *semidirect product.* The projection $\mathcal{V} \ltimes \mathcal{N} \to \mathcal{V}$ is a morphism of groupoids, and has a section of groupoids $S : \mathcal{V} \to \mathcal{V} \ltimes \mathcal{N}$, $S(f) = (f, \mathrm{id}\, b(f))$.

Conversely, let $T : \mathcal{G} \to \mathcal{K}$ be a morphism of groupoids over $\mathcal{P}$. Then $\mathcal{G}$ acts on the kernel $\mathcal{N}$ of T by the adjoint action: $n \leftharpoonup g = g^{-1} n g$. If there is a section $S : \mathcal{K} \to \mathcal{G}$, then $\mathcal{G} \simeq \mathcal{K} \ltimes \mathcal{N}$.

The structure theorem of groupoids can be phrased in this language as the isomorphism of groupoids $\mathcal{G} \simeq \mathcal{R} \times \mathcal{G}^{\mathrm{bundle}}$ where $\mathcal{R}$ is the groupoid associated to the equivalence relation $\approx$.

1.7. **Actions of matched pairs of groupoids.**

Let $(\mathcal{V}, \mathcal{H})$ be a matched pair of groupoids over $\mathcal{P}$.

Definition 1.8. [AA]. A (set-theoretic) *representation* of $(\mathcal{V}, \mathcal{H})$ is a triple $(\mathcal{A}, \rightharpoonup, |\,|)$, where

- $\mathcal{A}$ is a quiver over $\mathcal{P}$,
- $\rightharpoonup : \mathcal{H}\,{}_r\times_{\mathfrak{s}} \mathcal{A} \to \mathcal{A}$ is a left action of $\mathcal{H}$ on $\mathfrak{s}$, and
- $|\,| : \mathcal{A} \to \mathcal{V}$ is a morphism of quivers over $\mathcal{P}$, called the grading, such that

$$|x \rightharpoonup a| = x \rightharpoonup |a|, \qquad (x, a) \in \mathcal{H}\,{}_r\times_{\mathfrak{s}} \mathcal{A}. \tag{1.10}$$

We shall say simply that "$\mathcal{A}$ is a representation of $(\mathcal{V}, \mathcal{H})$".

Morphisms of representations of $(\mathcal{V}, \mathcal{H})$ are morphisms of quivers intertwining the actions of $\mathcal{H}$ and preserving the grading $|\,|$. Thus representations of $(\mathcal{V}, \mathcal{H})$ form a category $\operatorname{Rep}(\mathcal{V}, \mathcal{H})$; this is a monoidal subcategory of $\operatorname{Quiv}(\mathcal{P})$. Namely, if $\mathcal{A}$ and $\mathcal{B}$ are two representations of $(\mathcal{V}, \mathcal{H})$, then $\mathcal{A}\,{}_{\mathfrak{e}}\times_{\mathfrak{s}} \mathcal{B}$ is also a representation of $(\mathcal{V}, \mathcal{H})$, with respect to the action and grading given by

$$x \rightharpoonup (a, b) = (x \rightharpoonup a, (x \leftharpoonup |a|) \rightharpoonup b), \tag{1.11}$$

$$|(a, b)| = |a||b|, \tag{1.12}$$

$x \in \mathcal{H}$, $(a, b) \in \mathcal{A}\,{}_{\mathfrak{e}}\times_{\mathfrak{s}} \mathcal{B}$. See [AA] for details.

Remark 1.9. The groupoid $\mathcal{V}$ itself becomes a representation of $(\mathcal{V}, \mathcal{H})$, in fact the final object in the category $\operatorname{Rep}(\mathcal{V}, \mathcal{H})$, via the maps $p = t$, $q = b : \mathcal{V} \to \mathcal{P}$, the identity grading $|g| = g$, and the action $\mathcal{H}\,{}_r\times_t \mathcal{V} \xrightarrow{\rightharpoonup} \mathcal{V}$ from the definition of matched pair. See [AA] for more examples.

Remark 1.10. We can state the preceding notion in terms of the associated vacant double groupoid. A representation of $(\mathcal{V}, \mathcal{H})$ is the same as a left action of the groupoid $\mathfrak{B} \rightrightarrows \mathcal{V}$. In fact, we have:

(a). Let $\mathcal{V}$ be a quiver over $\mathcal{P}$ and let $p : \mathcal{E} \to \mathcal{V}$ be a map. Then $\mathcal{E}$ is a quiver over $\mathcal{P}$ with source $\mathfrak{s} \circ p$ and end $\mathfrak{e} \circ p$. Moreover p is a morphism of quivers. We have a functor from the category of sets over $\mathcal{V}$ to $\operatorname{Quiv}(\mathcal{P})$.

(b). Let $\mathcal{V}$ be a groupoid over $\mathcal{P}$. Then the category of sets over $\mathcal{V}$ is monoidal; if $p : \mathcal{E} \to \mathcal{V}$ and $q : \mathcal{F} \to \mathcal{V}$ are maps, then the tensor product is $\mathcal{E}\,{}_{\mathfrak{e}}\times_{\mathfrak{s}} \mathcal{F}$, with grading (1.12). The functor in (a) is monoidal too.

(c). Let $(\mathcal{V}, \mathcal{H})$ be a matched pair of groupoids over $\mathcal{P}$. If $\mathcal{A}$ is a representation of $(\mathcal{V}, \mathcal{H})$, then we define a left action of the groupoid $\mathfrak{B} \rightrightarrows \mathcal{V}$ on the map $|\,| : \mathcal{A} \to \mathcal{V}$ by the rule

$$(x, g) \rightharpoonup a = x \rightharpoonup a, \qquad \text{if } g = |a|.$$

Conversely, a left action of the groupoid $\mathfrak{B} \rightrightarrows \mathcal{V}$ on a map $|\,| : \mathcal{A} \to \mathcal{V}$ defines a representation of $(\mathcal{V}, \mathcal{H})$ by the same rule. We omit the straightforward verifications.

However there is no obvious translation of the monoidal structure on $\operatorname{Rep}(\mathcal{V}, \mathcal{H})$ to the actions of $\mathfrak{B} \rightrightarrows \mathcal{V}$. In other words we can not extend the definition of representations of matched pairs to arbitrary double groupoids.

Lemma 1.11. *If $\mathcal{A}$ is a representation of $(\mathcal{V},\mathcal{H})$, then $\mathcal{A}^{\mathrm{op}}$ is a representation of $(\mathcal{V},\mathcal{H})$ with*

$$x \rightharpoonup a^{-1} = \big((x \leftharpoonup |a|^{-1}) \rightharpoonup a\big)^{-1}, \qquad |a^{-1}| = |a|^{-1}, \qquad x \in \mathcal{H}, a \in \mathcal{A}. \tag{1.13}$$

Proof. Straightforward, using formula (1.17) in [AA]. □

Since disjoint unions of representations are again representations, we conclude that $\mathcal{D}\mathcal{A}$ is a representation of $(\mathcal{V},\mathcal{H})$.

2. The quiver-theoretical braid equation

In this section we introduce our main object of study– the quiver-theoretical braid equation. We establish some basic properties and present methods to construct examples.

2.1. The quiver-theoretical quantum Yang-Baxter equation.

Let $\mathcal{A}$ be a quiver and let $\sigma : \mathcal{A}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}} \mathcal{A} \to \mathcal{A}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}} \mathcal{A}$ be an isomorphism of quivers. We set $\mathcal{A}^n := \mathcal{A}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}} \mathcal{A}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}} \dots {}_{\mathfrak{e}}\!\times_{\mathfrak{s}} \mathcal{A}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}} \mathcal{A}$, n-times; and $\sigma_{i,i+1} := \mathrm{id}_{\mathcal{A}^{i-1}} \times \sigma \times \mathrm{id}_{\mathcal{A}^{n-i-1}} : \mathcal{A}^n \to \mathcal{A}^n$, an isomorphism of quivers.

A *solution of the quiver-theoretical braid equation over $\mathcal{P}$* (a *"solution"* or a *"braided quiver"*, for short) is a pair formed by a quiver $\mathcal{A}$ and an isomorphism of quivers $\sigma : \mathcal{A}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}} \mathcal{A} \to \mathcal{A}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}} \mathcal{A}$ such that

$$(\sigma \times \mathrm{id})(\mathrm{id} \times \sigma)(\sigma \times \mathrm{id}) = (\mathrm{id} \times \sigma)(\sigma \times \mathrm{id})(\mathrm{id} \times \sigma) : \mathcal{A}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}} \mathcal{A}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}} \mathcal{A} \to \mathcal{A}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}} \mathcal{A}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}} \mathcal{A}. \tag{2.1}$$

A solution $(\mathcal{A},\sigma)$ is a *symmetry* when $\sigma^2 = \mathrm{id}$. If $(\mathcal{A},\sigma)$ is a solution, then the braid group $\mathbb{B}_n$ acts by automorphisms of quivers on $\mathcal{A}^n$ for any $n \geq 2$; if $(\mathcal{A},\sigma)$ is a symmetry, this action descends to an action of the symmetric group $\mathbb{S}_n$.

Analogously, a *solution of the quiver-theoretical quantum Yang-Baxter equation over $\mathcal{P}$* is a pair formed by a quiver $\mathcal{A}$ and an isomorphism of quivers $R : \mathcal{A}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}} \mathcal{A} \to \mathcal{A}\,{}_{\mathfrak{s}}\!\times_{\mathfrak{e}} \mathcal{A}$ such that

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} : \mathcal{A}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}} \mathcal{A}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}} \mathcal{A} \to \mathcal{A}\,{}_{\mathfrak{s}}\!\times_{\mathfrak{e}} \mathcal{A}\,{}_{\mathfrak{s}}\!\times_{\mathfrak{e}} \mathcal{A}. \tag{2.2}$$

Note that R and both members of the equality (2.2) are isomorphisms between *different* quivers. In fact, the precise meaning of the members of (2.2) is given by the commutativity of the diagrams:

$$\begin{array}{ccc} \mathcal{A}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}} \mathcal{A}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}} \mathcal{A} & \xrightarrow{R_{23}R_{13}R_{12}} & \mathcal{A}\,{}_{\mathfrak{s}}\!\times_{\mathfrak{e}} \mathcal{A}\,{}_{\mathfrak{s}}\!\times_{\mathfrak{e}} \mathcal{A} \\ {\scriptstyle R_{12}}\downarrow & & \uparrow{\scriptstyle R_{23}} \\ (\mathcal{A}\,{}_{\mathfrak{s}}\!\times_{\mathfrak{e}} \mathcal{A})\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}} \mathcal{A} & \xrightarrow{R_{13}} & \mathcal{A}\,{}_{\mathfrak{s}}\!\times_{\mathfrak{e}} (\mathcal{A}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}} \mathcal{A}), \end{array} \qquad \begin{array}{ccc} \mathcal{A}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}} \mathcal{A}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}} \mathcal{A} & \xrightarrow{R_{12}R_{13}R_{23}} & \mathcal{A}\,{}_{\mathfrak{s}}\!\times_{\mathfrak{e}} \mathcal{A}\,{}_{\mathfrak{s}}\!\times_{\mathfrak{e}} \mathcal{A} \\ {\scriptstyle R_{23}}\downarrow & & \uparrow{\scriptstyle R_{12}} \\ \mathcal{A}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}} (\mathcal{A}\,{}_{\mathfrak{s}}\!\times_{\mathfrak{e}} \mathcal{A}) & \xrightarrow{R_{13}} & (\mathcal{A}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}} \mathcal{A})\,{}_{\mathfrak{s}}\!\times_{\mathfrak{e}} \mathcal{A}. \end{array}$$

As usual, there is a bijective correspondence between solutions of the quiver-theoretical quantum Yang-Baxter equation and solutions of the quiver-theoretical braid equation over $\mathcal{P}$. Namely, let $\sigma : \mathcal{A}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}} \mathcal{A} \to \mathcal{A}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}} \mathcal{A}$ be an isomorphism of quivers, let $\tau : \mathcal{A}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}} \mathcal{A} \to \mathcal{A}\,{}_{\mathfrak{s}}\!\times_{\mathfrak{e}} \mathcal{A}$ be given by (1.4), and let $R := \tau\sigma$. Then R is a solution of the quiver-theoretical quantum Yang-Baxter equation if and only if σ is a solution of the quiver-theoretical braid equation.

2.2. Non-degenerate solutions.

Let $(\mathcal{A},\sigma)$ be a braided quiver. We define maps $\rightharpoonup, \leftharpoonup, \rightharpoondown, \leftharpoondown : \mathcal{A}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}}\mathcal{A} \to \mathcal{A}$ by

$$\sigma(x,y) = (x \rightharpoonup y, x \leftharpoonup y), \tag{2.3}$$

$$\sigma^{-1}(x,y) = (x \rightharpoondown y, x \leftharpoondown y), \tag{2.4}$$

$(x,y) \in \mathcal{A}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}}\mathcal{A}$. Clearly,

$$\mathfrak{s}(x) = \mathfrak{s}(x \rightharpoonup y), \qquad \mathfrak{e}(x \rightharpoonup y) = \mathfrak{s}(x \leftharpoonup y), \qquad \mathfrak{e}(x \leftharpoonup y) = \mathfrak{e}(y), \qquad (x,y) \in \mathcal{A}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}}\mathcal{A}. \tag{2.5}$$

We shall use below the relations between $\rightharpoonup, \leftharpoonup, \rightharpoondown, \leftharpoondown$, like

$$x = (x \rightharpoonup g) \rightharpoondown (x \leftharpoonup g), \quad g = (x \rightharpoonup g) \leftharpoondown (x \leftharpoonup g), \quad y = (y \rightharpoondown h) \rightharpoonup (y \leftharpoondown h), \quad h = (y \rightharpoondown h) \leftharpoonup (y \leftharpoondown h), \tag{2.6}$$

for composable x, g, y, h.

The braid equation (2.1) can be restated as $(\mathrm{id}\times\sigma)(\sigma\times\mathrm{id})(\mathrm{id}\times\sigma)^{-1} = (\sigma\times\mathrm{id})^{-1}(\mathrm{id}\times\sigma)(\sigma\times\mathrm{id})$. The equality of the first two components in this identity, specialized at $(h,f,u) \in \mathcal{A}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}}\mathcal{A}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}}\mathcal{A}$, gives

$$h \rightharpoonup (f \rightharpoondown u) = (h \rightharpoonup f) \rightharpoondown [(h \leftharpoonup f) \rightharpoonup u], \tag{2.7}$$

$$[h \leftharpoonup (f \rightharpoondown u)] \rightharpoonup (f \leftharpoondown u) = (h \rightharpoonup f) \leftharpoondown [(h \leftharpoonup f) \rightharpoonup u]. \tag{2.8}$$

Remark 2.1. Let $\mathcal{C}$ be any tensor category. Then we can define a solution of the braid equation in $\mathcal{C}$ as a pair (V,c), where V is an object of $\mathcal{C}$ and $c : V\otimes V \to V \otimes V$ is an invertible arrow satisfying $(c\otimes\mathrm{id})(\mathrm{id}\otimes c)(c\otimes\mathrm{id}) = (\mathrm{id}\otimes c)(c\otimes\mathrm{id})(\mathrm{id}\otimes c)$. Assume further that $\mathcal{C}$ is *rigid*. Then we say that a solution (V,c) is *rigid* if the map $c^\flat : V^*\otimes V \to V\otimes V^*$ is invertible, where

$$c^\flat = (\mathrm{ev}_V \otimes \mathrm{id}_{V\otimes V^*})(\mathrm{id}_{V^*}\otimes c \otimes \mathrm{id}_{V^*})(\mathrm{id}_{V^*\otimes V}\otimes \mathrm{coev}_V).$$

The category $\mathrm{Quiv}(\mathcal{P})$ is not rigid but the analog of rigid solutions is given by the following definition.

Definition 2.2. A solution $(\mathcal{A},\sigma)$ is *non-degenerate* if

$$x \rightharpoonup \_ : \mathfrak{s}^{-1}(\mathfrak{e}(x)) \to \mathfrak{s}^{-1}(\mathfrak{s}(x)) \text{ and } \_ \leftharpoonup x : \mathfrak{e}^{-1}(\mathfrak{s}(x)) \to \mathfrak{e}^{-1}(\mathfrak{e}(x)) \tag{2.9}$$

are bijections for any $x \in \mathcal{A}$.

We next introduce the structure groupoid of a solution, a generalization of definitions in [ESS, LYZ1, S]. It plays the role of the FRT-bialgebra in this context.

Definition 2.3. The *structure groupoid* of the braided quiver $(\mathcal{A},\sigma)$ is the groupoid $\mathbb{G}_{\mathcal{A}}$ generated by $\mathcal{A}$ with relations

$$xy = (x \rightharpoonup y)(x \leftharpoonup y), \qquad (x,y) \in \mathcal{A}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}}\mathcal{A}. \tag{2.10}$$

Equivalently, it can be defined as the groupoid generated by $\mathcal{A}$ with relations

$$xy = (x \rightharpoondown y)(x \leftharpoondown y), \qquad (x,y) \in \mathcal{A}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}}\mathcal{A}. \tag{2.11}$$

Note that if $\rightharpoonup : \mathcal{A}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}}\mathcal{A} \to \mathcal{A}$ is a map such that $x \rightharpoonup \_ : \mathfrak{s}^{-1}(\mathfrak{e}(x)) \to \mathfrak{s}^{-1}(\mathfrak{s}(x))$ is a bijection for any $x \in \mathcal{A}$ then it extends to a left action of the free groupoid $F(\mathcal{A})$ on $\mathfrak{s} : \mathcal{A} \to \mathcal{P}$. Similarly, a map $\leftharpoonup$ with the analogous property induces a right action of $F(\mathcal{A})$ on $\mathfrak{e} : \mathcal{A} \to \mathcal{P}$.

Given an isomorphism of quivers $\sigma : \mathcal{A}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}}\mathcal{A} \to \mathcal{A}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}}\mathcal{A}$, we can still define maps $\rightharpoonup$ and $\leftharpoonup$ by (2.3); then we can further define the groupoid $\mathbb{G}_{\mathcal{A}}$ generated by $\mathcal{A}$ with relations (2.10). In other words, whether σ is a solution or not does not play any role in the definition of $\mathbb{G}_{\mathcal{A}}$.

Lemma 2.4. *Let $\mathcal{A}$ be a quiver and $\sigma : \mathcal{A}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}}\mathcal{A} \to \mathcal{A}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}}\mathcal{A}$ be an isomorphism of quivers. Then $(\mathcal{A},\sigma)$ is a non-degenerate solution if and only if*

(2.12) $\rightharpoonup$ *extends to a left action of $\mathbb{G}_{\mathcal{A}}$ on $\mathfrak{s} : \mathcal{A} \to \mathcal{P}$;*

(2.13) $\leftharpoonup$ *extends to a right action of $\mathbb{G}_{\mathcal{A}}$ on $\mathfrak{e} : \mathcal{A} \to \mathcal{P}$;*

(2.14) $(x \rightharpoonup y) \leftharpoonup ((x \leftharpoonup y) \rightharpoonup z) = (x \leftharpoonup (y \rightharpoonup z)) \rightharpoonup (y \leftharpoonup z)$, *for all $(x, y, z) \in \mathcal{A}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}}\mathcal{A}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}}\mathcal{A}$.*

Proof. We fix $(x, y, z) \in \mathcal{A}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}}\mathcal{A}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}}\mathcal{A}$ and compute:

$$(\sigma \times \mathrm{id})(\mathrm{id} \times\sigma)(\sigma \times \mathrm{id})(x, y, z) = ((x \rightharpoonup y) \rightharpoonup ((x \leftharpoonup y) \rightharpoonup z)\,, (x \rightharpoonup y) \leftharpoonup ((x \leftharpoonup y) \rightharpoonup z), (x \leftharpoonup y) \leftharpoonup z),$$
$$(\mathrm{id} \times\sigma)(\sigma \times \mathrm{id})(\mathrm{id} \times\sigma)(x, y, z) = (x \rightharpoonup (y \rightharpoonup z), (x \leftharpoonup (y \rightharpoonup z)) \rightharpoonup (y \leftharpoonup z), (x \leftharpoonup (y \rightharpoonup z)) \leftharpoonup (y \leftharpoonup z)).$$

Then $(\mathcal{A},\sigma)$ is a solution iff 3 equalities hold, the second one being (2.14). In presence of non-degeneracy, the first of these equalities is equivalent to (2.12) and the third to (2.13). □

If $(\mathcal{A},\sigma)$ is a non-degenerate solution, then we shall denote by

(2.15) $$h^{-1} \rightharpoonup \_,\ \text{respectively}\ \_ \leftharpoonup g^{-1},\ \text{the inverse of}\ h \rightharpoonup \_,\ \text{respectively}\ \_ \leftharpoonup g.$$

Let $(\mathcal{A},\sigma)$ be a solution. Then $(\mathcal{A},\sigma^{-1})$ is a solution. Furthermore, if $\vartheta : \mathcal{A}^{\mathrm{op}}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}}\mathcal{A}^{\mathrm{op}} \to \mathcal{A}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}}\mathcal{A}$ is given by (1.3), then $(\mathcal{A}^{\mathrm{op}}, \vartheta^{-1}\sigma\vartheta)$, as well as $(\mathcal{A}^{\mathrm{op}}, \vartheta^{-1}\sigma^{-1}\vartheta)$, are solutions. We denote $\sigma^* = \vartheta^{-1}\sigma^{-1}\vartheta$ and $\sigma^*(x^{-1}, y^{-1}) = (x^{-1} \rightharpoonup y^{-1}, x^{-1} \leftharpoonup y^{-1})$; thus

(2.16) $$x^{-1} \rightharpoonup y^{-1} = (y \leftharpoondown x)^{-1}, \qquad x^{-1} \leftharpoonup y^{-1} = (y \rightharpoondown x)^{-1}.$$

Let $\mathbb{G}_{\mathcal{A}^{\mathrm{op}}}$ be the structure groupoid of $(\mathcal{A}^{\mathrm{op}}, \sigma^*)$. Then the map $\mathcal{A} \to \mathcal{A}^{\mathrm{op}}$, $x \mapsto x^{-1}$, induces an isomorphism of groupoids $\mathbb{G}_{\mathcal{A}} \to \mathbb{G}_{\mathcal{A}^{\mathrm{op}}}$, *cf.* (2.10). We shall identify $\mathbb{G}_{\mathcal{A}} = \mathbb{G}_{\mathcal{A}^{\mathrm{op}}}$ via this isomorphism in what follows.

Lemma 2.5. *Let $(\mathcal{A},\sigma)$ be a non-degenerate solution. Then*

(a) $(\mathcal{A}^{\mathrm{op}}, \sigma^*)$ is non-degenerate.

(b) $(\mathcal{A}, \sigma^{-1})$ is non-degenerate.

(c). We have in $\mathbb{G}_{\mathcal{A}}$:

(2.17) $$xy^{-1} = \left(x \rightharpoonup y^{-1}\right)\left(x \leftharpoonup y^{-1}\right),$$

(2.18) $$x^{-1}z = \left(x^{-1} \rightharpoonup z\right)\left(x^{-1} \leftharpoonup z\right),$$

$x, y, z \in \mathcal{A}$, $\mathfrak{s}(x) = \mathfrak{s}(z)$, $\mathfrak{e}(x) = \mathfrak{e}(y)$.

By (a), there are actions $\rightharpoonup\ : \mathbb{G}_{\mathcal{A}}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}}\mathcal{A}^{\mathrm{op}} \to \mathcal{A}^{\mathrm{op}}$, $\leftharpoonup\ : \mathcal{A}^{\mathrm{op}}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}}\mathbb{G}_{\mathcal{A}} \to \mathcal{A}^{\mathrm{op}}$.

Proof. (a). We define $\rightharpoonup\ : \mathcal{A}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}}\mathcal{A}^{\mathrm{op}} \to \mathcal{A}^{\mathrm{op}}$, $\leftharpoonup\ : \mathcal{A}^{\mathrm{op}}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}}\mathcal{A} \to \mathcal{A}^{\mathrm{op}}$ by

(2.19) $$x \rightharpoonup g^{-1} = ((x \leftharpoonup g^{-1}) \rightharpoonup g)^{-1}, \qquad h^{-1} \leftharpoonup y = (h \leftharpoonup (h^{-1} \rightharpoonup y))^{-1}.$$

We compute

$$\begin{aligned} x^{-1} \rightharpoonup (x \rightharpoonup g^{-1}) &= x^{-1} \rightharpoonup ((x \leftharpoonup g^{-1}) \rightharpoonup g)^{-1} = \left(((x \leftharpoonup g^{-1}) \rightharpoonup g) \leftharpoondown x\right)^{-1} \\ &= \left(((x \leftharpoonup g^{-1}) \rightharpoonup g) \leftharpoondown \left((x \leftharpoonup g^{-1}) \leftharpoonup g\right)\right)^{-1} = g^{-1}, \end{aligned}$$

and also

$$x \rightharpoonup (x^{-1} \rightharpoonup g^{-1}) = x \rightharpoonup (g \leftharpoondown x)^{-1} = \left((x \leftharpoonup (g \leftharpoondown x)^{-1}) \rightharpoonup (g \leftharpoondown x)\right)^{-1} = ((x \rightharpoondown g) \rightharpoonup (g \leftharpoondown x))^{-1} = g^{-1}.$$

This means that $x^{-1} \rightharpoonup \_ : \mathcal{A}^{\mathrm{op}} \to \mathcal{A}^{\mathrm{op}}$ is the partial inverse of $x \rightharpoonup \_ : \mathcal{A}^{\mathrm{op}} \to \mathcal{A}^{\mathrm{op}}$. Similarly for the right action, and (a) is proved. Then (b) follows from (a) by (2.16).

(c). By definition of $\mathbb{G}_{\mathcal{A}}$, $(x \leftharpoonup y^{-1})y = \left((x \leftharpoonup y^{-1}) \rightharpoonup y\right)\left((x \leftharpoonup y^{-1}) \leftharpoonup y\right) = \left((x \leftharpoonup y^{-1}) \rightharpoonup y\right) x$, thus $xy^{-1} = \left((x \leftharpoonup y^{-1}) \rightharpoonup y\right)^{-1}\left(x \leftharpoonup y^{-1}\right) = \left(x \rightharpoonup y^{-1}\right)\left(x \leftharpoonup y^{-1}\right)$, proving (2.17). The proof of (2.18) is similar. □

Putting together (2.9), (2.15), (2.16) and (2.19), we have maps $\rightharpoonup, \leftharpoonup : \mathcal{DA}\,{}_{\mathfrak{e}}\times_{\mathfrak{s}}\mathcal{DA} \to \mathcal{DA}$. We can then define $\overline{\sigma} : \mathcal{DA}\,{}_{\mathfrak{e}}\times_{\mathfrak{s}}\mathcal{DA} \to \mathcal{DA}\,{}_{\mathfrak{e}}\times_{\mathfrak{s}}\mathcal{DA}$ by

$$\overline{\sigma}(x, y) = (x \rightharpoonup y, x \leftharpoonup y), \qquad (x, y) \in \mathcal{DA}\,{}_{\mathfrak{e}}\times_{\mathfrak{s}}\mathcal{DA}. \tag{2.20}$$

Lemma 2.6. *If* $(\mathcal{A}, \sigma)$ *is a non-degenerate solution then* $(\mathcal{DA}, \overline{\sigma})$ *is a non-degenerate solution.*

Proof. Once the validity of the braid equation is established, the rigidity will be clear by construction. Now it is necessary to verify the equality (2.1) on 8 subsets of $\mathcal{DA}\,{}_{\mathfrak{e}}\times_{\mathfrak{s}}\mathcal{DA}$. It holds in $\mathcal{A}\,{}_{\mathfrak{e}}\times_{\mathfrak{s}}\mathcal{A}\,{}_{\mathfrak{e}}\times_{\mathfrak{s}}\mathcal{A}$ by hypothesis and it holds in $\mathcal{A}^{\mathrm{op}}\,{}_{\mathfrak{e}}\times_{\mathfrak{s}}\mathcal{A}^{\mathrm{op}}\,{}_{\mathfrak{e}}\times_{\mathfrak{s}}\mathcal{A}^{\mathrm{op}}$ by Lemma 2.5 (a). Since $((\mathcal{A}^{\mathrm{op}})^{\mathrm{op}}, (\sigma^*)^*) = (\mathcal{A}, \sigma)$ we are reduced to three cases.

Case I. $(\sigma \times \mathrm{id})(\mathrm{id} \times \sigma)(\sigma \times \mathrm{id}) \overset{?}{=} (\mathrm{id} \times \sigma)(\sigma \times \mathrm{id})(\mathrm{id} \times \sigma) : \mathcal{A}\,{}_{\mathfrak{e}}\times_{\mathfrak{s}}\mathcal{A}^{\mathrm{op}}\,{}_{\mathfrak{e}}\times_{\mathfrak{s}}\mathcal{A}^{\mathrm{op}} \to \mathcal{A}^{\mathrm{op}}\,{}_{\mathfrak{e}}\times_{\mathfrak{s}}\mathcal{A}^{\mathrm{op}}\,{}_{\mathfrak{e}}\times_{\mathfrak{s}}\mathcal{A}$.

We fix $(x, y^{-1}, z^{-1}) \in \mathcal{A}\,{}_{\mathfrak{e}}\times_{\mathfrak{s}}\mathcal{A}^{\mathrm{op}}\,{}_{\mathfrak{e}}\times_{\mathfrak{s}}\mathcal{A}^{\mathrm{op}}$ and compute both sides of the desired equality:

$$\begin{aligned} LHS &= ((x \rightharpoonup y^{-1}) \rightharpoonup \left((x \leftharpoonup y^{-1}) \rightharpoonup z^{-1}\right), (x \rightharpoonup y^{-1}) \leftharpoonup ((x \leftharpoonup y^{-1}) \rightharpoonup z^{-1}), (x \leftharpoonup y^{-1}) \leftharpoonup z^{-1}), \\ RHS &= (x \rightharpoonup (y^{-1} \rightharpoonup z^{-1}), (x \leftharpoonup (y^{-1} \rightharpoonup z^{-1})) \rightharpoonup (y^{-1} \leftharpoonup z^{-1}), (x \leftharpoonup (y^{-1} \rightharpoonup z^{-1})) \leftharpoonup (y^{-1} \leftharpoonup z^{-1})). \end{aligned}$$

The first components of the LHS and the RHS are equal by (2.17), while the last are equal by definition of $\mathbb{G}_{\mathcal{A}^{\mathrm{op}}}$. It remains to show the equality of the middle components:

$$(x \rightharpoonup y^{-1}) \leftharpoonup ((x \leftharpoonup y^{-1}) \rightharpoonup z^{-1}) \overset{?}{=} (x \leftharpoonup (y^{-1} \rightharpoonup z^{-1})) \rightharpoonup (y^{-1} \leftharpoonup z^{-1}). \tag{2.21}$$

Now RHS of (2.21) $= \left[x^{-1} \leftharpoonup \left(x \rightharpoonup (y^{-1} \rightharpoonup z^{-1})\right)\right]^{-1} \rightharpoonup (y^{-1} \leftharpoonup z^{-1})$ by (2.19). Thus, we are reduced to prove

$$\begin{aligned} y^{-1} \leftharpoonup z^{-1} &\overset{?}{=} \left[x^{-1} \leftharpoonup \left(x \rightharpoonup (y^{-1} \rightharpoonup z^{-1})\right)\right] \rightharpoonup \left((x \rightharpoonup y^{-1}) \leftharpoonup ((x \leftharpoonup y^{-1}) \rightharpoonup z^{-1})\right) \\ &= \left[x^{-1} \leftharpoonup \left((x \rightharpoonup y^{-1}) \rightharpoonup ((x \leftharpoonup y^{-1}) \rightharpoonup z^{-1})\right)\right] \rightharpoonup \left((x \rightharpoonup y^{-1}) \leftharpoonup ((x \leftharpoonup y^{-1}) \rightharpoonup z^{-1})\right). \end{aligned}$$

We have $(f^{-1} \rightharpoonup g^{-1}) \leftharpoonup ((f^{-1} \leftharpoonup g^{-1}) \rightharpoonup h^{-1}) = (f^{-1} \leftharpoonup (g^{-1} \rightharpoonup h^{-1})) \rightharpoonup (g^{-1} \leftharpoonup h^{-1})$ by (2.14) applied to σ^* and suitable $f, g, h \in \mathcal{A}$. Filling this identity with $f = x$, $g^{-1} = x \rightharpoonup y^{-1}$ and $h^{-1} = (x \leftharpoonup y^{-1}) \rightharpoonup z^{-1}$, we obtain the desired equality.

Case II. $(\sigma \times \mathrm{id})(\mathrm{id} \times \sigma)(\sigma \times \mathrm{id}) \stackrel{?}{=} (\mathrm{id} \times \sigma)(\sigma \times \mathrm{id})(\mathrm{id} \times \sigma) : \mathcal{A}^{\mathrm{op}} {}_{\mathfrak{e}}\times_{\mathfrak{s}} \mathcal{A} {}_{\mathfrak{e}}\times_{\mathfrak{s}} \mathcal{A}^{\mathrm{op}} \to \mathcal{A}^{\mathrm{op}} {}_{\mathfrak{e}}\times_{\mathfrak{s}} \mathcal{A} {}_{\mathfrak{e}}\times_{\mathfrak{s}} \mathcal{A}^{\mathrm{op}}$.

We fix $(x^{-1}, y, z^{-1}) \in \mathcal{A}^{\mathrm{op}} {}_{\mathfrak{e}}\times_{\mathfrak{s}} \mathcal{A} {}_{\mathfrak{e}}\times_{\mathfrak{s}} \mathcal{A}^{\mathrm{op}}$ and compute both sides of the desired equality:

$$LHS = ((x^{-1} \rightharpoonup y) \rightharpoonup \left((x^{-1} \leftharpoonup y) \rightharpoonup z^{-1}\right), (x^{-1} \rightharpoonup y) \leftharpoonup ((x^{-1} \leftharpoonup y) \rightharpoonup z^{-1}), (x^{-1} \leftharpoonup y) \leftharpoonup z^{-1}),$$
$$RHS = (x^{-1} \rightharpoonup (y \rightharpoonup z^{-1}), (x^{-1} \leftharpoonup (y \rightharpoonup z^{-1})) \rightharpoonup (y \leftharpoonup z^{-1}), (x^{-1} \leftharpoonup (y \rightharpoonup z^{-1})) \leftharpoonup (y \leftharpoonup z^{-1})).$$

The first components of the LHS and the RHS are equal by (2.18), while the last are equal by (2.17). It remains to show the equality of the middle components:

$$(x^{-1} \rightharpoonup y) \leftharpoonup ((x^{-1} \leftharpoonup y) \rightharpoonup z^{-1}) \stackrel{?}{=} (x^{-1} \leftharpoonup (y \rightharpoonup z^{-1})) \rightharpoonup (y \leftharpoonup z^{-1}). \tag{2.22}$$

Now

$$\begin{aligned}
LHS \text{ of } (2.22) &= \left((x^{-1} \leftharpoonup y) \rightharpoonup y^{-1}\right)^{-1} \leftharpoonup ((x^{-1} \leftharpoonup y) \rightharpoonup z^{-1}) \\
&= \left\{\left((x^{-1} \leftharpoonup y) \rightharpoonup y^{-1}\right) \leftharpoonup \left((x^{-1} \rightharpoonup y) \rightharpoonup ((x^{-1} \leftharpoonup y) \rightharpoonup z^{-1})\right)\right\}^{-1} \\
&= \left\{\left((x^{-1} \leftharpoonup y) \rightharpoonup y^{-1}\right) \leftharpoonup \left(x^{-1} \rightharpoonup (y \rightharpoonup z^{-1})\right)\right\}^{-1} \\
&= \left\{\left((x^{-1} \leftharpoonup y) \leftharpoonup z^{-1})\right) \rightharpoonup \left(y^{-1} \leftharpoonup (y \rightharpoonup z^{-1})\right)\right\}^{-1} \\
&= \left\{\left((x^{-1} \leftharpoonup (y \rightharpoonup z^{-1})) \leftharpoonup (y \leftharpoonup z^{-1}))\right) \rightharpoonup \left(y^{-1} \leftharpoonup (y \rightharpoonup z^{-1})\right)\right\}^{-1} \\
&= \left\{\left((x^{-1} \leftharpoonup (y \rightharpoonup z^{-1})) \leftharpoonup (y \leftharpoonup z^{-1}))\right) \rightharpoonup \left(y \leftharpoonup z^{-1}\right)^{-1}\right\}^{-1} \\
&= (x^{-1} \rightharpoonup (y \rightharpoonup z^{-1})) \leftharpoonup (y \leftharpoonup z^{-1}) = RHS \text{ of } (2.22).
\end{aligned}$$

Here the first and the second equalities follow from (2.19); the third from (2.18); the fourth from (2.14) applied to σ^* filled with $f^{-1} = x^{-1} \leftharpoonup y$, $g = y$ and $h^{-1} = y \rightharpoonup z^{-1}$ as in the first step; the fifth from (2.17); and the last two from (2.19) again.

Case III. $(\sigma \times \mathrm{id})(\mathrm{id} \times \sigma)(\sigma \times \mathrm{id}) \stackrel{?}{=} (\mathrm{id} \times \sigma)(\sigma \times \mathrm{id})(\mathrm{id} \times \sigma) : \mathcal{A}^{\mathrm{op}} {}_{\mathfrak{e}}\times_{\mathfrak{s}} \mathcal{A}^{\mathrm{op}} {}_{\mathfrak{e}}\times_{\mathfrak{s}} \mathcal{A} \to \mathcal{A} {}_{\mathfrak{e}}\times_{\mathfrak{s}} \mathcal{A}^{\mathrm{op}} {}_{\mathfrak{e}}\times_{\mathfrak{s}} \mathcal{A}^{\mathrm{op}}$.

By Lemma 2.5 (b), σ^{-1} is rigid and hence the equality in case I holds for it. Then case III follows inverting this equality; indeed $\left(\sigma^{-1}\right)^* = (\sigma^*)^{-1}$. □

Now let $\mathcal{A}$ be a solution. We define maps $\sigma^{m,n} : \mathrm{Path}_m \mathcal{A} {}_{\mathfrak{e}}\times_{\mathfrak{s}} \mathrm{Path}_n \mathcal{A} \to \mathrm{Path}_n \mathcal{A} {}_{\mathfrak{e}}\times_{\mathfrak{s}} \mathrm{Path}\, \mathcal{A}_m$, $m, n \geq 0$, by

$$\sigma^{m,n} = \left(\sigma_{n,n+1} \dots \sigma_{2,3}\sigma_{1,2}\right)\left(\sigma_{n+1,n+2} \dots \sigma_{3,4}\sigma_{2,3}\right) \dots \left(\sigma_{n+m-1,n+m} \dots \sigma_{m+1,m+2}\sigma_{m,m+1}\right),$$

if $m, n > 0$; and by

$$\sigma^{0,n}(\mathrm{id}\, P, x) = (x, \mathrm{id}\, Q), \quad \sigma^{n,0}(x, \mathrm{id}\, Q) = (\mathrm{id}\, P, x)$$

if $x \in \mathrm{Path}_n \mathcal{A}$, $n \geq 0$, $\mathfrak{s}(x) = P$, $\mathfrak{e}(x) = Q$.

Therefore we have an isomorphism of quivers $\sigma^{\mathrm{P}} : \operatorname{Path}\mathcal{A}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}} \operatorname{Path}\mathcal{A} \to \operatorname{Path}\mathcal{A}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}} \operatorname{Path}\mathcal{A}$ by collecting together the maps $\sigma^{m,n}$; and, as usual, maps $\rightharpoonup, \leftharpoonup, \rightharpoondown, \leftharpoondown : \operatorname{Path}\mathcal{A}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}} \operatorname{Path}\mathcal{A} \to \operatorname{Path}\mathcal{A}$ given by (2.3).

Lemma 2.7. *If* $(\mathcal{A}, \sigma)$ *is a solution, then* $(\operatorname{Path}\mathcal{A}, \sigma^{\mathrm{P}})$ *is a solution.*

If $m, n, p \geq 0$ *and* $(u, v, w) \in \operatorname{Path}_m \mathcal{A}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}} \operatorname{Path}_n \mathcal{A}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}} \operatorname{Path}_p \mathcal{A}$*, then*

$$u \rightharpoonup (v \rightharpoonup w) = uv \rightharpoonup w, \tag{2.23}$$

$$u \leftharpoonup (v \leftharpoonup w) = u \leftharpoonup vw, \tag{2.24}$$

$$u \rightharpoonup vw = (u \rightharpoonup v)\left((u \leftharpoonup v) \rightharpoonup w\right), \tag{2.25}$$

$$uv \leftharpoonup w = (u \leftharpoonup (v \rightharpoonup w))\,(v \leftharpoonup w). \tag{2.26}$$

If σ *is non-degenerate, so is* σ^{P}*.*

Proof. The first claim follows from a well-known equality in the braid group. Also, $\sigma^{m+n,p} = (\mathrm{id} \times \sigma^{n,p})(\sigma^{m,n} \times \mathrm{id})$, which implies (2.23) and (2.26), and $\sigma^{m,n+p} = (\sigma^{m,n} \times \mathrm{id})(\mathrm{id} \times \sigma^{n,p})$, which implies (2.24) and (2.25). Finally, if $u = x_1 \dots x_n$ is a path of length n, then the inverse of $u \rightharpoonup \_$ is given by $x_n^{-1} \rightharpoonup \left(\dots (x_1^{-1} \rightharpoonup \_\right)$ by (2.23). Similarly $\_ \leftharpoonup u$ is invertible by (2.24). □

The next natural step is to show that $\mathbb{G}_{\mathcal{A}}$ has also the structure of braided quiver. To state this appropriately we study in the next subsection the notion of braided groupoid. We come back to the structure groupoid in Theorem 3.8.

2.3. Braided groupoids.

The notion of braided groupoid generalizes the notion of braided group introduced by Takeuchi [T] to reformulate results of Lu, Yan and Zhu [LYZ1].

Definition 2.8. A *braided groupoid* is a collection $(\mathcal{G}, \rightharpoonup, \leftharpoonup)$, where $\mathcal{G}$ is groupoid and

$$\mathcal{G} \xleftarrow{\ \leftharpoonup\ } \mathcal{G}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}} \mathcal{G} \xrightarrow{\ \rightharpoonup\ } \mathcal{G}$$

are a left and a right actions, such that $(\mathcal{G}, \mathcal{G})$, with $\rightharpoonup$, $\leftharpoonup$, form a matched pair of groupoids; see Definition 1.5, and

$$fg = (f \rightharpoonup g)(f \leftharpoonup g), \qquad (f, g) \in \mathcal{G}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}} \mathcal{G}. \tag{2.27}$$

A morphism of braided groupoids is a morphism of groupoids that preserves the actions $\rightharpoonup$, $\leftharpoonup$.

Let $\mathcal{G}$ be a braided groupoid. Then the maps $\iota_1, \iota_2 : \mathcal{G} \to \mathcal{G} \bowtie \mathcal{G}$ given by

$$\iota_1(g) = (g, \mathrm{id}\,\mathfrak{e}(g)), \qquad \iota_2(g) = (\mathrm{id}\,\mathfrak{s}(g), g) \tag{2.28}$$

are morphisms of groupoids.

There is some redundancy in the definition of braided groupoid that we study in the next lemma.

Lemma 2.9. *(a). Let* $\mathcal{G}$ *be a groupoid endowed with a left and a right actions* $\rightharpoonup$, $\leftharpoonup$ *such that* (2.27) *holds. Then* $\mathcal{G}$, $\mathcal{G}$, $\rightharpoonup$, $\leftharpoonup$ *form a matched pair of groupoids (and* $\mathcal{G}$ *is a braided groupoid).*

(b). Let $\mathcal{G}$, $\mathcal{G}$, $\rightharpoonup$, $\leftharpoonup$ *form a matched pair of groupoids. Then* (2.27) *holds iff the multiplication is a morphism of groupoids* $\mathcal{G} \bowtie \mathcal{G} \to \mathcal{G}$*.*

(c). Let $\mathcal{G}$ *be a groupoid endowed with a left action* $\rightharpoonup : \mathcal{G}\,{}_{\mathfrak{e}}\times_{\mathfrak{s}}\mathcal{G} \to \mathcal{G}$*. Let* $\leftharpoonup : \mathcal{G}\,{}_{\mathfrak{e}}\times_{\mathfrak{s}}\mathcal{G} \to \mathcal{G}$ *be given by* (2.27)*, i. e.,* $x \leftharpoonup y = (x \rightharpoonup y)^{-1}xy$*,* $(x, y) \in \mathcal{G}\,{}_{\mathfrak{e}}\times_{\mathfrak{s}}\mathcal{G}$*. Then* $\leftharpoonup$ *is a right action if and only if* (1.8) *holds. If this is the case, then* $\mathcal{G}$*,* $\mathcal{G}$*,* $\rightharpoonup$*,* $\leftharpoonup$ *form a matched pair of groupoids, and* $\mathcal{G}$ *is a braided groupoid.*

(d). Let $\mathcal{G}$ *be a groupoid endowed with a right action* $\leftharpoonup : \mathcal{G}\,{}_{\mathfrak{e}}\times_{\mathfrak{s}}\mathcal{G} \to \mathcal{G}$*. Let* $\rightharpoonup : \mathcal{G}\,{}_{\mathfrak{e}}\times_{\mathfrak{s}}\mathcal{G} \to \mathcal{G}$ *be given by* (2.27)*, i. e.,* $x \rightharpoonup y = xy(x \leftharpoonup y)^{-1}$*,* $(x, y) \in \mathcal{G}\,{}_{\mathfrak{e}}\times_{\mathfrak{s}}\mathcal{G}$*. Then* $\rightharpoonup$ *is a left action if and only if* (1.9) *holds. If this is the case, then* $\mathcal{G}$*,* $\mathcal{G}$*,* $\rightharpoonup$*,* $\leftharpoonup$ *form a matched pair of groupoids, and* $\mathcal{G}$ *is a braided groupoid.*

Proof. (a). It is clear that (1.7) holds. We check (1.8):

$$\begin{aligned}(f \rightharpoonup (gh))(f \leftharpoonup (gh)) &= fgh = (f \rightharpoonup g)(f \leftharpoonup g)h \\ &= (f \rightharpoonup g)[(f \leftharpoonup g) \rightharpoonup h][(f \leftharpoonup g) \leftharpoonup h] = (f \rightharpoonup g)[(f \leftharpoonup g) \rightharpoonup h](f \leftharpoonup (gh)).\end{aligned}$$

The proof of (1.9) is similar; thus (a) is valid. The proofs of (b), (c) and (d) are straightforward. □

The antipode $x \mapsto x^{-1}$ induces an isomorphism of quivers $\mathcal{G} \to \mathcal{G}^{\mathrm{op}}$. We check that the identities discussed in Lemma 2.5 are valid with respect to the antipode.

Lemma 2.10. *Let* $(\mathcal{G}, \rightharpoonup, \leftharpoonup)$ *be a braided groupoid.*

(a). The identities (2.15)*,* (2.16)*,* (2.19) *hold in* $\mathcal{G}$ *with respect to the antipode.*

(b). $(\mathcal{G}, \rightharpoondown, \leftharpoondown)$ *is also a braided groupoid.*

Proof. (a). The validity of (2.15) is clear since $\rightharpoonup$, $\leftharpoonup$ are actions. If $(x, g^{-1}) \in \mathcal{G}\,{}_{\mathfrak{e}}\times_{\mathfrak{s}}\mathcal{G}$, then

$$\mathrm{id}\,\mathfrak{s}(x) = x \rightharpoonup \mathrm{id}\,\mathfrak{e}(x) = x \rightharpoonup (g^{-1}g) = (x \rightharpoonup g^{-1})((x \leftharpoonup g^{-1}) \rightharpoonup g);$$

similarly for the other identity and (2.19) follows. Now let $\widetilde{\sigma} : \mathcal{A}\,{}_{\mathfrak{e}}\times_{\mathfrak{s}}\mathcal{A} \to \mathcal{A}\,{}_{\mathfrak{e}}\times_{\mathfrak{s}}\mathcal{A}$ be given by $\widetilde{\sigma}(x, y) = \left((y^{-1} \leftharpoonup x^{-1})^{-1}, (y^{-1} \rightharpoonup x^{-1})^{-1}\right)$, $(x, y) \in \mathcal{G}\,{}_{\mathfrak{e}}\times_{\mathfrak{s}}\mathcal{G}$. We compute:

$$\begin{aligned}\sigma\widetilde{\sigma}(x, y) &= \sigma\left((y^{-1} \leftharpoonup x^{-1})^{-1}, (y^{-1} \rightharpoonup x^{-1})^{-1}\right) \\ &= \left((y^{-1} \leftharpoonup x^{-1})^{-1} \rightharpoonup (y^{-1} \rightharpoonup x^{-1})^{-1}, (y^{-1} \leftharpoonup x^{-1})^{-1} \leftharpoonup (y^{-1} \rightharpoonup x^{-1})^{-1}\right).\end{aligned}$$

Now

$$\begin{aligned}(y^{-1} \leftharpoonup x^{-1})^{-1} \rightharpoonup (y^{-1} \rightharpoonup x^{-1})^{-1} &= (y \leftharpoonup (y^{-1} \rightharpoonup x)) \rightharpoonup ((y^{-1} \leftharpoonup x^{-1}) \rightharpoonup x) \\ &= \left((y \leftharpoonup (y^{-1} \rightharpoonup x))(y^{-1} \leftharpoonup x^{-1})\right) \rightharpoonup x \\ &= (yy^{-1} \leftharpoonup x^{-1}) \rightharpoonup x = x; \\ (y^{-1} \leftharpoonup x^{-1})^{-1} \leftharpoonup (y^{-1} \rightharpoonup x^{-1})^{-1} &= (y \leftharpoonup (y^{-1} \rightharpoonup x)) \leftharpoonup ((y^{-1} \leftharpoonup x^{-1}) \rightharpoonup x) \\ &= y \leftharpoonup \left((y^{-1} \rightharpoonup x)\left((y^{-1} \leftharpoonup x^{-1}) \rightharpoonup x\right)\right) \\ &= y \leftharpoonup \left(y^{-1} \rightharpoonup xx^{-1}\right) = y.\end{aligned}$$

Similarly, $\widetilde{\sigma}\sigma = \mathrm{id}$. Thus, (2.16) holds.

(b). We have $(x \rightharpoondown y)(x \leftharpoondown y) = ((x \rightharpoondown y) \rightharpoonup (x \leftharpoondown y))\,((x \rightharpoondown y) \leftharpoonup (x \leftharpoondown y)) = xy$. We claim that $\rightharpoondown$ is a left action. Let $(x, y, z) \in \mathcal{G}\,{}_{\mathfrak{e}}\times_{\mathfrak{s}}\mathcal{G}\,{}_{\mathfrak{e}}\times_{\mathfrak{s}}\mathcal{G}$. Then

$$x \rightharpoondown (y \rightharpoondown z) = x \rightharpoondown (y^{-1} \leftharpoonup z^{-1})^{-1} = \big((y^{-1} \leftharpoonup z^{-1}) \leftharpoonup x^{-1}\big)^{-1} = \big(y^{-1} \leftharpoonup z^{-1}x^{-1}\big)^{-1} = xy \rightharpoondown z.$$

Similarly, $\leftharpoondown$ is a right action. By Lemma 2.9 (a), $(\mathcal{G}, \rightharpoondown, \leftharpoondown)$ is a braided groupoid. □

2.4. **Braided groupoids are braided quivers.**

We next justify the name "braided groupoids".

Lemma 2.11. *Let $\mathcal{G}$ be a braided groupoid. Let $\sigma : \mathcal{G}\,{}_{\mathfrak{e}}\times_{\mathfrak{s}}\mathcal{G} \to \mathcal{G}\,{}_{\mathfrak{e}}\times_{\mathfrak{s}}\mathcal{G}$ be the map*

$$\sigma(f, g) = (f \rightharpoonup g, f \leftharpoonup g), \qquad (f, g) \in \mathcal{G}\,{}_{\mathfrak{e}}\times_{\mathfrak{s}}\mathcal{G}. \tag{2.29}$$

Then σ is a non-degenerate solution of the quiver-theoretical braid equation (2.1).

Proof. We fix $(f, g, h) \in \mathcal{G}\,{}_{\mathfrak{e}}\times_{\mathfrak{s}}\mathcal{G}\,{}_{\mathfrak{e}}\times_{\mathfrak{s}}\mathcal{G}$ and compute:

$$(\sigma \times \mathrm{id})(\mathrm{id} \times \sigma)(\sigma \times \mathrm{id})(f, g, h) = ((fg) \rightharpoonup h, (f \rightharpoonup g) \leftharpoonup ((f \leftharpoonup g) \rightharpoonup h), f \leftharpoonup (gh)),$$
$$(\mathrm{id} \times \sigma)(\sigma \times \mathrm{id})(\mathrm{id} \times \sigma)(f, g, h) = ((fg) \rightharpoonup h, (f \leftharpoonup (g \rightharpoonup h)) \rightharpoonup (g \leftharpoonup h), f \leftharpoonup (gh)).$$

Now we apply several times (2.27) and compute:

$$\begin{aligned} fgh &= (f \rightharpoonup g)(f \leftharpoonup g)h \\ &= (f \rightharpoonup g)[(f \leftharpoonup g) \rightharpoonup h][(f \leftharpoonup g) \leftharpoonup h] \\ &= [(f \rightharpoonup g) \rightharpoonup ((f \leftharpoonup g) \rightharpoonup h)][(f \rightharpoonup g) \leftharpoonup ((f \leftharpoonup g) \rightharpoonup h)][f \leftharpoonup (gh)] \\ &= [(fg) \rightharpoonup h][(f \rightharpoonup g) \leftharpoonup ((f \leftharpoonup g) \rightharpoonup h)][f \leftharpoonup (gh)] \end{aligned}$$

and also

$$\begin{aligned} fgh &= f(g \rightharpoonup h)(g \leftharpoonup h) \\ &= [f \rightharpoonup (g \rightharpoonup h)][f \leftharpoonup (g \rightharpoonup h)](g \leftharpoonup h) \\ &= [(fg) \rightharpoonup h][(f \leftharpoonup (g \rightharpoonup h)) \rightharpoonup (g \leftharpoonup h)][(f \leftharpoonup (g \rightharpoonup h)) \leftharpoonup (g \leftharpoonup h)] \\ &= [(fg) \rightharpoonup h][(f \leftharpoonup (g \rightharpoonup h)) \rightharpoonup (g \leftharpoonup h)][f \leftharpoonup (gh)]. \end{aligned}$$

Hence $(f \rightharpoonup g) \leftharpoonup ((f \leftharpoonup g) \rightharpoonup h) = (f \leftharpoonup (g \rightharpoonup h)) \rightharpoonup (g \leftharpoonup h)$, and σ is a solution of (2.1). Since $\rightharpoonup$, $\leftharpoonup$ are actions, σ is non-degenerate. □

Let us say that a sub-quiver $\mathcal{A}$ of a braided groupoid $\mathcal{G}$ is *invariant* if $\mathcal{A} \rightharpoonup \mathcal{A} \subset \mathcal{A}$, $\mathcal{A} \leftharpoonup \mathcal{A} \subset \mathcal{A}$, $\mathcal{A}^{-1} \rightharpoonup \mathcal{A} \subset \mathcal{A}$ and $\mathcal{A} \leftharpoonup \mathcal{A}^{-1} \subset \mathcal{A}$.

Corollary 2.12. *Let $\mathcal{A}$ be an invariant sub-quiver of a braided groupoid $\mathcal{G}$. Then $(\mathcal{A}, \sigma_{|\mathcal{A}\,{}_{\mathfrak{e}}\times_{\mathfrak{s}}\mathcal{A}})$ is a non-degenerate braided quiver.* □

Thus, braided groupoids and their invariant sub-quivers are naturally braided quivers. See Remark 3.13 below.

In the papers [ESS, LYZ1, S], braided structures on groups were described through suitable 1-cocycles. We show now that this description goes over also to groupoids but with group bundles as recipients of the 1-cocycles.

Definition 2.13. A 1-*cocycle groupoid datum* is a triple $(\mathcal{G}, \mathcal{N}, \pi)$ where $\mathcal{G}$ is a groupoid over $\mathcal{P}$, $\mathcal{N}$ is a group bundle over $\mathcal{P}$, provided with a right action $\leftharpoonup : \mathcal{N}\,{}_{p}\!\times_{\mathfrak{s}} \mathcal{G} \to \mathcal{N}$ by group bundle automorphisms; and $\pi : \mathcal{G} \to \mathcal{N}$ is a bijective 1-cocycle, i.e. it is a bijection with $p\pi = \mathfrak{s}$ and

$$\pi(fg) = (\pi(f) \leftharpoonup g)\pi(g), \qquad (f,g) \in \mathcal{G}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}} \mathcal{G}. \tag{2.30}$$

Theorem 2.14. *Let $\mathcal{G}$ be a groupoid. There is a bijective correspondence between*

(a) *Structures of braided groupoid $(\mathcal{G}, \sigma)$.*

(b) 1-*cocycle groupoid data $(\mathcal{G}, \mathcal{N}, \pi)$.*

In this correspondence, $\mathcal{G} \bowtie \mathcal{G} \simeq \mathcal{G} \ltimes \mathcal{N}$, and

$$\sigma(f,g) = \left(fg \left(\pi^{-1}\left(\pi(f) \leftharpoonup g\right)\right)^{-1}, \pi^{-1}\left(\pi(f) \leftharpoonup g\right)\right), \qquad (f,g) \in \mathcal{G}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}} \mathcal{G}. \tag{2.31}$$

Proof. (a) $\Longrightarrow$ (b). Let $\mathcal{N}$ be the kernel of the multiplication map $\mathcal{G} \bowtie \mathcal{G} \to \mathcal{G}$; identify $\mathcal{G}$ with $\mathcal{G}_1 :=$ the image of ι_1, cf. (2.28); let $\leftharpoonup$ be the restriction of the adjoint action. Finally, let $\pi : \mathcal{G} \to \mathcal{N}$ be defined by $\pi(g) = (g^{-1}, g)$, $g \in \mathcal{G}$. Then $(\mathcal{G}, \mathcal{N}, \pi)$ is a 1-cocycle groupoid datum. Indeed,

$$\begin{aligned}
(\pi(f) \leftharpoonup g)\pi(g) &= (g^{-1}, \operatorname{id}\mathfrak{s}(g))(f^{-1}, f)(g, \operatorname{id}\mathfrak{e}(g))(g^{-1}, g) \\
&= (g^{-1}(\operatorname{id}\mathfrak{s}(g) \rightharpoonup f^{-1}), (\operatorname{id}\mathfrak{s}(g) \leftharpoonup f^{-1})f)(g(\operatorname{id}\mathfrak{e}(g) \rightharpoonup g^{-1}), (\operatorname{id}\mathfrak{e}(g) \leftharpoonup g^{-1})g^{-1}) \\
&= (g^{-1}f^{-1}, f)(\operatorname{id}\mathfrak{s}(g), g) = (g^{-1}f^{-1}(f \rightharpoonup \operatorname{id}\mathfrak{s}(g)), (f \leftharpoonup \operatorname{id}\mathfrak{s}(g))g) \\
&= (g^{-1}f^{-1}, fg) = \pi(f,g).
\end{aligned}$$

(b) $\Longrightarrow$ (a). Consider the map $\psi : \mathcal{G}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}} \mathcal{G} \to \mathcal{G} \ltimes \mathcal{N}$, $\psi(f,g) = (fg, \pi(g))$, $(f,g) \in \mathcal{G}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}} \mathcal{G}$. Let $\mathcal{G}_1$, resp. $\mathcal{G}_2$, be the image of ι_1, resp. ι_2, as in (2.28). Then

$$\psi(\mathcal{G}_1) = \{(g, \operatorname{id}\mathfrak{e}(g)) : g \in \mathcal{G}\}, \qquad \psi(\mathcal{G}_2) = \{(g, \pi(g)) : g \in \mathcal{G}\}$$

are subgroupoids of $\mathcal{G} \ltimes \mathcal{N}$ isomorphic to $\mathcal{G}$, and they form an exact factorization of $\mathcal{G} \ltimes \mathcal{N}$. Transporting the structure back to $\mathcal{G}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}} \mathcal{G}$ via ψ, we have a groupoid structure on this, which is isomorphic to $\mathcal{G}_1 \bowtie \mathcal{G}_2 \simeq \mathcal{G} \bowtie \mathcal{G}$. A straightforward computation shows that the induced actions $\mathcal{G} \xleftarrow{\leftharpoonup} \mathcal{G}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}} \mathcal{G} \xrightarrow{\rightharpoonup} \mathcal{G}$ are explicitly given by

$$f \rightharpoonup g = fg\left(\pi^{-1}\left(\pi(f) \leftharpoonup g\right)\right)^{-1}, \qquad f \leftharpoonup g = \pi^{-1}\left(\pi(f) \leftharpoonup g\right), \qquad (f,g) \in \mathcal{G}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}} \mathcal{G}. \tag{2.32}$$

Hence (2.27) holds, and the corresponding solution is given by (2.31).

Finally, it is not difficult to see that these constructions are inverse of each other. □

It follows from (2.32) that

$$\pi(f \leftharpoonup g) = \pi(f) \leftharpoonup g, \qquad (f,g) \in \mathcal{G}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}} \mathcal{G}. \tag{2.33}$$

A *symmetric groupoid* is a braided groupoid such that the corresponding σ is a symmetry. The next characterization of symmetric groupoids generalizes results from [ESS, LYZ1].

Proposition 2.15. *A braided groupoid $\mathcal{G}$ is symmetric if and only if the corresponding $\mathcal{N}$ is abelian.*

Proof. Let $(f,g) \in \mathcal{G}\,{}_{\mathfrak{e}}\times_{\mathfrak{s}}\,\mathcal{G}$. Then

$$\begin{aligned}\pi(f \leftharpoonup g)\pi(g) &= (\pi(f) \leftharpoonup g)\pi(g) = \pi(fg) = \pi\left((f \rightharpoonup g)(f \leftharpoonup g)\right)\\ &= \left(\pi(f \rightharpoonup g) \leftharpoonup (f \leftharpoonup g)\right)\pi(f \leftharpoonup g) = \pi\left((f \rightharpoonup g) \leftharpoonup (f \leftharpoonup g)\right)\pi(f \leftharpoonup g),\end{aligned}$$

by (2.33), (2.30) and (2.27). This says that $\mathcal{N}$ is abelian if and only if $g = (f \rightharpoonup g) \leftharpoonup (f \leftharpoonup g)$ for any $(f,g) \in \mathcal{G}\,{}_{\mathfrak{e}}\times_{\mathfrak{s}}\,\mathcal{G}$. But $\sigma^2(f,g) = \sigma(f \rightharpoonup g, f \leftharpoonup g) = ((f \rightharpoonup g) \rightharpoonup (f \leftharpoonup g), (f \rightharpoonup g) \leftharpoonup (f \leftharpoonup g))$. Hence, if $\mathcal{G}$ is symmetric, then $\mathcal{N}$ is abelian. If $\mathcal{N}$ is abelian, then $fg = \left((f \rightharpoonup g) \rightharpoonup (f \leftharpoonup g)\right)\left((f \rightharpoonup g) \leftharpoonup (f \leftharpoonup g)\right) = \left((f \rightharpoonup g) \rightharpoonup (f \leftharpoonup g)\right) g$, and thus $\mathcal{G}$ is symmetric. □

We next define the subgroup bundles $\Gamma_r :=$ kernel of $\leftharpoonup$, $\Gamma_l :=$ kernel of $\rightharpoonup$ and $\Gamma := \Gamma_r \cap \Gamma_l$ of the braided groupoid $\mathcal{G}$. Hence

$$\Gamma = \{v \in \mathcal{G} : \; v \rightharpoonup w = w, \quad z \leftharpoonup v = z, \quad \forall w, z \in \mathcal{G}, \quad \mathfrak{s}(w) = \mathfrak{e}(v) = \mathfrak{s}(v) = \mathfrak{e}(z)\}. \tag{2.34}$$

By (2.27), we also have

$$\begin{aligned}\Gamma_l &= \{m \in \mathcal{G} : m \leftharpoonup y = y^{-1}my, \quad \forall y \in \mathcal{G}, \quad \mathfrak{s}(m) = \mathfrak{e}(m) = \mathfrak{s}(y)\},\\ \Gamma_r &= \{n \in \mathcal{G} : \; x \rightharpoonup n = xnx^{-1}, \quad \forall x \in \mathcal{G}, \quad \mathfrak{e}(x) = \mathfrak{s}(n) = \mathfrak{e}(n)\}.\end{aligned}$$

Lemma 2.16. *Let $\mathcal{G}$ be a braided groupoid.*

(a) *The wide subgroup bundle Γ defined above is abelian and normal.*

(b) *If $\Lambda \subset \Gamma$ is a normal subgroup bundle of $\mathcal{G}$, then $\mathcal{G}/\Lambda$ is a braided groupoid, with braiding inherited from $\mathcal{G}$ via the canonical projection.*

Proof. (a). It follows at once from (2.27) that Γ is abelian, and it is clearly normal.

(b). One checks that $\Gamma_l\,{}_{\mathfrak{e}}\times_{\mathfrak{s}}\,\mathcal{P}$ and $\mathcal{P}\,{}_{\mathfrak{e}}\times_{\mathfrak{s}}\,\Gamma_r$ are normal subgroup bundles of $\mathcal{G} \bowtie \mathcal{G}$; hence $\Gamma\,{}_{\mathfrak{e}}\times_{\mathfrak{s}}\,\Gamma$, and *a fortiori* $\Lambda\,{}_{\mathfrak{e}}\times_{\mathfrak{s}}\,\Lambda$, are normal subgroup bundles of $\mathcal{G} \bowtie \mathcal{G}$. Then the quotient $\mathcal{G} \bowtie \mathcal{G}/\Lambda\,{}_{\mathfrak{e}}\times_{\mathfrak{s}}\,\Lambda$ factors as a product of the wide subgroupoids $\mathcal{G}/\Lambda\,{}_{\mathfrak{e}}\times_{\mathfrak{s}}\,\mathcal{P}$ and $\mathcal{P}\,{}_{\mathfrak{e}}\times_{\mathfrak{s}}\,\mathcal{G}/\Lambda$; this factorization induces a matched pair structure, and hence a structure of braided groupoid, on $\mathcal{G}/\Lambda$. □

We close this subsection with an application of Theorem 2.14; this is a generalization of the examples in [LYZ1, Section 3], [WX].

Let $(\mathcal{V}, \mathcal{H})$ be a matched pair of groupoids. Recall that the restricted product of $\mathcal{V}$ and $\mathcal{H}$ is the groupoid $\mathcal{V} \boxtimes \mathcal{H} := \{(g,x) \in \mathcal{V} \times \mathcal{H} : \mathfrak{s}(g) = \mathfrak{s}(x), \quad \mathfrak{e}(g) = \mathfrak{e}(x)\}$, with component-wise product [AA].

Proposition 2.17. *Let $\mathcal{G} := \mathcal{V} \boxtimes \mathcal{H}$, $\mathcal{N} := (\mathcal{V} \bowtie \mathcal{H})^{\mathrm{bundle}}$; let $\pi : \mathcal{G} \to \mathcal{N}$ and $\leftharpoonup \; : \mathcal{N}\,{}_{p}\times_{\mathfrak{s}}\,\mathcal{G} \to \mathcal{N}$ be given by*

$$\pi(g,x) = g^{-1}x, \tag{2.35}$$

$$d \leftharpoonup (g,x) = g^{-1}dg, \tag{2.36}$$

$(g,x) \in \mathcal{V}\boxtimes\mathcal{H}$, $d \in (\mathcal{V} \bowtie \mathcal{H})^{\text{bundle}}$. *Then* $(\mathcal{G},\mathcal{N},\pi)$ *is a* 1*-cocycle groupoid datum. Thus,* $(\mathcal{V}\boxtimes\mathcal{H},\sigma)$, *where*

$$\sigma\left((g,x),(h,y)\right) = \left((x \rightharpoonup h, xy(x \leftharpoonup h)^{-1}\right), \left((x \rightharpoonup h)^{-1}gh, x \leftharpoonup h\right)\right), \tag{2.37}$$

$(g,x),(h,y) \in \mathcal{V}\boxtimes\mathcal{H}$, $\mathfrak{e}(x) = \mathfrak{s}(h)$, *is a braided groupoid.*

Proof. A straightforward verification shows that $(\mathcal{G},\mathcal{N},\pi)$ is a 1-cocycle groupoid datum. The explicit formula (2.37) follows from (2.31) once we show that the actions $\rightharpoonup$, $\leftharpoonup$ of $\mathcal{V}\boxtimes\mathcal{H}$ are given by

$$(g,x) \rightharpoonup (h,y) = \left(x \rightharpoonup h, xy(x \leftharpoonup h)^{-1}\right), \qquad (g,x) \leftharpoonup (h,y) = \left((x \rightharpoonup h)^{-1}gh, x \leftharpoonup h\right), \tag{2.38}$$

$(g,x),(h,y) \in \mathcal{V}\boxtimes\mathcal{H}$, $\mathfrak{e}(x) = \mathfrak{s}(h)$. By (2.32), we have

$$\begin{aligned}(g,x) \leftharpoonup (h,y) &= \pi^{-1}\left(\pi(g,x) \leftharpoonup (h,y)\right) = \pi^{-1}\left(g^{-1}x \leftharpoonup (h,y)\right) = \pi^{-1}\left(h^{-1}g^{-1}xh\right) \\ &= \pi^{-1}\left(h^{-1}g^{-1}(x \rightharpoonup h)(x \leftharpoonup h)\right) = \left((x \rightharpoonup h)^{-1}gh, x \leftharpoonup h\right).\end{aligned}$$

This shows the second equality in (2.38), and implies the first by (2.32). □

3. Characterizations of braided quivers

3.1. LYZ-pairs.

We begin with a categorical way of constructing braided quivers. Let $(\mathcal{V},\mathcal{H})$ be a matched pair of groupoids and recall the definition of representation of $(\mathcal{V},\mathcal{H})$ in Subsection 1.7.

Definition 3.1. ([AA], inspired in [LYZ2, T]). Let $\kappa : \mathcal{V} \to \mathcal{H}$ be a morphism of groupoids. We shall say that κ is a *rotation* if

$$y\kappa(g) = \kappa(y \rightharpoonup g)(y \leftharpoonup g) \qquad \text{for all } g \in \mathcal{V},\ y \in \mathcal{H},\ t(g) = r(y). \tag{3.1}$$

Let $\lambda : \mathcal{V} \bowtie \mathcal{H} \to \mathcal{H}$, given by $\lambda(g,x) = \kappa(g)x$. Then κ is a rotation if and only if λ is a morphism of groupoids, see [AA].

A *LYZ-pair*[1] is a pair (ξ,η) of rotations $\mathcal{V} \to \mathcal{H}$ such that

$$\eta(g) \rightharpoonup f = gf\left(\xi(f)^{-1} \rightharpoonup g^{-1}\right). \tag{3.2}$$

for every f and g in $\mathcal{V}$ with $b(g) = t(f)$.

Theorem 3.2. [AA]. *Structures of braided category on* $\operatorname{Rep}(\mathcal{V},\mathcal{H})$ *are parameterized by LYZ-pairs. If* $\mathcal{A},\mathcal{B}$ *are representations of* $(\mathcal{V},\mathcal{H})$ *and* (ξ,η) *is a LYZ-pair, then the induced braiding* $\sigma_{\mathcal{A},\mathcal{B}} : \mathcal{A}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}} \mathcal{B} \to \mathcal{B}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}} \mathcal{A}$ *is given by*

$$\sigma_{\mathcal{A},\mathcal{B}}(a,b) = \Big(\eta(|a|) \rightharpoonup b, \left(\xi(|b|)^{-1} \leftharpoonup |a|^{-1}\right) \rightharpoonup a\Big), \quad (a,b) \in \mathcal{A}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}} \mathcal{B}. \tag{3.3}$$

□

It can be easily shown that

$$\sigma^{-1}_{\mathcal{B},\mathcal{A}}(a,b) = \Big(\xi(|a|) \rightharpoonup b, \left(\eta(|b|)^{-1} \leftharpoonup |a|^{-1}\right) \rightharpoonup a\Big), \quad (a,b) \in \mathcal{A}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}} \mathcal{B}. \tag{3.4}$$

[1]LYZ pairs are called “matched pairs of rotations” in [AA].

Corollary 3.3. *Let $(\mathcal{V},\mathcal{H})$ be a matched pair of groupoids and let (ξ,η) be a LYZ-pair. If $\mathcal{A}$ is a representation of $(\mathcal{V},\mathcal{H})$ then $(\mathcal{A},\sigma_{\mathcal{A},\mathcal{A}})$ is a braided quiver.*

Proof. The statement follows from a well-known general result in braided categories. □

As we shall see in the next Subsection, any non-degenerate braided quiver arises in this way. To this end we shall need the following result that generalizes [T, Prop. 5.1].

Theorem 3.4. *Let $\mathcal{G}$ be a braided groupoid and let $\mathcal{G}\bowtie\mathcal{G}$ be the corresponding diagonal groupoid with respect to $\rightharpoonup$, $\leftharpoonup$.*

Let $\hookrightarrow\, : \mathcal{G}\bowtie\mathcal{G}\;{}_{\mathfrak{e}}\times_{\mathfrak{s}}\mathcal{G}\to\mathcal{G}$, $\hookleftarrow\, : \mathcal{G}\bowtie\mathcal{G}\;{}_{\mathfrak{e}}\times_{\mathfrak{s}}\mathcal{G}\to\mathcal{G}\bowtie\mathcal{G}$, and $\mathrm{in}_1,\mathrm{in}_2:\mathcal{G}\to\mathcal{G}\bowtie\mathcal{G}$ be given by

$$(g,h)\hookrightarrow f = g\rightharpoondown(h\rightharpoonup f), \tag{3.5}$$

$$(g,h)\hookleftarrow f = (g\leftharpoondown(h\rightharpoonup f), h\leftharpoonup f), \tag{3.6}$$

$$\mathrm{in}_1(f) = (f,\mathrm{id}\,\mathfrak{e}(f)), \tag{3.7}$$

$$\mathrm{in}_2(f) = (\mathrm{id}\,\mathfrak{s}(f), f), \tag{3.8}$$

$(g,h)\in\mathcal{G}\bowtie\mathcal{G}$, $f\in\mathcal{G}$, $\mathfrak{e}(h)=\mathfrak{s}(f)$. Then $(\mathcal{G},\mathcal{G}\bowtie\mathcal{G})$, with $\hookrightarrow$, $\hookleftarrow$, is a matched pair of groupoids, and $(\mathrm{in}_1,\mathrm{in}_2)$ is a LYZ-pair for it.

As we already observed in Remark 1.9, $\mathcal{G}$ is a representation of $(\mathcal{G},\mathcal{G}\bowtie\mathcal{G})$; the corresponding braiding as in Corollary 3.3 coincides with the original braiding of $\mathcal{G}$.

Proof. We first check that $\hookrightarrow$ is a left action. If $(g,h,f,k,\ell)\in\mathcal{G}\;{}_{\mathfrak{e}}\times_{\mathfrak{s}}\mathcal{G}\;{}_{\mathfrak{e}}\times_{\mathfrak{s}}\mathcal{G}\;{}_{\mathfrak{e}}\times_{\mathfrak{s}}\mathcal{G}\;{}_{\mathfrak{e}}\times_{\mathfrak{s}}\mathcal{G}$, then

$$\begin{aligned}
(g,h)\hookrightarrow((f,k)\hookrightarrow\ell) &= (g,h)\hookrightarrow(f\rightharpoondown(k\rightharpoonup\ell)) = g\rightharpoondown[h\rightharpoonup(f\rightharpoondown(k\rightharpoonup\ell))]\,;\\
(g,h)(f,k)\hookrightarrow\ell &= (g(h\rightharpoonup f),(h\leftharpoonup f)k)\hookrightarrow\ell = [g(h\rightharpoonup f)]\rightharpoondown[((h\leftharpoonup f)k)\rightharpoonup\ell]\\
&= g\rightharpoondown\{(h\rightharpoonup f)\rightharpoondown[(h\leftharpoonup f)\rightharpoonup(k\rightharpoonup\ell)]\}\,.
\end{aligned}$$

Thus $\hookrightarrow$ is a left action by (2.7). We next check that $\hookleftarrow$ is a right action. If g,hf,k are as above, then

$$\begin{aligned}
((g,h)\hookleftarrow f)\hookleftarrow k &= (g\leftharpoondown(h\rightharpoonup f),h\leftharpoonup f)\hookleftarrow k\\
&= (((g\leftharpoondown(h\rightharpoonup f))\leftharpoondown((h\leftharpoonup f)\rightharpoonup k)\,,(h\leftharpoonup f)\leftharpoonup k)\\
&= (g\leftharpoondown((h\rightharpoonup f)\,((h\leftharpoonup f)\rightharpoonup k))\,,h\leftharpoonup fk)\\
&= (g\leftharpoondown(h\rightharpoonup fk)\,,h\leftharpoonup fk) = (g,h)\hookleftarrow fk.
\end{aligned}$$

We next check the compatibility conditions, (1.7) being clear. We verify (1.8); let g,h,f,k as before. Then

$$\begin{aligned}
((g,h)\hookrightarrow f)\,[((g,h)\hookleftarrow f)\hookrightarrow k] &= [g\rightharpoondown(h\rightharpoonup f)]\,[(g\leftharpoondown(h\rightharpoonup f),h\leftharpoonup f)\hookrightarrow k]\\
&= [g\rightharpoondown(h\rightharpoonup f)]\,\{[(g\leftharpoondown(h\rightharpoonup f)]\rightharpoondown[(h\leftharpoonup f)\rightharpoonup k]\}\\
&= g\rightharpoondown[(h\rightharpoonup f)((h\leftharpoonup f)\rightharpoonup k)]\\
&= g\rightharpoondown(h\rightharpoonup fk) = (g,h)\hookrightarrow fk.
\end{aligned}$$

If g, h, f, k, ℓ are as above, then the left-hand side of (1.9) is

$$\begin{aligned}(g,h)(f,k) \hookleftarrow \ell &= (g(h \rightharpoonup f), (h \leftharpoonup f)k) \hookleftarrow \ell \\ &= ((g(h \rightharpoonup f)) \leftharpoondown [((h \leftharpoonup f)k) \rightharpoonup \ell]\,, ((h \leftharpoonup f)k) \leftharpoonup \ell)\,,\end{aligned}$$

whose first component is

$$\begin{aligned}(g(h \rightharpoonup f)) \leftharpoondown [((h \leftharpoonup f)k) \rightharpoonup \ell] &= \{g \leftharpoondown [(h \rightharpoonup f) \rightharpoondown (((h \leftharpoonup f)k) \rightharpoonup \ell)]\}\,\{(h \rightharpoonup f) \leftharpoondown [((h \leftharpoonup f)k) \rightharpoonup \ell]\} \\ &= \{g \leftharpoondown [(h \rightharpoonup f) \rightharpoondown [(h \leftharpoonup f) \rightharpoonup (k \rightharpoonup \ell)]]\} \\ &\qquad \{(h \rightharpoonup f) \leftharpoondown [(h \leftharpoonup f) \rightharpoonup (k \rightharpoonup \ell)]\}\,.\end{aligned}$$

On the other hand, the first component of the right-hand side of (1.9):

$$((g,h) \hookleftarrow ((f,k) \hookrightarrow \ell))\,((f,k) \hookleftarrow \ell) = (g \leftharpoondown [h \rightharpoonup [f \rightharpoondown (k \rightharpoonup \ell)]]\,, h \leftharpoonup [f \rightharpoondown (k \rightharpoonup \ell)])\,(f \leftharpoondown (k \rightharpoonup \ell), k \leftharpoonup \ell)$$

is

$$\{g \leftharpoondown [h \rightharpoonup [f \rightharpoondown (k \rightharpoonup \ell)]]\}\,\{[h \leftharpoonup [f \rightharpoondown (k \rightharpoonup \ell)]] \rightharpoonup [f \leftharpoondown (k \rightharpoonup \ell)]\}\,,$$

and we have equality with the former because of (2.7) and (2.8). Finally, the second component of the right-hand side of (1.9) is

$$\begin{aligned}\{[h \leftharpoonup [f \rightharpoondown (k \rightharpoonup \ell)]] \leftharpoonup [f \leftharpoondown (k \rightharpoonup \ell)]\}\,(k \leftharpoonup \ell) &= \{h \leftharpoonup [[f \rightharpoondown (k \rightharpoonup \ell)]\,[f \leftharpoondown (k \rightharpoonup \ell)]]\}\,(k \leftharpoonup \ell) \\ &= \{h \leftharpoonup [f(k \rightharpoonup \ell)]\}\,(k \leftharpoonup \ell) \\ &= \{(h \leftharpoonup f) \leftharpoonup (k \rightharpoonup \ell)\}\,(k \leftharpoonup \ell) \\ &= ((h \leftharpoonup f)k) \leftharpoonup \ell.\end{aligned}$$

Thus, we have shown the validity of (1.9).

We claim now that the maps in_1 and in_2 are rotations. If g, h, f are as above, then

$$\begin{aligned}(g,h)\,\mathrm{in}_1(f) &= (g,h)\,(f, \mathrm{id}\,\mathfrak{e}(f)) = (g(h \rightharpoonup f), (h \leftharpoonup f))\,; \\ \mathrm{in}_1\,(((g,h) \hookrightarrow f)\,((g,h) \hookleftarrow f) &= (g \rightharpoondown (h \rightharpoonup f), \mathrm{id}\,\mathfrak{e}(g \rightharpoondown (h \rightharpoonup f)))\,(g \leftharpoondown (h \rightharpoonup f), h \leftharpoonup f) \\ &= ((g \rightharpoondown (h \rightharpoonup f))\,(g \leftharpoondown (h \rightharpoonup f))\,, h \leftharpoonup f) = (g(h \rightharpoonup f), (h \leftharpoonup f))\,;\end{aligned}$$

$$\begin{aligned}(g,h)\,\mathrm{in}_2(f) &= (g,h)\,(\mathrm{id}\,\mathfrak{s}(f), f) = (g, hf)\,; \\ \mathrm{in}_2\,(((g,h) \hookrightarrow f)\,((g,h) \hookleftarrow f) &= (\mathrm{id}\,\mathfrak{s}(g \rightharpoondown (h \rightharpoonup f)), g \rightharpoondown (h \rightharpoonup f))\,(g \leftharpoondown (h \rightharpoonup f), h \leftharpoonup f) \\ &= ((g \rightharpoondown (h \rightharpoonup f)) \rightharpoonup (g \leftharpoondown (h \rightharpoonup f))\,, [(g \rightharpoondown (h \rightharpoonup f)) \leftharpoonup (g \leftharpoondown (h \rightharpoonup f))]\,(h \leftharpoonup f)) \\ &= (g, (h \rightharpoonup f)(h \leftharpoonup f))\,,\end{aligned}$$

where in the last equality we have used (2.6).

We next verify the condition (3.2).

$$\begin{aligned} gf\left(\mathrm{in}_1(f)^{-1} \rightharpoonup g^{-1}\right) &= gf\left((f^{-1}, \mathrm{id}\,\mathfrak{s}(f)) \hookrightarrow g^{-1}\right) = gf\left(f^{-1} \rightharpoondown g^{-1}\right) \\ &= (g \rightharpoonup f)(g \leftharpoonup f)\left(f^{-1} \rightharpoondown g^{-1}\right) = g \rightharpoonup f; \\ \mathrm{in}_2(g) \hookrightarrow f &= (\mathrm{id}\,\mathfrak{s}(g), g) \hookrightarrow f = g \rightharpoonup f. \end{aligned}$$

We have proved that $(\mathrm{in}_1, \mathrm{in}_2)$ is a LYZ-pair for $(\mathcal{G}, \mathcal{G} \bowtie \mathcal{G})$. □

Remark 3.5. Let $\mathcal{G}$ be a braided groupoid. Let $\hookrightarrow\, : \mathcal{G} \bowtie \mathcal{G}\, {}_{\mathfrak{e}}\times_{\mathfrak{s}} \mathcal{G} \to \mathcal{G}$, $\hookleftarrow\, : \mathcal{G} \bowtie \mathcal{G}\, {}_{\mathfrak{e}}\times_{\mathfrak{s}} \mathcal{G} \to \mathcal{G} \bowtie \mathcal{G}$, be given by

$$\begin{aligned} (g, h) \hookrightarrow f &= gh \rightharpoonup f, \\ (g, h) \hookleftarrow f &= (g \leftharpoonup (h \rightharpoonup f), h \leftharpoonup f), \end{aligned}$$

$(g, h) \in \mathcal{G} \bowtie \mathcal{G}$, $f \in \mathcal{G}$, $\mathfrak{e}(h) = \mathfrak{s}(f)$. Then $(\mathcal{G}, \mathcal{G} \bowtie \mathcal{G})$, with $\hookrightarrow$, $\hookleftarrow$, is a matched pair of groupoids. We shall not need this result in the sequel, thus we leave the proof to the reader.

Remark 3.6. Let $(\mathcal{V}, \mathcal{H})$ be a matched pair of groupoids and let (ξ, η) be a LYZ-pair. Then there is a structure of braided groupoid on $\mathcal{V}$ such that $(\mathcal{V}, \mathcal{V} \bowtie \mathcal{V})$, with $(\mathrm{in}_1, \mathrm{in}_2)$, "covers" $(\mathcal{V}, \mathcal{H})$ with (ξ, η). Compare with [T, Section 5]. We shall not need this result in the sequel, so we do not discuss it in detail.

Lemma 3.7. *Let $\mathcal{G}$ be a braided groupoid and let $\mathcal{A}$ be a quiver.*

(i). There is a bijective correspondence between

(a) *Left actions $\hookrightarrow$ of $\mathcal{G} \bowtie \mathcal{G}$ on $\mathcal{A}$.*

(b) *Pairs $(\rightharpoonup, \rightharpoondown)$ of left actions of $\mathcal{G}$ on $\mathcal{A}$ such that*

$$h \rightharpoonup (f \rightharpoondown y) = (h \rightharpoonup f) \rightharpoondown [(h \leftharpoonup f) \rightharpoonup y]\,, \tag{3.9}$$

Assume that $\mathcal{A}$ is a representation of $(\mathcal{G}, \mathcal{G} \bowtie \mathcal{G})$.

(ii). The left action $\rightharpoondown$ is compatible with the definition (2.4).

(iii). The map $|\ |$ is a morphism of braided quivers. Therefore it preserves $\rightharpoonup$, $\leftharpoonup$, $\rightharpoondown$, $\leftharpoondown$.

(iv). The left action $\hookrightarrow$ of $\mathcal{G} \bowtie \mathcal{G}$ on $\mathcal{A}$ induces a left action of $\mathcal{G}$ on $\mathcal{A}\, {}_{\mathfrak{e}}\times_{\mathfrak{s}} \mathcal{A}$ by

$$g \hookrightarrow (x, y) := (g \rightharpoonup x, (g \leftharpoonup |x|) \rightharpoondown y), \tag{3.10}$$

$g \in \mathcal{G}$, $(x, y) \in \mathcal{A}\, {}_{\mathfrak{e}}\times_{\mathfrak{s}} \mathcal{A}$, where $\rightharpoonup$, $\rightharpoondown$ are as in (b).

We shall denote by $\nabla : \mathcal{G} \to \mathfrak{aut}\,(\mathcal{A}\, {}_{\mathfrak{e}}\times_{\mathfrak{s}} \mathcal{A})$ the map induced by the action (3.10).

Proof. (i). (b) $\Longrightarrow$ (a). The correspondence is given by

$$(g, h) \hookrightarrow x = g \rightharpoondown (h \rightharpoonup x), \tag{3.11}$$

$(g,h) \in \mathcal{G} \bowtie \mathcal{G}$, $x \in \mathcal{A}$, $\mathfrak{e}(h) = \mathfrak{s}(x)$. We check that $\hookrightarrow$ is a left action. If $(g,h),(f,k) \in \mathbb{G}_{\mathcal{A}} \bowtie \mathbb{G}_{\mathcal{A}}$, $x \in \mathcal{A}$, $\mathfrak{e}(k) = \mathfrak{s}(x)$, then

$$\begin{aligned}(g,h) \hookrightarrow ((f,k) \hookrightarrow x) &= (g,h) \hookrightarrow (f \rightharpoondown (k \rightharpoonup x)) = g \rightharpoondown [h \rightharpoonup (f \rightharpoondown (k \rightharpoonup x))]\,;\\ (g,h)(f,k) \rightharpoonup x &= (g(h \rightharpoonup f), (h \leftharpoonup f)k) \rightharpoonup x = (g(h \rightharpoonup f)) \rightharpoondown [((h \leftharpoonup f)k) \rightharpoonup x]\\ &= g \rightharpoondown \{(h \rightharpoonup f) \rightharpoondown [(h \leftharpoonup f) \rightharpoonup (k \rightharpoonup x)]\}\,.\end{aligned}$$

Letting $y = k \rightharpoonup x$, we are reduced to (3.9). (a) $\implies$ (b) is similar.

(ii). We have $a \rightharpoondown b = \mathrm{in}_1(|a|) \hookrightarrow b = |a| \rightharpoondown b$, the first equality by (3.4).

(iii). We have $\sigma(a,b) = \Big(\mathrm{in}_2(|a|) \hookrightarrow b, \big(\mathrm{in}_1(|b|)^{-1} \hookleftarrow |a|^{-1}\big) \hookrightarrow a\Big) = \Big(|a| \rightharpoonup b, \big(|b|^{-1} \leftharpoondown |a|^{-1}\big) \rightharpoondown a\Big)$, hence $|\ |$ is a morphism of braided quivers.

(iv). Straightforward using (1.10). □

3.2. **Groupoid-theoretical characterization of braided quivers.**

We first state a characterization of the structure groupoid by a universal property, a generalization of [LYZ1, Th. 9]. Let $(\mathcal{A}, \sigma)$ be a non-degenerate braided quiver and let $\iota : \mathcal{A} \to \mathbb{G}_{\mathcal{A}}$ be the canonical map.

Theorem 3.8. *(a). There is a unique structure of braided groupoid on $\mathbb{G}_{\mathcal{A}}$ such that ι is a morphism of braided quivers.*

(b). The braided groupoid $\mathbb{G}_{\mathcal{A}}$– with the structure in (a)– is universal in the following sense. If $\mathcal{G}$ is a braided groupoid and $\varphi : \mathcal{A} \to \mathcal{G}$ is a morphism of braided quivers, then there is a unique morphism of braided groupoids $\widehat{\varphi} : \mathbb{G}_{\mathcal{A}} \to \mathcal{G}$ such that $\varphi = \widehat{\varphi}\iota$.

Proof. (a). By Lemmas 2.6 and 2.7, there is a structure of a braided quiver on $\mathrm{Path}(\mathcal{DA})$. We first claim that it descends to the free groupoid $F(\mathcal{A})$. Let $(u,v) \in \mathrm{Path}(\mathcal{DA})\,{}_{\mathfrak{e}}{\times}_{\mathfrak{s}}\,\mathrm{Path}(\mathcal{DA})$ and let u', v' be elementary reductions of u, v, respectively. By (2.23) and (2.24), we have $u \rightharpoonup v = u' \rightharpoonup v$ and $u \leftharpoonup v = u \leftharpoonup v'$. Now let $x \in \mathcal{DA}$, $\mathfrak{s}(x) = \mathfrak{e}(u)$. Then

$$u \rightharpoonup (xx^{-1}) = (u \rightharpoonup x)\left((u \leftharpoonup x) \rightharpoonup x^{-1}\right) = (u \rightharpoonup x)(u \rightharpoonup x)^{-1},$$

the first equality by (2.25) and the second by (2.19), since $\mathrm{Path}(\mathcal{DA}) = \mathrm{Path}(\mathcal{DA})^{\mathrm{op}}$ as braided quivers. This implies that $u \rightharpoonup v \sim u \rightharpoonup v'$. Similarly $u \leftharpoonup v \sim u' \leftharpoonup v$. In conclusion, $F(\mathcal{A})$ is a braided quiver with structure inherited from $\mathrm{Path}(\mathcal{DA})$.

We next claim that this descends to the structure groupoid $\mathbb{G}_{\mathcal{A}}$. To see this, we observe that the kernel of the canonical map $F(\mathcal{A}) \to \mathbb{G}_{\mathcal{A}}$ is the subgroup bundle of $F(\mathcal{A})$ given by

$$\mathcal{N} = \left\{n_1^{\pm 1} \dots n_k^{\pm 1} : n_i = x_i y_i (x_i \leftharpoonup y_i)^{-1} (x_i \rightharpoonup y_i)^{-1},\ \text{for some } (x_i, y_i) \in \mathcal{A}\,{}_{\mathfrak{e}}{\times}_{\mathfrak{s}}\,\mathcal{A}\right\}.$$

It is indeed clear that $\mathcal{N}$ is the subgroup bundle generated by the elements of the form $< x, y >:= xy(x \leftharpoonup y)^{-1}(x \rightharpoonup y)^{-1}$, so it remains to check that it is normal. Let $(u, x, y) \in \mathcal{A} {}_{\mathfrak{e}}\times_{\mathfrak{s}} \mathcal{A} {}_{\mathfrak{e}}\times_{\mathfrak{s}} \mathcal{A}$. Then

$$\begin{aligned} u < x, y > u^{-1} &= uxy(x \leftharpoonup y)^{-1}(x \rightharpoonup y)^{-1}u^{-1} =< ux, y > (ux \rightharpoonup y)(ux \leftharpoonup y)(x \leftharpoonup y)^{-1}(x \rightharpoonup y)^{-1}u^{-1} \\ &=< ux, y > (ux \rightharpoonup y)(u \leftharpoonup (x \rightharpoonup y))(x \rightharpoonup y)^{-1}u^{-1} =< ux, y >< u, x \rightharpoonup y >^{-1} \in \mathcal{N}, \end{aligned}$$

where we have used (2.23) and (2.25). This implies that $\mathcal{N}$ is normal. We have $n \rightharpoonup y = y$, $x \leftharpoonup n = x$, by (2.12) and (2.13), and hence $n \leftharpoonup y \equiv y^{-1}ny \mod \mathcal{N}$, $x \rightharpoonup n \equiv xnx^{-1} \mod \mathcal{N}$, if $(x, n, y) \in \mathcal{A} {}_{\mathfrak{e}}\times_{\mathfrak{s}} \mathcal{N} {}_{\mathfrak{e}}\times_{\mathfrak{s}} \mathcal{A}$. Therefore, the maps $\rightharpoonup$, $\leftharpoonup$ descend to well-defined maps $\rightharpoonup, \leftharpoonup : \mathbb{G}_{\mathcal{A}} {}_{\mathfrak{e}}\times_{\mathfrak{s}} \mathbb{G}_{\mathcal{A}} \to \mathbb{G}_{\mathcal{A}}$; these define a map $\sigma : \mathbb{G}_{\mathcal{A}} {}_{\mathfrak{e}}\times_{\mathfrak{s}} \mathbb{G}_{\mathcal{A}} \to \mathbb{G}_{\mathcal{A}} {}_{\mathfrak{e}}\times_{\mathfrak{s}} \mathbb{G}_{\mathcal{A}}$, and this is clearly a solution.

We next claim that $\mathbb{G}_{\mathcal{A}}$ is a braided groupoid. Indeed, (2.27) follows by induction using (2.23), (2.24), (2.25) and (2.26). The structure is unique since $\iota(\mathcal{A})$ generates $\mathbb{G}_{\mathcal{A}}$ as a groupoid.

Finally, the proof of (b) is straightforward. □

We are now ready to prove the main result of this paper.

Definition 3.9. A *structural pair* is a pair $(\mathcal{G}, \mathcal{A})$, where $\mathcal{G}$ is a braided groupoid and $\mathcal{A}$ is a representation of $(\mathcal{G}, \mathcal{G} \bowtie \mathcal{G})$ such that

(3.12) The image $|\mathcal{A}|$ generates the groupoid $\mathcal{G}$.

(3.13) The map $\nabla : \mathcal{G} \to \mathfrak{aut}\,(\mathcal{D}\mathcal{A} {}_{\mathfrak{e}}\times_{\mathfrak{s}} \mathcal{D}\mathcal{A})$ induced by the left action is injective.

Theorem 3.10. *There is a bijective correspondence between*

(a) *non-degenerate braided quivers; and*

(b) *structural pairs.*

Proof. Let $(\mathcal{A}, \sigma)$ be a non-degenerate braided quiver. By Lemma 2.10 (b) and Theorem 3.8 the structure groupoid of $(\mathcal{A}, \sigma^{-1})$ coincides with $\mathbb{G}_{\mathcal{A}}$. Thus we have left and right actions $\rightharpoondown : \mathbb{G}_{\mathcal{A}} {}_{\mathfrak{e}}\times_{\mathfrak{s}} \mathcal{A} \to \mathcal{A}$, $\leftharpoondown : \mathcal{A} {}_{\mathfrak{e}}\times_{\mathfrak{s}} \mathbb{G}_{\mathcal{A}} \to \mathcal{A}$. We define a left action $\hookrightarrow : \mathbb{G}_{\mathcal{A}} \bowtie \mathbb{G}_{\mathcal{A}} {}_{\mathfrak{e}}\times_{\mathfrak{s}} \mathcal{A} \to \mathcal{A}$ by

$$(g, h) \hookrightarrow x = g \rightharpoondown (h \rightharpoonup x), \tag{3.14}$$

$(g, h) \in \mathbb{G}_{\mathcal{A}} \bowtie \mathbb{G}_{\mathcal{A}}$, $x \in \mathcal{A}$, $\mathfrak{e}(h) = \mathfrak{s}(x)$. We check that $\hookrightarrow$ is a left action. By Lemma 3.7, it is enough to verify (3.9): $g \rightharpoonup (h \rightharpoondown x) \overset{?}{=} (g \rightharpoonup h) \rightharpoondown [(g \leftharpoonup h) \rightharpoonup x]$ for g, h, x as above. We know that this is true if $g, h \in \mathcal{D}\mathcal{A}$ by (2.7) applied to $\overline{\sigma}$, *cf.* Lemma 2.6. The set $I_1 = \{h \in \mathbb{G}_{\mathcal{A}} : \text{(3.9) holds for all } g \in \mathcal{D}\mathcal{A}, x \in \mathcal{A}\}$ is closed under multiplication. Indeed, if $h, k \in I_1$, then

$$\begin{aligned} g \rightharpoonup ((hk) \rightharpoondown x) &= g \rightharpoonup (h \rightharpoondown (k \rightharpoondown x)) = (g \rightharpoonup h) \rightharpoondown [(g \leftharpoonup h) \rightharpoonup (k \rightharpoondown x)] \\ &= (g \rightharpoonup h) \rightharpoondown \big[((g \leftharpoonup h) \rightharpoonup k) \rightharpoondown [((g \leftharpoonup h) \leftharpoonup k) \rightharpoonup x] \big] \\ &= ((g \rightharpoonup h)\, ((g \leftharpoonup h) \rightharpoonup k)) \rightharpoondown \big[(g \leftharpoonup hk) \rightharpoonup x \big] = (g \rightharpoonup hk) \rightharpoondown [(g \leftharpoonup hk) \rightharpoonup x]. \end{aligned}$$

Hence $I_1 = \mathbb{G}_{\mathcal{A}}$. Similarly, $I_2 = \{g \in \mathbb{G}_{\mathcal{A}} : \text{(3.9) holds for all } h \in \mathbb{G}_{\mathcal{A}}, x \in \mathcal{A}\}$ is closed under multiplication and hence equals $\mathbb{G}_{\mathcal{A}}$. Thus $\hookrightarrow$ is a left action. The map $\iota : \mathcal{A} \to \mathbb{G}_{\mathcal{A}}$ preserves the actions $\rightharpoonup$, $\leftharpoonup$, $\rightharpoondown$, $\leftharpoondown$; hence it verifies (1.10) because of the definitions (3.5), (3.14). Thus $\mathcal{A}$ is a representation of the matched pair $(\mathbb{G}_{\mathcal{A}}, \mathbb{G}_{\mathcal{A}} \bowtie \mathbb{G}_{\mathcal{A}})$.

Let Λ be the normal subgroup bundle of $\mathbb{G}_{\mathcal{A}}$ given by the intersection of the kernels of the actions $\rightharpoonup$ and $\leftharpoonup$ on $\mathcal{DA}$. We define the *reduced structure groupoid*

$$\mathcal{G}_{\mathcal{A}} := \mathbb{G}_{\mathcal{A}}/\Lambda. \tag{3.15}$$

We claim that Λ is a subgroup bundle of the normal subgroup bundle Γ of $\mathbb{G}_{\mathcal{A}}$ defined by (2.34). Indeed, ι extends to a morphism of braided quivers $\iota : \mathcal{DA} \to \mathbb{G}_{\mathcal{A}}$, and it preserves the actions $\rightharpoonup, \leftharpoonup, \rightharpoondown, \leftharpoondown$. We have

$$v \in \Lambda \implies v \rightharpoonup \iota(x) = \iota(x), \quad \iota(y) \leftharpoonup v = \iota(y), \quad x, y \in \mathcal{DA}, \quad \mathfrak{s}(x) = \mathfrak{e}(v) = \mathfrak{s}(v) = \mathfrak{e}(y).$$

If $v \in \Lambda$ and $w \in \operatorname{Path}(\mathcal{DA})$, say $w = x_1 \dots x_n$ with $x_i \in \mathcal{DA}$, $1 \le i \le n$, $\mathfrak{e}(v) = \mathfrak{s}(w)$, then we see by induction on n that $v \rightharpoonup w = w$. Similarly for the right action $\leftharpoonup$, and thus $\Lambda \subset \Gamma$.

Next $\mathcal{G}_{\mathcal{A}}$ is a braided groupoid by Lemma 2.16 and the action (3.14) induces an action of $\mathcal{G}_{\mathcal{A}} \bowtie \mathcal{G}_{\mathcal{A}}$ on $\mathcal{A}$ by definition of Λ. Let $\| \ \| : \mathcal{A} \to \mathcal{G}_{\mathcal{A}}$ be given by $\|x\| =$ class of $\iota(x)$. The map $\| \ \|$ also preserves the actions $\rightharpoonup, \leftharpoonup, \rightharpoondown, \leftharpoondown$, hence it verifies (1.10). Thus $\mathcal{A}$ is a representation of the matched pair $(\mathcal{G}_{\mathcal{A}}, \mathcal{G}_{\mathcal{A}} \bowtie \mathcal{G}_{\mathcal{A}})$.

We claim that $(\mathcal{G}_{\mathcal{A}}, \mathcal{A})$ is a structural pair. Indeed, condition (3.12) is clear since $\iota(\mathcal{A})$ generates $\mathbb{G}_{\mathcal{A}}$, and condition (3.13) follows at once from the definition of Λ.

Conversely, if $(\mathcal{G}, \mathcal{A})$ is a structural pair, then $\mathcal{A}$ has a structure of braided quiver by Corollary 3.3.

Let us finally check that both constructions are reciprocal. If $\mathcal{A}$ is a braided quiver and $(a, b) \in \mathcal{A}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}} \mathcal{A}$, then we compute the braiding arising from $\mathcal{G}_{\mathcal{A}}$ by (3.3):

$$\begin{aligned}
\sigma_{\mathcal{A},\mathcal{A}}(a, b) &= \Big(\mathrm{in}_2(\|a\|) \hookrightarrow b, \big(\mathrm{in}_1(\|b\|)^{-1} \hookleftarrow \|a\|^{-1}\big) \hookrightarrow a\Big) \\
&= \Big((\mathrm{id}\,\mathfrak{s}(a), \|a\|) \hookrightarrow b, \big((\|b\|^{-1}, \mathrm{id}\,\mathfrak{s}(b)) \hookleftarrow \|a\|^{-1}\big) \hookrightarrow a\Big) \\
&= \Big(\mathrm{id}\,\mathfrak{s}(a) \rightharpoondown (\|a\| \rightharpoonup b), \big(\|b\|^{-1} \leftharpoondown \|a\|^{-1}, \mathrm{id}\,\mathfrak{s}(a)\big) \hookrightarrow a\Big) \\
&= \Big(a \rightharpoonup b, \big(b^{-1} \leftharpoondown a^{-1}\big) \rightharpoondown a\Big) = \Big(a \rightharpoonup b, \big(b^{-1} \rightharpoondown a^{-1}\big)^{-1}\Big) \\
&= \Big(a \rightharpoonup b, a \leftharpoonup b\Big) = \sigma(a, b).
\end{aligned}$$

Conversely, let $(\mathcal{G}, \mathcal{A})$ be a structural pair. By Theorem 3.8 there is a unique morphism of braided groupoids $\psi : \mathbb{G}_{\mathcal{A}} \to \mathcal{G}$ such that $\psi(\iota(a)) = |a|$, $a \in \mathcal{A}$; by condition (3.12), ψ is surjective. We claim that $\operatorname{Ker} \psi = \Lambda$. It is enough to show that the following diagram commutes:

$$\begin{array}{ccc}
\mathbb{G}_{\mathcal{A}} & \xrightarrow{\ \psi\ } & \mathcal{G} \\
& {\scriptstyle \triangledown} \searrow \quad \swarrow {\scriptstyle \triangledown} & \\
& \mathfrak{aut}\,\mathcal{DA}. &
\end{array}$$

Let $(g, x, y) \in \mathbb{G}_{\mathcal{A}}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}} \mathcal{DA}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}} \mathcal{DA}$. Then

$$\psi(g) \hookrightarrow (x, y) = (\psi(g) \rightharpoonup x, (\psi(g) \leftharpoonup |x|) \rightharpoondown y) = (\psi(g) \rightharpoonup x, \psi(g \leftharpoonup \iota(x)) \rightharpoondown y),$$

the second equality since ψ is a morphism of solutions. Hence, we are reduced to prove that $\psi(g) \rightharpoonup x \overset{?}{=} g \rightharpoonup x$, $z \leftharpoonup \psi(g) \overset{?}{=} z \leftharpoonup g$, if $(z, g, x) \in \mathcal{DA} {}_{\mathfrak{e}}\!\times_{\mathfrak{s}} \mathbb{G}_{\mathcal{A}} {}_{\mathfrak{e}}\!\times_{\mathfrak{s}} \mathcal{DA}$. If $g \in \iota(\mathcal{DA})$, the identities hold because of the definition of ψ. Then the identities always hold. □

Remark 3.11. The notion of "structural pair" is a generalization and reformulation of the notion of "faithful bijective 7-uple" in [S]; Theorem 3.10 is a generalization of Soloviev's Theorem [S, Th. 2.7]. In our formulation, Theorem 2.7 in [S] reads as follows:

"There is a bijective correspondence between non-degenerate braided sets and pairs (G, A), where G is a braided group and A is a representation of the matched pair $(G, G \bowtie G)$ such that (3.12) and (3.13) hold."

Remark 3.12. If the braided quiver $\mathcal{A}$ is finite, then the reduced structure groupoid $\mathcal{G}_{\mathcal{A}}$ is finite by condition (3.13).

Remark 3.13. Let us say that a non-degenerate braided quiver is *faithful* if the map ι is injective. In this case, $\Lambda = \Gamma$ as in the proof of Theorem 3.10, and $\| \ \|$ is also injective. Thus, faithful non-degenerate braided quivers are in bijective correspondence with pairs $(\mathcal{G}, \mathcal{A})$ where $\mathcal{G}$ is a braided quiver and $\mathcal{A}$ is an invariant sub-quiver that generates $\mathcal{G}$.

Remark 3.14. The structural pair of a rack $(X, \triangleright)$ is $(\operatorname{Inn}_{\triangleright} X, \phi)$ where the group $\operatorname{Inn}_{\triangleright} X$ is braided via the adjoint representation and $\phi : X \to \operatorname{Inn}_{\triangleright} X$ is the map $\phi(x)(y) = x \triangleright y$, see [AG].

Remark 3.15. An explicit group-theoretical description of matched pairs of groupoids can be found in [AN, Th. 2.16], see also [AM]. It follows from this description that a finite braided groupoid is roughly determined by a group D with two subgroups V and H such that:

- there is a bijection $\mathcal{P} \simeq V \backslash D / H$;
- V and H are isomorphic;
- V intersects trivially any conjugate of H,

plus some other data in terms of functions $V \times \mathcal{P} \times V \to \mathcal{P}$, etc. See [MM] for more details and explicit examples.

3.3. **Rack bundles.**

Let $p : \mathcal{L} \to \mathcal{P}$ be a loop bundle. If $\sigma : \mathcal{L} {}_p\!\times_p \mathcal{L} \to \mathcal{L} {}_p\!\times_p \mathcal{L}$ is a solution, and $\mathcal{L}_P$ is the fiber of $P \in \mathcal{P}$, then the restriction $\sigma_P : \mathcal{L}_P \times \mathcal{L}_P \to \mathcal{L}_P \times \mathcal{L}_P$ is a solution to the set-theoretical braid equation. In other words, a solution $(\mathcal{L}, \sigma)$ with $\mathcal{L}$ a loop bundle is the same as a bundle of solutions of the set-theoretical braid equation. For instance, a *rack bundle* is a pair $(\mathcal{L}, \triangleright)$ where $\mathcal{L}$ is a loop bundle and $\triangleright = (\triangleright_P)_{P \in \mathcal{P}}$, where $\triangleright_P$ is a structure of rack in the fiber $\mathcal{L}_P$, $P \in \mathcal{P}$; we omit the subscript in what follows. (See *e. g.* [AG] for information on racks). This means that

$$\phi_x : \mathcal{L}_P \to \mathcal{L}_P, \quad \phi_x(y) = x \triangleright y, \quad \text{is a bijection for all } x \in \mathcal{L}_P,\ P \in \mathcal{P}, \tag{3.16}$$

$$x \triangleright (y \triangleright z) = (x \triangleright y) \triangleright (x \triangleright z) \qquad x, y, z \in \mathcal{L}_P,\ P \in \mathcal{P}. \tag{3.17}$$

Remark 3.16. Let $p : \mathcal{L} \to \mathcal{P}$ be a loop bundle and let $\triangleright : \mathcal{L}\,{}_p{\times}_p\,\mathcal{L} \to \mathcal{L}$ be a morphism of quivers. Let $c : \mathcal{L}\,{}_p{\times}_p\,\mathcal{L} \to \mathcal{L}\,{}_p{\times}_p\,\mathcal{L}$ be given by

$$c(x, y) = (x \triangleright y, x), \qquad (x, y) \in \mathcal{L}\,{}_p{\times}_p\,\mathcal{L}. \tag{3.18}$$

Then c is a non-degenerate solution if and only if $(\mathcal{L}, \triangleright)$ is a rack bundle.

Observe also that if $c : \mathcal{A}\,{}_{\mathfrak{e}}{\times}_{\mathfrak{s}}\,\mathcal{A} \to \mathcal{A}\,{}_{\mathfrak{e}}{\times}_{\mathfrak{s}}\,\mathcal{A}$ is an isomorphism of quivers defined by a formula analogous to (3.18) and with $(\mathfrak{s}(x), x \triangleright \_\,, \mathfrak{e}(x)) \in \mathfrak{aut}\,\mathfrak{s}$ for any $x \in \mathcal{A}$, then $\mathcal{A}$ is a loop bundle. Indeed, given $x \in \mathcal{A}$ there exists $y \in \mathcal{A}$ such that $x \triangleright y = x$ but then $\mathfrak{s}(x) = \mathfrak{e}(x)$.

If $(\mathcal{L}, \triangleright)$ is a rack bundle, then we set

$$\mathfrak{aut}_{\,\triangleright}\,\mathcal{L} := \{(P, x, Q) : P, Q \in \mathcal{P},\ \text{and } x : \mathcal{E}_Q \to \mathcal{E}_P \text{ is an isomorphism of racks}\}.$$

Example 3.17. If $\mathcal{N}$ is a group bundle, define $\triangleright : \mathcal{N}\,{}_p{\times}_p\,\mathcal{N} \to \mathcal{N}$ by $x \triangleright y := xyx^{-1}$; then $(\mathcal{N}, \triangleright)$ is a rack bundle.

3.4. The derived solution. Rack-theoretical characterization of braided quivers.

Let $\mathcal{A}$, $\widetilde{\mathcal{A}}$ be quivers and $\sigma : \mathcal{A}\,{}_{\mathfrak{e}}{\times}_{\mathfrak{s}}\,\mathcal{A} \to \mathcal{A}\,{}_{\mathfrak{e}}{\times}_{\mathfrak{s}}\,\mathcal{A}$, $\widetilde{\sigma} : \widetilde{\mathcal{A}}\,{}_{\mathfrak{e}}{\times}_{\mathfrak{s}}\,\widetilde{\mathcal{A}} \to \widetilde{\mathcal{A}}\,{}_{\mathfrak{e}}{\times}_{\mathfrak{s}}\,\widetilde{\mathcal{A}}$ be isomorphisms of quivers. We say that $(\mathcal{A}, \sigma)$ and $(\widetilde{\mathcal{A}}, \widetilde{\sigma})$ are *equivalent* if there exists a family of bijections $U^n : \mathcal{A}^n \to \widetilde{\mathcal{A}}^n$ such that $U^n \sigma_{i,i+1} = \widetilde{\sigma}_{i,i+1} U^n$, for all $n \geq 2$, $1 \leq i \leq n-1$.

Remark 3.18. If $(\mathcal{A}, \sigma)$ and $(\widetilde{\mathcal{A}}, \widetilde{\sigma})$ are equivalent and $(\mathcal{A}, \sigma)$ is a solution, then $(\widetilde{\mathcal{A}}, \widetilde{\sigma})$ is also a solution and the U^n's intertwine the corresponding actions of the braid group $\mathbb{B}_n$.

Definition 3.19. A 1-*cocycle quiver datum* is a collection $(\mathcal{A}, \mathcal{L}, \varphi, \mu)$ where $\mathcal{A}$ is a quiver over $\mathcal{P}$, $\mathcal{L}$ is a rack bundle over $\mathcal{P}$, $\varphi : \mathcal{A} \to \mathfrak{aut}\,\mathcal{L}$ is an injective morphism of quivers over $\mathcal{P}$, and $\mu : \mathcal{L} \to \mathcal{A}^{\mathfrak{e}}$ is an isomorphism of bundles over $\mathcal{P}$, subject to the cocycle condition (3.21) below.

For simplicity of the notation, we shall identify $\mathcal{A}$ with a sub-quiver of $\mathfrak{aut}\,\mathcal{L}$, and denote the inverse $\mu^{-1} : \mathcal{A}^{\mathfrak{e}} \to \mathcal{L}$ by $\mu^{-1}(x) = \overline{x}$.

To state the cocycle condition, we define first $\leftharpoonup\, : \mathcal{A}\,{}_{\mathfrak{e}}{\times}_{\mathfrak{s}}\,\mathfrak{aut}\,\mathcal{L} \to \mathcal{A}$ by

$$x \leftharpoonup y = \mu y^{-1}(\overline{x}), \qquad (x, y) \in \mathcal{A}\,{}_{\mathfrak{e}}{\times}_{\mathfrak{s}}\,\mathfrak{aut}\,\mathcal{L}. \tag{3.19}$$

In other words, this is the natural right action of $\mathfrak{aut}\,\mathcal{L}$ on $\mathcal{L}$ pulled back to $\mathcal{A}$ via μ. Clearly $\mathfrak{e}(x \leftharpoonup y) = \mathfrak{e}(y)$. By restriction we have $\leftharpoonup\, : \mathcal{A}\,{}_{\mathfrak{e}}{\times}_{\mathfrak{s}}\,\mathcal{A} \to \mathcal{A}$. We then define $\rightharpoonup\, : \mathcal{A}\,{}_{\mathfrak{e}}{\times}_{\mathfrak{s}}\,\mathcal{A} \to \mathcal{A}$ by

$$x \rightharpoonup y = \mu\,(\overline{x \leftharpoonup y} \triangleright \overline{y}) \leftharpoonup (x \leftharpoonup y)^{-1}, \qquad (x, y) \in \mathcal{A}\,{}_{\mathfrak{e}}{\times}_{\mathfrak{s}}\,\mathcal{A}. \tag{3.20}$$

This map is well-defined and $\mathfrak{e}(x \rightharpoonup y) = \mathfrak{s}(x \leftharpoonup y)$. *The cocycle condition is*

$$x \rightharpoonup y = xy(x \leftharpoonup y)^{-1}, \qquad (x, y) \in \mathcal{A}\,{}_{\mathfrak{e}}{\times}_{\mathfrak{s}}\,\mathcal{A}. \tag{3.21}$$

Note that (3.21) implies $\mathfrak{s}(x \rightharpoonup y) = \mathfrak{s}(x)$. Note also that (3.20) is equivalent to

$$(z \leftharpoonup y^{-1}) \rightharpoonup y = \mu(\overline{z} \triangleright \overline{y}) \leftharpoonup z^{-1}, \qquad (z, y) \in \mathcal{A}\,{}_{\mathfrak{e}}{\times}_{\mathfrak{s}}\,\mathcal{A}.$$

This is in turn equivalent to (3.22) below.

Now we are ready to state the main result of this subsection.

Theorem 3.20. *Let $\mathcal{A}$ be a quiver. There is a bijective correspondence between*

(a) *structures of non-degenerate braided quiver $(\mathcal{A}, \sigma)$;*

(b) 1-*cocycle quiver data $(\mathcal{A}, \mathcal{L}, \varphi, \mu)$.*

Proof. The proof of "(a) $\implies$ (b)" is given by the next lemma that generalizes results from [S, LYZ1].

Lemma 3.21. *Let $(\mathcal{A}, \sigma)$ be a non-degenerate braided quiver and let $\mathcal{L} := \mathcal{A}^{\mathfrak{e}}$ be the loop bundle as in (1.2). Let $\triangleright : \mathcal{L}\, {}_p{\times}_p\, \mathcal{L} \to \mathcal{L}$ be the morphism of bundles, resp. $\varphi : \mathcal{A} \to \mathfrak{aut}\, \mathcal{L}$ the morphism of quivers, defined by*

$$\overline{x} \triangleright \overline{y} = \overline{((x \leftharpoonup y^{-1}) \rightharpoonup y) \leftharpoonup x}, \tag{3.22}$$

$$\varphi_y(\overline{x}) = \overline{x \leftharpoonup y^{-1}}, \tag{3.23}$$

$x, y \in \mathcal{A}$, $\mathfrak{e}(x) = \mathfrak{e}(y)$.

(a). φ induces a morphism of groupoids $\varphi : \mathbb{G}_{\mathcal{A}} \to \mathfrak{aut}_{\,\triangleright}\mathcal{L}$.

(b). $(\mathcal{L}, \triangleright)$ is a rack bundle. If c is given by (3.18), then c is a solution, called the derived solution *of σ. The solutions σ and c are equivalent.*

Proof. Note that (3.22) and (3.23) are well-defined since $\mathfrak{e}(x) = \mathfrak{e}(y)$. By (2.13), φ induces a morphism of groupoids $\varphi : \mathbb{G}_{\mathcal{A}} \to \mathfrak{aut}\, \mathcal{L}$.

(a). We have to show $\varphi_y(\overline{x} \triangleright \overline{z}) \overset{?}{=} \varphi_y(\overline{x}) \triangleright \varphi_y(\overline{z})$ if $\mathfrak{e}(x) = \mathfrak{e}(y) = \mathfrak{e}(z)$. Now

$$\begin{aligned}
\varphi_y(\overline{x}) \triangleright \varphi_y(\overline{z}) &= \overline{x \leftharpoonup y^{-1}} \triangleright \overline{z \leftharpoonup y^{-1}} \\
&= \overline{(((x \leftharpoonup y^{-1}) \leftharpoonup (z \leftharpoonup y^{-1})^{-1}) \rightharpoonup (z \leftharpoonup y^{-1})) \leftharpoonup (x \leftharpoonup y^{-1})} \\
&= \overline{(((x \leftharpoonup y^{-1}) \leftharpoonup yz^{-1}(z \rightharpoonup y^{-1})) \rightharpoonup (z \leftharpoonup y^{-1})) \leftharpoonup (x \leftharpoonup y^{-1})} \\
&= \overline{(((x \leftharpoonup z^{-1}) \leftharpoonup (z \rightharpoonup y^{-1})) \rightharpoonup (z \leftharpoonup y^{-1})) \leftharpoonup (x \leftharpoonup y^{-1})} \\
&= \overline{(((x \leftharpoonup z^{-1}) \rightharpoonup z) \leftharpoonup (((x \leftharpoonup z^{-1}) \leftharpoonup z) \rightharpoonup y^{-1})) \leftharpoonup (x \leftharpoonup y^{-1})} \\
&= \overline{((x \leftharpoonup z^{-1}) \rightharpoonup z) \leftharpoonup (x \rightharpoonup y^{-1})(x \leftharpoonup y^{-1})} \\
&= \overline{((x \leftharpoonup z^{-1}) \rightharpoonup z) \leftharpoonup xy^{-1}} \\
&= \varphi_y(\overline{x} \triangleright \overline{z}).
\end{aligned}$$

Here the third, fourth, sixth and seventh equalities are by (2.13), and the fifth by (2.14).

(b). Let $U^n : \mathcal{A}^n \to \mathcal{L}^n$ be defined inductively by

$$U^2(x_1, x_2) = (\overline{x_1 \leftharpoonup x_2}, \overline{x_2}), \tag{3.24}$$

$$U^{n+1} = Q_n(U^n \times \mathrm{id}), \text{ where } Q_n(\overline{x_1}, \dots, \overline{x_n}, x_{n+1}) = (\overline{x_1 \leftharpoonup x_{n+1}}, \dots, \overline{x_n \leftharpoonup x_{n+1}}, \overline{x_{n+1}}). \tag{3.25}$$

We claim that

$$c_{i,i+1} U^n = U^n \sigma_{i,i+1}, \qquad n \geq 2,\ 1 \leq i \leq n-1. \tag{3.26}$$

If $n = 2$, then

$$\begin{aligned} c\,U^2(x_1, x_2) &= c(\overline{x_1 \leftharpoondown x_2}, \overline{x_2}) = (\overline{x_1 \leftharpoondown x_2} \triangleright \overline{x_2}, \overline{x_1 \leftharpoondown x_2}) \\ &= (\overline{(((x_1 \leftharpoondown x_2) \leftharpoondown x_2^{-1}) \rightharpoonup x_2)} \leftharpoondown (x_1 \leftharpoondown x_2), \overline{x_1 \leftharpoondown x_2}) \\ &= (\overline{(x_1 \rightharpoonup x_2) \leftharpoondown (x_1 \leftharpoondown x_2)}, \overline{x_1 \leftharpoondown x_2}) \\ &= U^2(x_1 \rightharpoonup x_2, x_1 \leftharpoondown x_2) = U^2\sigma(x_1, x_2). \end{aligned}$$

Assume that (3.26) is true for n and let i be such that $1 \le i \le n-1$. Then

$$c_{i,i+1}U^{n+1} = c_{i,i+1}Q_n(U^n \times \mathrm{id}) = Q_n c_{i,i+1}(U^n \times \mathrm{id}) = Q_n(U^n \times \mathrm{id})\sigma_{i,i+1} = U^{n+1}\sigma_{i,i+1};$$

here the second equality follows from part (a). So, let $i = n$. We claim that $c_{n,n+1}Q_n(Q^{n-1} \times \mathrm{id}) = Q_n(Q^{n-1} \times \mathrm{id})\sigma_{n,n+1}$; indeed

$$\begin{aligned} c_{n,n+1}Q_n(Q^{n-1} \times \mathrm{id})(\overline{x_1}, \dots, \overline{x_{n-1}}, x_n, x_{n+1}) &= c_{n,n+1}Q_n(\overline{x_1 \leftharpoondown x_n}, \dots, \overline{x_{n-1} \leftharpoondown x_n}, \overline{x_n}, x_{n+1}) \\ &= c_{n,n+1}(\overline{(x_1 \leftharpoondown x_n) \leftharpoondown x_{n+1}}, \dots, \overline{(x_{n-1} \leftharpoondown x_n) \leftharpoondown x_{n+1}}, \overline{x_n \leftharpoondown x_{n+1}}, \overline{x_{n+1}}) \\ &= (\overline{x_1 \leftharpoondown x_n x_{n+1}}, \dots, \overline{x_{n-1} \leftharpoondown x_n x_{n+1}}, \overline{(x_n \rightharpoonup x_{n+1}) \leftharpoondown (x_n \leftharpoondown x_{n+1})}, \overline{x_n \leftharpoondown x_{n+1}}), \end{aligned}$$

and this equals

$$\begin{aligned} &Q_n(Q^{n-1} \times \mathrm{id})\sigma_{n,n+1}(\overline{x_1}, \dots, \overline{x_{n-1}}, x_n, x_{n+1}) = Q_n(Q^{n-1} \times \mathrm{id})(\overline{x_1}, \dots, \overline{x_{n-1}}, x_n \rightharpoonup x_{n+1}, x_n \leftharpoondown x_{n+1}) \\ &= (\overline{x_1 \leftharpoondown (x_n \rightharpoonup x_{n+1})(x_n \leftharpoondown x_{n+1})}, \dots, \overline{(x_n \rightharpoonup x_{n+1}) \leftharpoondown (x_n \leftharpoondown x_{n+1})}, \overline{x_n \leftharpoondown x_{n+1}}) \end{aligned}$$

because of (2.13). Then

$$\begin{aligned} c_{n,n+1}U^{n+1} &= c_{n,n+1}Q_n(Q^{n-1} \times \mathrm{id})(U^{n-1} \times \mathrm{id}) \\ &= Q_n(Q^{n-1} \times \mathrm{id})\sigma_{n,n+1}(U^{n-1} \times \mathrm{id}) = Q_n(Q^{n-1} \times \mathrm{id})(U^{n-1} \times \mathrm{id})\sigma_{n,n+1} = U^{n+1}\sigma_{i,i+1}. \end{aligned}$$

Hence σ and c are equivalent; then c is a solution by Remark 3.18, and $(\mathcal{L}, \triangleright)$ is a rack bundle by Remark 3.16. □

We now prove the implication "(b) $\implies$ (a)". This was implicit in [S], in the set-theoretical setting. See also [AG, Prop. 5.4 (3)].

Let us consider a collection $(\mathcal{A}, \mathcal{L}, \varphi, \mu)$ as in Definition 3.19 but without assuming the cocycle condition (3.21). Define $\leftharpoondown, \rightharpoonup \,: \mathcal{A}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}} \mathcal{A} \to \mathcal{A}$ by (3.19), (3.20); assume that $\mathfrak{s}(x \rightharpoonup y) = \mathfrak{s}(x)$. Let $\sigma : \mathcal{A}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}} \mathcal{A} \to \mathcal{A}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}} \mathcal{A}$ be given by (2.3): $\sigma(x, y) = (x \rightharpoonup y, x \leftharpoondown y)$, $(x, y) \in \mathcal{A}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}} \mathcal{A}$.

Lemma 3.22. *The map σ is a solution if and only if the condition* (3.21) *holds. If this happens, the solutions σ and c are equivalent, and σ is non-degenerate.*

Proof. Assume that (3.21) holds. Let $U^n : \mathcal{A}^n \to \mathcal{L}^n$ be defined by (3.24), (3.25). We claim that (3.26) holds in the present situation too. In fact, one can repeat the proof for $n = 2$ word by word, since (3.20) is equivalent to (3.22). Same for the proof of the inductive step, $i < n$ since $\mathcal{A} \subset \mathfrak{aut}_{\triangleright}\mathcal{L}$ by hypothesis. Finally, the proof of the inductive step, $i = n$ can also be repeated because condition (2.13) follows from (3.21). Hence σ and c are equivalent, and σ is a solution by Remark 3.18.

Conversely, assume that σ is a solution. Then (2.13) holds; writing this explicitly down, see (3.19), we get (3.21). □

This finishes the proof of the theorem. □

Remark 3.23. Let $(\mathcal{A},\sigma)$, $(\widetilde{\mathcal{A}},\widetilde{\sigma})$ be two non-degenerate braided quivers. A *morphism of braided quivers* is a morphism of quivers $T:\mathcal{A}\to\widetilde{\mathcal{A}}$ such that $T\times T$ intertwines σ and $\widetilde{\sigma}$; that is, such that

$$T(x\rightharpoonup y)=T(x)\rightharpoonup T(y), \tag{3.27}$$

$$T(x\leftharpoonup y)=T(x)\leftharpoonup T(y), \tag{3.28}$$

$(x,y)\in\mathcal{A}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}}\mathcal{A}$.

A morphism of quivers T is a morphism of braided quivers if and only $\overline{T}$ is a morphism of the associated rack bundles and (3.28) holds.

4. Linearization

Let $\Bbbk$ be a field. Let $\mathcal{X}$ be a set. We denote by $\Bbbk\mathcal{X}$ the $\Bbbk$-vector space with basis e_X, $X\in\mathcal{X}$.

Recall that $\mathcal{P}$ is our fixed basis of quivers and groupoids. We consider $\Bbbk\mathcal{P}$ as a (commutative, semisimple) algebra with multiplication $e_Pe_Q=\delta_{P,Q}\,e_P$, $P,Q\in\mathcal{P}$.

4.1. **The category of bimodules.**

The tensor category ${}_{\mathcal{P}}\mathcal{M}_{\mathcal{P}}$ of $\Bbbk\mathcal{P}$-bimodules, with tensor product $\otimes_{\Bbbk\mathcal{P}}$ and unit $\mathbf{1}=\Bbbk\mathcal{P}$, can be identified with the category of $\mathcal{P}\times\mathcal{P}$-graded vector spaces (with maps homogeneous of degree 0). If M is a $\mathcal{P}\times\mathcal{P}$-graded vector space, then the grading is denoted $M=\oplus_{P,Q\in\mathcal{P}}\,{}_PM_Q$, with ${}_PM_Q=e_PMe_Q$. If $m\in M$, then we set ${}_Pm_Q=e_Pme_Q$. The tensor product of $M,N\in{}_{\mathcal{P}}\mathcal{M}_{\mathcal{P}}$ is given by $M\otimes_{\Bbbk\mathcal{P}}N=\oplus_{P,Q\in\mathcal{P}}\left(\oplus_{R\in\mathcal{P}}\,{}_PM_R\otimes{}_RN_Q\right)$.

If $M\in{}_{\mathcal{P}}\mathcal{M}_{\mathcal{P}}$, then we set $M^*=\oplus_{P,Q\in\mathcal{P}}\,{}_P(M^*)_Q$ where ${}_P(M^*)_Q:=\operatorname{Hom}_{\Bbbk}({}_QM_P,\Bbbk)$. We shall consider the map $\operatorname{ev}:M^*\otimes_{\Bbbk\mathcal{P}}M\to\Bbbk\mathcal{P}$ given by

$$\operatorname{ev}\left(\sum_{P,Q,R\in\mathcal{P}}{}_P\alpha_R\otimes{}_Rm_Q\right)=\sum_{P\in\mathcal{P}}\left(\sum_{R\in\mathcal{P}}\langle{}_P\alpha_R,\,{}_Rm_P\rangle\right)e_P,$$

for ${}_P\alpha_R\in{}_P(M^*)_R=\operatorname{Hom}_{\Bbbk}({}_RM_P,\Bbbk)$, ${}_Rm_Q\in{}_RM_Q$.

Assume that $\mathcal{P}$ is finite. Let ${}_{\mathcal{P}}\mathfrak{m}_{\mathcal{P}}$ be the full tensor subcategory of ${}_{\mathcal{P}}\mathcal{M}_{\mathcal{P}}$ whose objects are the $\mathcal{P}\times\mathcal{P}$-graded vector spaces with finite-dimensional homogeneous subspaces. Let $M\in{}_{\mathcal{P}}\mathfrak{m}_{\mathcal{P}}$ and choose a basis ${}_Pm^i_Q$ of ${}_PM_Q$, i running in some index set $I(P,Q)$. Let ${}_P\alpha^i_Q$ be the corresponding dual basis. We consider the map $\operatorname{coev}:\Bbbk\mathcal{P}\to M\otimes_{\Bbbk\mathcal{P}}M^*$ given by

$$\operatorname{coev}(e_P):=\sum_{P,Q\in\mathcal{P}}\sum_{i\in I(P,Q)}{}_Pm^i_Q\otimes{}_P\alpha^i_Q\in{}_PM_Q\otimes\operatorname{Hom}_{\Bbbk}({}_PM_Q,\Bbbk)\subset{}_P(M^*\otimes_{\Bbbk\mathcal{P}}M)_P.$$

Then M^* is the dual of M and ${}_{\mathcal{P}}\mathfrak{m}_{\mathcal{P}}$ is rigid.

Remark 4.1. Assume that $\mathcal{P}$ is not finite. Then we define ${}_{\mathcal{P}}\mathfrak{m}_{\mathcal{P}}$ as the full tensor subcategory of ${}_{\mathcal{P}}\mathcal{M}_{\mathcal{P}}$ whose objects are the $\mathcal{P}\times\mathcal{P}$-graded vector spaces M with finite-dimensional homogeneous subspaces, and satisfying a condition of finite support: for any $P\in\mathcal{P}$, the sets

$$\operatorname{supp}{}_P(M)=\{Q\in\mathcal{P}:\,{}_PM_Q\neq0\}\text{ and }\operatorname{supp}(M)_P=\{Q\in\mathcal{P}:\,{}_QM_P\neq0\}$$

are both finite. Again ${}_{\mathcal{P}}\mathfrak{m}_{\mathcal{P}}$ is rigid.

There is a tensor functor Lin from the category Quiv($\mathcal{P}$) of quivers over $\mathcal{P}$ to ${}_{\mathcal{P}}\mathcal{M}_{\mathcal{P}}$ given by

$$\mathcal{A} \xrightarrow{\text{Lin}} \Bbbk\,\mathcal{A} = \oplus_{P,Q\in\mathcal{P}}\, {}_P(\Bbbk\,\mathcal{A})_Q, \quad \text{where } {}_P(\Bbbk\,\mathcal{A})_Q := \Bbbk\,\mathcal{A}(P,Q). \tag{4.1}$$

By abuse of notation, if $T : \mathcal{A} \to \mathcal{B}$ is a morphism of quivers, then we also denote by $T : \Bbbk\,\mathcal{A} \to \Bbbk\,\mathcal{B}$ the linear map $\operatorname{Lin} T$; that is, $T(e_f) = e_{T(f)}$, $f \in \mathcal{A}$.

If $M \in {}_{\mathcal{P}}\mathcal{M}_{\mathcal{P}}$, then choose a basis $\mathcal{A}(P,Q)$ of ${}_QM_P$, $P,Q \in \mathcal{P}$. The union $\mathcal{A} := \coprod_{P,Q\in\mathcal{P}} \mathcal{A}(P,Q)$ is a quiver and $M \simeq \Bbbk\,\mathcal{A}$ in ${}_{\mathcal{P}}\mathcal{M}_{\mathcal{P}}$.

4.2. **Solutions of the braid equation in the category of bimodules.**

Let $\mathcal{A}$ be a quiver over $\mathcal{P}$, let $\sigma : \mathcal{A}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}}\mathcal{A} \to \mathcal{A}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}}\mathcal{A}$ be an isomorphism of quivers and let $\mathbf{q} : \mathcal{A}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}}\mathcal{A} \to \Bbbk^{\times}$ be a function. Let $\sigma^{\mathbf{q}} : \Bbbk\,\mathcal{A} \otimes \Bbbk\,\mathcal{A} \to \Bbbk\,\mathcal{A} \otimes \Bbbk\,\mathcal{A}$ be given by

$$\sigma^{\mathbf{q}}(e_x \otimes e_y) = \mathbf{q}_{x,y}\,\sigma(e_x \otimes e_y) = \mathbf{q}_{x,y}\, e_{x \rightharpoonup y} \otimes e_{x \leftharpoonup y}, \qquad (x,y) \in \mathcal{A}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}}\mathcal{A}. \tag{4.2}$$

Lemma 4.2. *$\sigma^{\mathbf{q}}$ is a solution of the braid equation in ${}_{\mathcal{P}}\mathcal{M}_{\mathcal{P}}$ if and only if $(\mathcal{A},\sigma)$ a braided quiver and*

$$\mathbf{q}_{x,y}\mathbf{q}_{x \leftharpoonup y,z}\mathbf{q}_{x \rightharpoonup y,(x \leftharpoonup y) \rightharpoonup z} = \mathbf{q}_{y,z}\mathbf{q}_{x,y \rightharpoonup z}\mathbf{q}_{x \leftharpoonup (y \rightharpoonup z),y \leftharpoonup z}, \qquad (x,y,z) \in \mathcal{A}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}}\mathcal{A}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}}\mathcal{A}. \tag{4.3}$$

Proof. Straightforward. □

Remark 4.3. Let $(\mathcal{A},\sigma)$ be a braided quiver. Let Γ be an abelian group denoted multiplicatively. A map $\mathbf{q} : \mathcal{A}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}}\mathcal{A} \to \Gamma$ is called a *2-cocycle* if it satisfies (4.3). The space of all 2-cocycles, which contains all constant functions, is denoted $Z^2(\mathcal{A},\Gamma)$.

We say that two functions $\mathbf{q}, \widetilde{\mathbf{q}} : \mathcal{A}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}}\mathcal{A} \to \Gamma$ are *cohomologous* if there exists a function $\mathbf{u} : \mathcal{A} \to \Gamma$ such that

$$\mathbf{q}_{x,y}\mathbf{u}_{x \rightharpoonup y}\mathbf{u}_{x \leftharpoonup y} = \widetilde{\mathbf{q}}_{x,y}\mathbf{u}_x\mathbf{u}_x, \qquad (x,y,z) \in \mathcal{A}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}}\mathcal{A}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}}\mathcal{A}. \tag{4.4}$$

This is an equivalence relation. Furthermore, if $\mathbf{q}$ and $\widetilde{\mathbf{q}}$ are cohomologous and $\mathbf{q}$ is a 2-cocycle, then $\widetilde{\mathbf{q}}$ is also a 2-cocycle. The quotient of $Z^2(\mathcal{A},\Gamma)$ by this equivalence relation is denoted $H^2(\mathcal{A},\Gamma)$.

Let $\mathbf{u} : \mathcal{A} \to \Bbbk^{\times}$ be a function and let $\phi_{\mathbf{u}} : \Bbbk\,\mathcal{A} \to \Bbbk\,\mathcal{A}$ be the linear map given by $\phi_{\mathbf{u}}(e_x) = \mathbf{u}_x e_x$, $x \in \mathcal{A}$. Let $\mathbf{q}$, $\widetilde{\mathbf{q}}$ be 2-cocycles with values in $\Bbbk^{\times}$ related by (4.4). Then the solutions $\sigma^{\mathbf{q}}$ and $\sigma^{\widetilde{\mathbf{q}}}$ are intertwined by $\phi_{\mathbf{u}}$. Therefore, the computation of $H^2(\mathcal{A},\Bbbk^{\times})$ is desirable.

The previous considerations, in the set-theoretical case, are well-known. A brief discussion is in [AG, Lemma 5.7]; according to M. Graña, P. Etingof was aware of this. A definition of the full cohomology of set-theoretical solutions is given in [CES].

Remark 4.4. Let (M,c) be a solution of the braid equation in ${}_{\mathcal{P}}\mathcal{M}_{\mathcal{P}}$ of the form $(\Bbbk\,\mathcal{A},\sigma^{\mathbf{q}})$. The braided quiver $(\mathcal{A},\sigma)$ is *not* determined by (M,c), see [AG].

Assume again that $\mathcal{P}$ is finite. Recall the definition of rigid in Remark 2.1.

Lemma 4.5. *If $\mathcal{A}$ is finite and $\sigma^{\mathbf{q}}$ is a solution of the braid equation, then $\sigma^{\mathbf{q}}$ is rigid if and only if σ is non-degenerate.*

Proof. Assume for simplicity that $\mathbf{q} = 1$ and set $M = \Bbbk\,\mathcal{A}$, $c = \sigma^1$. Let $(\delta_x)_{x\in\mathcal{A}}$ be the basis of M^* dual to $\mathcal{A}$. Then

$$c^\flat(\delta_x \otimes e_y) = \sum_{z\in\mathcal{A}\colon \mathfrak{s}(z)=\mathfrak{e}(y),\, \mathfrak{e}(y\rightharpoonup z)=\mathfrak{e}(x)} \delta_{x, y\rightharpoonup z}\, e_{y\leftharpoonup z} \otimes \delta_z, \qquad x, y \in \mathcal{A},\, \mathfrak{s}(x) = \mathfrak{s}(y).$$

Hence, if $y \rightharpoonup \_\_$ is bijective, then $c^\flat(\delta_x \otimes e_y) = e_{y\leftharpoonup(y^{-1}\rightharpoonup x)} \otimes \delta_{y^{-1}\rightharpoonup x}$, or equivalently $c^\flat(\delta_{y\rightharpoonup u} \otimes e_y) = e_{y\leftharpoonup u} \otimes \delta_u$. Thus, if σ is non-degenerate, then $c^\flat$ is an isomorphism.

For the converse, observe that $\sigma^\flat(\delta_x \otimes y) = 0$ if x is not in the image of $y \rightharpoonup \_\_$. Thus if $c^\flat$ is an isomorphism then $y \rightharpoonup \_\_$ is surjective, and *a fortiori* bijective for all y. Finally if $y \leftharpoonup u = t \leftharpoonup u$, then $c^\flat(\delta_{y\rightharpoonup u} \otimes e_y) = c^\flat(\delta_{t\rightharpoonup u} \otimes e_t)$, which implies $y = t$. □

4.3. **Face models.**

A Yang-Baxter (or star-triangular) face model is essentially the same as a solution of the braid equation in the category of $\Bbbk\mathcal{P}$-bimodules, see [H2] and references therein. Thus, braided quivers equipped with 2-cocycles with values in $\Bbbk^\times$ give rise to Yang-Baxter face models. For completeness we restate the results of the preceding subsection in the language of face models. We begin by a definition inspired by [H2].

Definition 4.6. Let us first state that a *quiver* in a category $\mathcal{C}$ is a pair of arrows $\mathfrak{s}, \mathfrak{e} : \mathfrak{A} \to \mathfrak{P}$ in $\mathcal{C}$.

A *double quiver* is a quiver in the category Quiv of all quivers. That is, in the "vertical and horizontal" notation, a double quiver is a pair of morphisms of quivers $t, b : \mathfrak{B} \to \mathfrak{H}$, where $\mathfrak{B}$ and $\mathfrak{H}$ are quivers in the usual sense: $l, r : \mathfrak{B} \to \mathfrak{V}$, $l, r : \mathfrak{H} \to \mathcal{P}$, and t, b should preserve l, r:

$$tr = rt, \qquad tl = lt, \qquad br = rb, \qquad bl = lb. \tag{4.5}$$

In short, a double quiver is a collection of sets and maps

$$\begin{array}{ccc} \mathfrak{B} & \overset{t,b}{\rightrightarrows} & \mathfrak{H} \\ l,r \downdownarrows & & \downdownarrows l,r \\ \mathfrak{V} & \underset{t,b}{\rightrightarrows} & \mathcal{P} \end{array}$$

satisfying (4.5). By abuse of notation we say that $(\mathfrak{B}, \mathfrak{V}, \mathfrak{H})$ is a double quiver over $\mathcal{P}$; or alternatively that $\mathfrak{B}$ is a double quiver with sides in $\mathfrak{V}$ and $\mathfrak{H}$; or that $\mathfrak{B}$ is a double quiver with sides in $\mathcal{A}$ in case $\mathfrak{V} = \mathfrak{H} = \mathcal{A}$. An element B of $\mathfrak{B}$ is called an *oriented box* and depicted as a box

$$B = l \underset{b}{\overset{t}{\square}} r$$

where $t = t(B)$, $b = b(B)$, $r = r(B)$, $l = l(B)$, and the four corners are $tl(B)$, $tr(B)$, $bl(B)$, $br(B)$. In this picture, we keep in mind the orientations top-to-bottom and left-to-right. Morphisms of double quivers, or of double quivers over $\mathcal{P}$, are defined in the standard way.

Let $\mathcal{P}$ be a set and $\mathfrak{V}$, $\mathfrak{H}$ be quivers over $\mathcal{P}$ denoted vertically and horizontally, respectively. The *coarse double quiver* with sides in $\mathfrak{V}$ and $\mathfrak{H}$ is the collection $(\mathfrak{V} \boxplus \mathfrak{H}, \mathfrak{V}, \mathfrak{H})$ where $\mathfrak{V} \boxplus \mathfrak{H}$ is the set of all quadruples $\begin{pmatrix} & x & \\ f & & g \\ & y & \end{pmatrix}$ with $x, y \in \mathfrak{H}$, $f, g \in \mathfrak{V}$ such that

$$l(x) = t(f), \quad r(x) = t(g), \quad l(y) = b(f), \quad r(y) = b(g). \tag{4.6}$$

Such a quadruple is called a *face*. We omit the obvious description of the arrows.

If $(\mathfrak{B}, \mathfrak{V}, \mathfrak{H})$ is a double quiver over $\mathcal{P}$, then there are maps $\Theta : \mathfrak{B} \to \mathfrak{V} \boxplus \mathfrak{H}$, $\Xi : \mathfrak{B} \to \mathfrak{H}\,{}_r{\times}_l\,\mathfrak{V}$ given by

$$\Theta \begin{pmatrix} & x & \\ f & \square & g \\ & y & \end{pmatrix} = \begin{pmatrix} & x & \\ f & & g \\ & y & \end{pmatrix}, \qquad \Xi \begin{pmatrix} & x & \\ f & \square & g \\ & y & \end{pmatrix} = (x, g), \qquad \begin{matrix} & x & \\ f & \square & g \\ & y & \end{matrix} \in \mathfrak{B}.$$

Clearly Θ is a morphism of double quivers.

We shall say that $(\mathfrak{B}, \mathfrak{V}, \mathfrak{H})$ is *thin* if Θ is injective (any box is determined by its sides) and Ξ is surjective.

We shall say that $(\mathfrak{B}, \mathfrak{V}, \mathfrak{H})$ is *vacant* if Ξ is bijective.

We now attach a vacant double quiver to any braided quiver.

Definition 4.7. Let $\mathcal{A}$ be a braided quiver. The associated vacant double quiver with sides in $\mathcal{A}$ is the collection $\mathcal{A} \bowtie \mathcal{A}$ of faces of the shape

$$\begin{matrix} & x & \\ x \rightharpoonup g & \square & g, \\ & x \leftharpoonup g & \end{matrix} \qquad (x, g) \in \mathcal{A}\,{}_{\mathfrak{e}}{\times}_{\mathfrak{s}}\,\mathcal{A}.$$

This is well defined by (2.5).

We next recall the definition of face models, see for example [H2] and references therein.

Definition 4.8. Let $\mathcal{A}$ be a quiver. A *face model* on $\mathcal{A}$ is a pair $(\mathfrak{P}, \mathbf{w})$, where $\mathfrak{P}$ is a thin double quiver with sides in $\mathcal{A}$ and $\mathbf{w} : \mathfrak{P} \to \Bbbk^\times$ is a function.

A face model $(\mathfrak{P}, \mathbf{w})$ induces a linear map $c^{\mathbf{w}} : \Bbbk\mathcal{A} \otimes_{\Bbbk\mathcal{P}} \Bbbk\mathcal{A} \to \Bbbk\mathcal{A} \otimes_{\Bbbk\mathcal{P}} \Bbbk\mathcal{A}$ by

$$c^{\mathbf{w}}(e_x \otimes e_g) = \sum \mathbf{w} \begin{pmatrix} & x & \\ f & \square & g \\ & y & \end{pmatrix} e_f \otimes e_y, \qquad (x, g) \in \mathcal{A}\,{}_{\mathfrak{e}}{\times}_{\mathfrak{s}}\,\mathcal{A}, \tag{4.7}$$

where the sum is over all the pairs $(f, y) \in \mathcal{A}\,{}_{\mathfrak{e}}{\times}_{\mathfrak{s}}\,\mathcal{A}$ such that $\begin{matrix} & x & \\ f & \square & g \\ & y & \end{matrix} \in \mathfrak{P}$.

A *star-triangular face model* is face model $(\mathfrak{P}, \mathbf{w})$ such that $(\Bbbk\mathcal{A}, c^{\mathbf{w}})$ is a solution to the braid equation in ${}_{\mathcal{P}}\mathcal{M}_{\mathcal{P}}$.

Remark 4.9. (Hayashi, [H3]). Any solution to the braid equation in ${}_{\mathcal{P}}\mathcal{M}_{\mathcal{P}}$ arises as $(\Bbbk\mathcal{A}, c^{\mathbf{w}})$ for some face model $(\mathfrak{P}, \mathbf{w})$; see the considerations in Subsection 4.1.

By Lemma 4.2, any pair $(\mathcal{A}, \mathbf{q})$ where $\mathcal{A}$ is a braided quiver and $\mathbf{q}$ is a 2-cocycle with values in $\Bbbk^{\times}$ gives rise to a star-triangular face model $(\mathcal{A} \bowtie \mathcal{A}, \mathbf{w})$. Namely, set

$$\mathbf{w}\left(\begin{array}{ccc} & x & \\ x \rightharpoonup g & \square & g \\ & x \leftharpoonup g & \end{array}\right) = \mathbf{q}_{x,g}, \qquad (x,g) \in \mathcal{A}\,{}_{\mathfrak{e}}\times_{\mathfrak{s}}\mathcal{A}.$$

There exist of course star-triangular face models that do *not* arise from braided quivers.

4.4. **Quasitriangular quantum groupoids.**

Let (M, c) be any solution of the braid equation in ${}_{\mathcal{P}}\mathcal{M}_{\mathcal{P}}$. By a generalization of the FRT-construction, Hayashi has shown the existence of a coquasitriangular weak bialgebra $B(M, c)$ such that M is a $B(M, c)$-comodule and c arises from the coquasitriangular structure. If (M, c) is in addition rigid, then once can even produce a weak Hopf algebra $Hc(M, c)$ with this property [H1]. Assume that (M, c) is of the form $(\Bbbk\mathcal{A}, \sigma)$, where $(\mathcal{A}, \sigma)$ is a braided quiver. We give an alternative construction of a quasitriangular weak Hopf algebra realizing $(\Bbbk\mathcal{A}, \sigma)$ as above.

Let H be a weak Hopf algebra or quantum groupoid in the sense of [BNS, BS], see also [NV]. The category ${}_{H}\mathcal{M}$ of left H-modules is a monoidal category with tensor product $\otimes N := \Delta(1) M \otimes_{\Bbbk} N$, $M, N \in {}_{H}\mathcal{M}$ [BS, NTV].

Let $(\mathcal{V}, \mathcal{H})$ be a matched pair of groupoids over $\mathcal{P}$, with $\mathcal{V}$, $\mathcal{H}$ and $\mathcal{P}$ are finite sets. Let $\Bbbk(\mathcal{V}, \mathcal{H})$ be the corresponding weak Hopf algebra introduced in [AN], with notation of [AA]. We know:

- the category ${}_{\Bbbk(\mathcal{V},\mathcal{H})}\mathcal{M}$ of left $\Bbbk(\mathcal{V}, \mathcal{H})$-modules can be tensorially embedded into ${}_{\mathcal{P}}\mathcal{M}_{\mathcal{P}}$ [AA];
- if $\mathcal{A}$ is a representation of $(\mathcal{V}, \mathcal{H})$, then $\Bbbk\mathcal{A}$ is naturally a left module over ${}_{\Bbbk(\mathcal{V},\mathcal{H})}\mathcal{M}$, and $\operatorname{Rep}(\mathcal{V}, \mathcal{H})$ can be tensorially embedded into ${}_{\Bbbk(\mathcal{V},\mathcal{H})}\mathcal{M}$ [AA, Prop. 5.6];
- let (ξ, η) be a LYZ-pair for $(\mathcal{V}, \mathcal{H})$. Let $\mathcal{R}_{\xi,\eta} \in \Bbbk(\mathcal{V}, \mathcal{H}) \otimes_{\Bbbk} \Bbbk(\mathcal{V}, \mathcal{H})$ be the universal R-matrix constructed in [AA, (5.8)]. Then $(\Bbbk(\mathcal{V}, \mathcal{H}), \mathcal{R}_{\xi,\eta})$ is a quasitriangular quantum groupoid– in the sense of [BS, NTV]– by [AA, Th. 5.9]. Let $\mathcal{A}$ be a representation of $(\mathcal{V}, \mathcal{H})$ and let $\sigma_{\mathcal{A},\mathcal{A}}$ be the corresponding solution. By construction, the linearization $\sigma_{\mathcal{A},\mathcal{A}} : \Bbbk\mathcal{A} \otimes \Bbbk\mathcal{A} \to \Bbbk\mathcal{A} \otimes \Bbbk\mathcal{A}$ is a solution of the braid equation in ${}_{\mathcal{P}}\mathcal{M}_{\mathcal{P}}$ that arises also from $\mathcal{R}_{\xi,\eta}$ and the induced structure of $\Bbbk(\mathcal{V}, \mathcal{H})$-module on $\Bbbk\mathcal{A}$.

Combining these remarks with Theorem 3.10, we conclude:

Proposition 4.10. *Let $(\mathcal{A}, \sigma)$ be a finite non-degenerate quiver. Then the linearization $\Bbbk\mathcal{A}$ is a module over the weak Hopf algebra $\Bbbk(\mathcal{G}_{\mathcal{A}}, \mathcal{G}_{\mathcal{A}} \bowtie \mathcal{G}_{\mathcal{A}})$ and σ arises from the universal R-matrix $\mathcal{R}_{\mathrm{in}_1,\mathrm{in}_2}$.* □

5. Appendix
by Mitsuhiro Takeuchi

Proposition 5.1. *If $\mathcal{A}$ is a quiver over $\mathcal{P}$, there is a matched pair of groupoids $(\mathcal{V}(\mathcal{A}), \mathcal{H}(\mathcal{A}))$ such that there is a one-to-one correspondence between representations of $(\mathcal{V}, \mathcal{H})$ on $\mathcal{A}$ and morphisms of groupoids $(\mathcal{V}, \mathcal{H}) \to (\mathcal{V}(\mathcal{A}), \mathcal{H}(\mathcal{A}))$.*

Proof. The matched pair $(\mathcal{V}(\mathcal{A}), \mathcal{H}(\mathcal{A}))$ is constructed as follows. $\mathcal{V}(\mathcal{A})$ is the free groupoid generated by $\mathcal{A}$. For $Q \in \mathcal{P}$, let $\mathcal{X}_Q$ be the set of all paths in $\mathcal{V}(\mathcal{A})$ beginning with Q:

$$Q \xrightarrow{a_1} \quad \xrightarrow{a_2} \dots \xrightarrow{a_n}, \qquad a_i \in \mathcal{V}(\mathcal{A}).$$

For $P, Q \in \mathcal{P}$, let $\mathcal{H}(\mathcal{A})(P, Q)$ be the set of all bijections $f : \mathcal{X}_Q \to \mathcal{X}_P$ satisfying the following conditions:

(1) f preserves the length of paths, hence we can write

$$f(a_1, a_2, \dots, a_n) = (b_1, b_2, \dots, b_n).$$

(2) a_i is in $\mathcal{A}$ iff b_i is in $\mathcal{A}$.
(3) a_i is an identity iff b_i is an identity.
(4) $f(a_1, \dots, a_i a_{i+1}, \dots, a_n) = (b_1, \dots, b_i b_{i+1}, \dots, b_n)$, $1 \le i < n$.
(5) $f(a_1, a_2, \dots, a_{n-1}) = (b_1, b_2, \dots, b_{n-1})$, if $n > 1$.

The groupoid $\mathcal{H}(\mathcal{A})$ is defined by arrows $\mathcal{H}(\mathcal{A})(P, Q)$ with obvious composition. In particular, we have $f(a_1) = b_1$. We may write $b_1 = f \rightharpoonup a_1$. This gives a left action of $\mathcal{H}(\mathcal{A})$ on $\mathcal{A}$ by (2). There is a canonical quiver map $\mathcal{A} \to \mathcal{V}(\mathcal{A})$. If we fix f and a_1 as above, the correspondence

$$(a_2, a_3, \dots, a_n) \mapsto (b_2, b_3 \dots, b_n)$$

satisfies conditions 2) – 5). Hence it determines an element of $\mathcal{H}(\mathcal{A})$. If we denote this map by $f \leftharpoonup a_1$, we get a matched pair of groupoids $(\mathcal{V}(\mathcal{A}), \mathcal{H}(\mathcal{A}))$ and a canonical representation on $\mathcal{A}$. The one-to-one correspondence between representations of $(\mathcal{V}, \mathcal{H})$ on $\mathcal{A}$ and morphisms of groupoids $(\mathcal{V}, \mathcal{H}) \to (\mathcal{V}(\mathcal{A}), \mathcal{H}(\mathcal{A}))$ is constructed easily. □

As an important application of this proposition, we have a version of the so-called FRT construction for matched pairs of groupoids. We consider quadruples $(\mathcal{V}, \mathcal{H}, \xi, \eta)$ where $(\mathcal{V}, \mathcal{H})$ is a matched pair of groupoids and (ξ, η) is a matched pair of rotations on it. In view of Theorem 3.2, this is equivalent to specifying a braiding structure on $\mathrm{Rep}(\mathcal{V}, \mathcal{H})$. A morphism

$$(\alpha, \beta) : (\mathcal{V}_1, \mathcal{H}_1, \xi_1, \eta_1) \to (\mathcal{V}_2, \mathcal{H}_2, \xi_2, \eta_2)$$

means a morphism $(\alpha, \beta) : (\mathcal{V}_1, \mathcal{H}_1) \to (\mathcal{V}_2, \mathcal{H}_2)$ of matched pairs of groupoids such that the following diagram commutes.

$$\begin{array}{ccc} \mathcal{V}_1 & \xrightarrow{\xi_1, \eta_1} & \mathcal{H}_1 \\ {\scriptstyle\beta}\uparrow & & \downarrow{\scriptstyle\alpha} \\ \mathcal{V}_2 & \xrightarrow{\xi_2, \eta_2} & \mathcal{H}_2. \end{array}$$

This is equivalent to saying that the functor $\mathrm{Res}_\alpha^\beta : \mathrm{Rep}(\mathcal{V}_2, \mathcal{H}_2) \to \mathrm{Rep}(\mathcal{V}_1, \mathcal{H}_1)$ preserves the braiding structure.

Theorem 5.2. *Let $\mathcal{A}$ be a non-degenerate braided quiver. There is a quadruple $(\mathbb{G}_\mathcal{A}, \mathbb{H}_\mathcal{A}, \xi_\mathcal{A}, \eta_\mathcal{A})$ with a canonical representation ρ_0 on $\mathcal{A}$ satisfying the following properties.*

(a) *The matched pair of rotations $(\xi_\mathcal{A}, \eta_\mathcal{A})$ induces the braiding on $\mathcal{A}$.*

(b) *Let $(\mathcal{V}, \mathcal{H}, \xi, \eta)$ be a general quadruple. Let ρ be a representation of $(\mathcal{V}, \mathcal{H})$ on $\mathcal{A}$ such that (ξ, η) induces the braiding on $\mathcal{A}$. Then, there is a unique morphism of quadruples*

$$(\alpha, \beta) : (\mathcal{V}, \mathcal{H}, \xi, \eta) \to (\mathbb{G}_\mathcal{A}, \mathbb{H}_\mathcal{A}, \xi_\mathcal{A}, \eta_\mathcal{A})$$

such that $(\mathcal{A}, \rho) = \mathrm{Res}_\alpha^\beta(\mathcal{A}, \rho_0)$.

(c) *We have a one-to-one correspondence between data $\mathcal{V}, \mathcal{H}, \xi, \eta, \rho$ as in (b) and morphisms of quadruples $(\alpha, \beta) : (\mathcal{V}, \mathcal{H}, \xi, \eta) \to (\mathbb{G}_\mathcal{A}, \mathbb{H}_\mathcal{A}, \xi_\mathcal{A}, \eta_\mathcal{A})$.*

Proof. Recall the structure groupoid $\mathbb{G}_\mathcal{A}$ of $(\mathcal{A}, \sigma)$, Definition 2.3; it is a quotient of $\mathcal{V}(\mathcal{A})$ and we write $a \sim b$ if the clases of $a, b \in \mathcal{V}(\mathcal{A})$ are equal in $\mathbb{G}_\mathcal{A}$.

Let $\mathbb{H}_\mathcal{A}$ be the maximal subgroupoid of $\mathcal{H}(\mathcal{A})$ which is compatible with the defining relations of $\mathbb{G}_\mathcal{A}$. If $P, Q \in \mathcal{P}$, $\mathbb{H}_\mathcal{A}(P, Q)$ consists of all $f \in \mathcal{H}(\mathcal{A})(P, Q)$ such that for all $(a_1, a_2, \dots, a_n)$ and $(b_1, b_2, \dots, b_n)$ in $\mathcal{X}_Q$, $(a_1, a_2, \dots, a_n) \sim (b_1, b_2, \dots, b_n)$ iff $f(a_1, a_2, \dots, a_n) \sim f(b_1, b_2, \dots, b_n)$. Then $(\mathbb{G}_\mathcal{A}, \mathbb{H}_\mathcal{A})$ is a sub-matched pair of $(\mathcal{V}(\mathcal{A}), \mathcal{H}(\mathcal{A}))$. We define a LYZ-pair (ξ, η) for $(\mathbb{G}_\mathcal{A}, \mathbb{H}_\mathcal{A})$ as follows.

Let $a \in \mathbb{G}_\mathcal{A}(P, Q)$ and $(b_1, b_2, \dots, b_n) \in \mathcal{X}_Q$. Make the following diagrams by means of matched pairs $(\mathcal{V}(\mathcal{A}), \mathbb{G}_\mathcal{A})$ and $(\mathbb{G}_\mathcal{A}, \mathcal{V}(\mathcal{A}))$.

$$\begin{array}{ccc} & \xrightarrow{a} & \\ c_1\downarrow & & \downarrow b_1 \\ & \xrightarrow{a_1} & \\ c_2\downarrow & & \downarrow b_2 \\ & \xrightarrow{a_2} & \\ \dots & & \dots \\ & \xrightarrow{a_{n-1}} & \\ c_n\downarrow & & \downarrow b_n \\ & \xrightarrow{a_n} & \end{array}, \qquad \begin{array}{ccccccc} & \xrightarrow{d_1} & & \xrightarrow{d_2} & \dots & \xrightarrow{d_n} & \\ a\downarrow & & \downarrow \alpha_1 & & \downarrow \alpha_2 \ \downarrow \alpha_{n-1} & & \downarrow \alpha_n \\ & \xrightarrow{b_1} & & \xrightarrow{b_2} & \dots & \xrightarrow{b_n} & \end{array}$$

where c_i, a_i, d_i and α_i are defined inductively by

$$c_i = a_{i-1} \rightharpoonup b_i, \quad a_i = a_{i-1} \leftharpoonup b_i, \quad a_0 = a \quad \text{and} \quad \alpha_{i-1} = d_i \rightharpoonup \alpha_i, \quad b_i = d_i \leftharpoonup \alpha_1, \quad \alpha_0 = a.$$

The maps

$$(b_1, b_2, \ldots, b_n) \mapsto (c_1, c_2, \ldots, c_n) \quad \text{and} \quad (d_1, d_2, \ldots, d_n)$$

belong to $\mathbb{H}_{\mathcal{A}}(P, Q)$. Let $\eta(a)$ and $\xi(a)$ be these maps. Then $\xi, \eta : \mathbb{G}_{\mathcal{A}} \to \mathbb{H}_{\mathcal{A}}$ are groupoid maps giving rise to a LYZ-pair which induces the original braiding σ on $\mathcal{A}$.

□

In view of the universality (b), (c) above, one may call the matched pair $(\mathbb{G}_{\mathcal{A}}, \mathbb{H}_{\mathcal{A}})$ the *FRT-construction for matched pairs of groupoids.*

Relations of this construction with the construction in the text are explained as follows. If $\mathcal{A}$ is a non-degenerate braided quiver, the structure groupoid $\mathbb{G}_{\mathcal{A}}$ has a braided structure (Theorem 3.8). We have a matched pair of groupoids $(\mathbb{G}_{\mathcal{A}}, \mathbb{G}_{\mathcal{A}} \bowtie \mathbb{G}_{\mathcal{A}})$ which has a canonical LYZ-pair $(\mathrm{in}_1, \mathrm{in}_2)$ (Theorem 3.4). Thus we have a quadruple $(\mathbb{G}_{\mathcal{A}}, \mathbb{G}_{\mathcal{A}} \bowtie \mathbb{G}_{\mathcal{A}}, \mathrm{in}_1, \mathrm{in}_2)$. Further there is a canonical representation of this matched pair on $\mathcal{A}$ which induces the original braiding on $\mathcal{A}$ (Theorem 3.10). Comparing with Theorem 5.2 above, we conclude that there is a unique morphism of groupoids

$$\alpha : \mathbb{G}_{\mathcal{A}} \bowtie \mathbb{G}_{\mathcal{A}} \to \mathbb{H}_{\mathcal{A}}$$

such that $(\mathrm{id}, \alpha) : (\mathbb{G}_{\mathcal{A}}, \mathbb{G}_{\mathcal{A}} \bowtie \mathbb{G}_{\mathcal{A}}) \to (\mathbb{G}_{\mathcal{A}}, \mathbb{H}_{\mathcal{A}})$ is a morphism of matched pairs and that $\xi_{\mathcal{A}} = \alpha\, \mathrm{in}_1$ and $\eta_{\mathcal{A}} = \alpha\, \mathrm{in}_2$.

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