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Representations of matched pairs of groupoids and applications to weak Hopf algebras

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Abstract. We introduce the category of set-theoretic representations of a matched pair of groupoids. This is a monoidal category endowed with a monoidal functor $f\ell$ to the category of quivers over the common base of the groupoids in the matched pair. We study monoidal functors between two such categories of representations which preserve the functor $f\ell$. We show that the centralizer of such a monoidal functor is the category of representations of a new matched pair, which we construct explicitly. We introduce the notions of *double* of a matched pair of groupoids and *generalized double* of a morphism of matched pairs. We show that the centralizer of $f\ell$ is the category of representations of the dual matched pair, and the centralizer of the identity functor (the center) is the category of representations of the double. We use these constructions to classify the braidings in the category of representations of a matched pair. Such braidings are parametrized by certain groupoid-theoretic structures which we call *matched pairs of rotations*. Finally, we express our results in terms of the weak Hopf algebra associated to a matched pair of groupoids. A matched pair of rotations gives rise to a quasitriangular structure for the associated weak Hopf algebra. The Drinfeld double of the weak Hopf algebra of a matched pair is the weak Hopf algebra associated to the double matched pair.

Introduction

An exact factorization of a group Σ is a pair of subgroups F, G, such that $\Sigma = FG$ and $F \cap G = \{1\}$. Such a factorization may be described without reference to the group Σ, by means of a pair of actions of the factors F and G on each other satisfying certain axioms: a matched pair of groups [**7, 12, 16**].

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Exact factorizations $\mathcal{D} = \mathcal{V}\mathcal{H}$ of groupoids are equivalent to structures of matched pairs of groupoids on the pair $\mathcal{V}$, $\mathcal{H}$, and to structures of vacant double groupoids [**11, 2**]. In this paper we work with matched pairs of groupoids, while the notions from double category theory are implicit in our notations and illustrations. Thus, a matched pair of groupoids $(\mathcal{V}, \mathcal{H})$ consists of a pair of groupoids $\mathcal{V}$ and $\mathcal{H}$ with a common base (set of objects) $\mathcal{P}$ together with actions $\triangleright$ and $\triangleleft$ of each of them on the other satisfying certain simple axioms (Definition 1.1). This is summarized by the following illustration

$$\begin{tikzcd} P \arrow[r, bend left, "x"] \arrow[d, bend right, "x \triangleright g"'] & Q \arrow[d, bend left, "g"] \\ R \arrow[r, bend right, "x \triangleleft g"'] & S \end{tikzcd}$$

where g is an element (arrow) of the *vertical* groupoid $\mathcal{V}$ and x is an element of the *horizontal* groupoid $\mathcal{H}$.

A quiver over $\mathcal{P}$ is a set $\mathcal{E}$ equipped with two maps $p, q : \mathcal{E} \to \mathcal{P}$. We introduce the notion of representations of a matched pair of groupoids. A representation of $(\mathcal{V}, \mathcal{H})$ is a quiver $\mathcal{E}$ over $\mathcal{P}$ together with an action of $\mathcal{H}$ and a grading over $\mathcal{V}$ which are compatible (Definition 2.1). The matched pair structure allows us to construct a monoidal structure on the category $\operatorname{Rep}(\mathcal{V}, \mathcal{H})$ of representations. The forgetful functor $f\ell : \operatorname{Rep}(\mathcal{V}, \mathcal{H}) \to \operatorname{Quiv}(\mathcal{P})$ to the category of quivers over $\mathcal{P}$ is monoidal.

We introduce a notion of morphisms between matched pairs (Definition 1.11). Associated to a morphism $(\alpha, \beta) : (\mathbb{V}, \mathbb{H}) \to (\mathcal{V}, \mathcal{H})$ there is a monoidal functor $\operatorname{Res}_\alpha^\beta : \operatorname{Rep}(\mathcal{V}, \mathcal{H}) \to \operatorname{Rep}(\mathbb{V}, \mathbb{H})$, called the restriction along (α, β). Any restriction functor preserves the forgetful functors $f\ell$ (Section 2.3). Our first main result states that any monoidal functor $\operatorname{Rep}(\mathcal{V}, \mathcal{H}) \to \operatorname{Rep}(\mathbb{V}, \mathbb{H})$ which preserves $f\ell$ must be a restriction functor (Theorem 2.10).

In Section 3 we undertake the study of centralizers of such monoidal functors. This includes the center of a category $\operatorname{Rep}(\mathcal{V}, \mathcal{H})$ as the particular case when the monoidal functor is the identity. Our second main result (Theorem 3.1) states that any such centralizer is again the category of representations of a (new) matched pair, which we call a generalized double. In particular, the center of $\operatorname{Rep}(\mathcal{V}, \mathcal{H})$ is the category of representations of the double $D(\mathcal{V}, \mathcal{H})$ of $(\mathcal{V}, \mathcal{H})$, and the centralizer of $f\ell : \operatorname{Rep}(\mathcal{V}, \mathcal{H}) \to \operatorname{Quiv}(\mathcal{P})$ is the category of representations of the dual of $(\mathcal{V}, \mathcal{H})$ (Corollaries 3.4 and 3.3).

Doubles and generalized doubles are introduced and studied in Section 1.4. The explicit description is given in Theorem 1.15 and the functoriality of the construction in Proposition 1.21. We also show that the constructions of generalized doubles and that of duals commute (Proposition 1.23).

The classification of braidings in the monoidal category $\operatorname{Rep}(\mathcal{V}, \mathcal{H})$ is accomplished in Section 4. By general results (recalled in Section 3.1), braidings on $\operatorname{Rep}(\mathcal{V}, \mathcal{H})$ are in bijective correspondence with monoidal sections of the canonical functor $Z\big(\operatorname{Rep}(\mathcal{V}, \mathcal{H})\big) \to \operatorname{Rep}(\mathcal{V}, \mathcal{H})$. The latter is the restriction functor along a canonical morphism of matched pairs $(\iota, \pi) : (\mathcal{V}, \mathcal{H}) \to D(\mathcal{V}, \mathcal{H})$. Theorem 2.10 reduces the classification to the description of all sections of (ι, π). The result is that braidings are in bijective correspondence with *matched pairs of rotations* for $(\mathcal{V}, \mathcal{H})$.

A rotation for $(\mathcal{V},\mathcal{H})$ is a morphism η from the vertical groupoid $\mathcal{V}$ to the horizontal groupoid $\mathcal{H}$ which is compatible with the actions $\triangleright$ and $\triangleleft$ (Definition 4.1). Suppose η and ξ are rotations. Starting from a pair of composable arrows (g,f) in $\mathcal{V}$, one may rotate g with η and f^{-1} with ξ, and build the following diagram:

$$\begin{tikzcd}
 & P \arrow[d, bend right, "g"'] & R \arrow[l, bend right, "\xi(f)^{-1}\triangleleft g^{-1}"'] \\
P \arrow[r, bend right, "\eta(g)"] \arrow[d, bend right, "\eta(g)\triangleright f"'] & Q \arrow[u, bend right, "g^{-1}"'] \arrow[d, bend right, "f"'] & S \arrow[l, bend right, "\xi(f)^{-1}"] \arrow[u, bend right, "\xi(f)^{-1}\triangleright g^{-1}"'] \\
R \arrow[r, bend right, "\eta(g)\triangleleft f"'] & S \arrow[u, bend right, "f^{-1}"'] &
\end{tikzcd}$$

There are then two ways of going from P to S via vertical arrows (through Q or through R). The pair (η,ξ) is a matched pair of rotations if these two ways coincide (Definition 4.3). The correspondence between braidings and matched pairs of rotations is settled in our third main result, Theorem 4.5.

From a matched pair of finite groups one may construct a Hopf algebra $\Bbbk(F,G)$, Takeuchi's *bismash product* [**7, 16, 13**], which is part of a Hopf algebra extension

$$1 \to \Bbbk^F \to \Bbbk(F,G) \to \Bbbk G \to 1\,.$$

On the other hand, a construction of a weak Hopf algebra (or quantum groupoid) $\Bbbk(\mathcal{V},\mathcal{H})$ out of a matched pair $(\mathcal{V},\mathcal{H})$ of finite groupoids appears in [**2**]. This fits in an extension

$$1 \to \Bbbk^{\mathcal{V}} \to \Bbbk(\mathcal{V},\mathcal{H}) \to \Bbbk\mathcal{H} \to 1\,.$$

We review this construction in Section 5.1. There is a monoidal functor from the category of representations of a matched pair $(\mathcal{V},\mathcal{H})$ to the category of modules over the weak Hopf algebra $\Bbbk(\mathcal{V},\mathcal{H})$ (the linearization functor). In Theorem 5.9, we show that a matched pair of rotations (η,ξ) for $(\mathcal{V},\mathcal{H})$ gives rise to an quasitriangular structure $\mathcal{R}$ for $\Bbbk(\mathcal{V},\mathcal{H})$, in the sense of [**15**]. Explicitly,

$$\mathcal{R} = \sum \big(\xi(f)^{-1}\triangleleft g^{-1}, g\big) \otimes \big(\eta(g), f\big)\,,$$

the sum being over the pairs (f,g) of composable arrows in the groupoid $\mathcal{V}$. We also provide an explicit formula for the corresponding Drinfeld element.

The construction of a weak Hopf algebra from a matched pair commutes with duals [**2**, Prop. 3.11]. We review this fact in Section 5.4, and we show that the construction also commutes with doubles. Theorem 5.10 provides an explicit description for the Drinfeld double of the weak Hopf algebra of a matched pair. In the case of matched pairs of groups, the Drinfeld double was calculated in [**3**].

The quasitriangular structures for $\Bbbk(\mathcal{V},\mathcal{H})$ that we find specialize for the case of groups to those constructed by Lu, Yan, and Zhu in [**10**]. Our matched pairs of rotations are in this case the *LYZ pairs* of [**17**]. This reference introduces a notation reminiscent of double categories, similar to ours. A different construction of these quasitriangular structures, still for the case of groups, was given by Masuoka (see [**17**]). Our approach, based on the calculation of the center of the category of representations, is different from both [**10**] and [**17**].

In [**9, 10**] it is shown that every Hopf algebra with a *positive* basis is of the from $\Bbbk(F,G)$ for some matched pair of groups (F,G), and that all positive quasitriangular structures for $\Bbbk(F,G)$ are of the form mentioned above. We leave for future work the question of whether these remain true for the case of groupoids.

The results of this paper are used in [**1**].

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1. Matched pairs of groupoids

1.1. Groupoids.

Let $\mathcal{P}$ be a set. Given maps $p : X \to \mathcal{P}$ and $q : Y \to \mathcal{P}$, let

$$X\,{}_p\!\times_q Y = \{(x,y) \in X \times Y : p(x) = q(y)\}\,. \tag{1.1}$$

A groupoid is a small category in which all arrows are invertible. It consists of a set of arrows $\mathcal{G}$, a set of objects $\mathcal{P}$ (called the *base*), source and end maps $\mathfrak{s},\mathfrak{e} : \mathcal{G} \to \mathcal{P}$, composition $m : \mathcal{G}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}} \mathcal{G} \to \mathcal{G}$, and identities $\mathrm{id} : \mathcal{P} \to \mathcal{G}$. We use $\mathcal{G}$ to denote both the groupoid and the set of arrows. We write $m(f,g) = fg$ (and not gf) for $(f,g) \in \mathcal{G}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}} \mathcal{G}$ and we usually identify $\mathcal{P}$ with the subset $\mathrm{id}(\mathcal{P})$ of $\mathcal{G}$. For $P,Q \in \mathcal{P}$ we let ${}_P\mathcal{G} = \{g \in \mathcal{G} : \mathfrak{s}(g) = P\}$, $\mathcal{G}_P = \{g \in \mathcal{G} : \mathfrak{e}(g) = P\}$, $\mathcal{G}(P,Q) = {}_P\mathcal{G} \cap \mathcal{G}_Q$, and $\mathcal{G}(P) = \mathcal{G}(P,P)$.

Alternatively, a groupoid may be defined as a set $\mathcal{G}$ with a partially defined associative product and partial units, whose elements are all invertible. The source and end maps are determined by $\mathfrak{s}(g) = gg^{-1}$, $\mathfrak{e}(g) = g^{-1}g$, and $P \in \mathcal{P}$ if and only if $P^2 = P$.

The opposite groupoid to $\mathcal{G}$ (where $\mathfrak{s}$ and $\mathfrak{e}$ are switched) is denoted $\mathcal{G}^{op}$.

A morphism of groupoids from $\mathcal{G}$ to $\mathcal{K}$ is a functor $\alpha : \mathcal{G} \to \mathcal{K}$. Equivalently, α is a map from $\mathcal{G}$ to $\mathcal{K}$ which preserves the product, and hence also the base, source and end maps. If $\mathcal{G}$ and $\mathcal{K}$ have the same base $\mathcal{P}$, we say that $\alpha : \mathcal{G} \to \mathcal{K}$ is a morphism of groupoids *over* $\mathcal{P}$ if the restriction $\mathcal{P} \to \mathcal{P}$ is the identity.

The category of groupoids over $\mathcal{P}$ has an initial object, the *discrete* groupoid with $\mathfrak{s} = \mathfrak{e} = m = \mathrm{id}$ are all the identity map $\mathcal{P} \to \mathcal{P}$, and a final one, the *coarse* groupoid $\mathcal{P} \times \mathcal{P}$, where $\mathfrak{s}(P,Q) = P$, $\mathfrak{e}(P,Q) = Q$, $\mathrm{id}(P) = (P,P)$ and $m\big((P,Q),(Q,R)\big) = (P,R)$.

A group bundle is a groupoid $\mathcal{N}$ with $\mathfrak{s} = \mathfrak{e}$; thus $\mathcal{N} = \coprod_{P \in \mathcal{P}} \mathcal{N}(P)$. Let $\alpha : \mathcal{G} \to \mathcal{K}$ be a morphism of groupoids over $\mathcal{P}$. The kernel of α is the group bundle $\mathcal{N} = \{g \in \mathcal{G} : \alpha(g) \in \mathcal{P}\}$. In this case $\mathcal{N}(P) = \ker(\alpha : \mathcal{G}(P) \to \mathcal{K}(P))$.

1.2. Actions and matched pairs of groupoids.

Let $\mathcal{G}$ be a groupoid with base $\mathcal{P}$. Let $p : \mathcal{E} \to \mathcal{P}$ be a map. A *left action* of $\mathcal{G}$ on p is a map $\triangleright : \mathcal{G}\,{}_{\mathfrak{e}}\!\times_p \mathcal{E} \to \mathcal{E}$ such that

(1.2) $p(g \triangleright e) = \mathfrak{s}(g)$,

(1.3) $g \triangleright (h \triangleright e) = gh \triangleright e$,

(1.4) $\mathrm{id}\, p(e) \triangleright e = e$, for all $g, h \in \mathcal{G}$, $e \in \mathcal{E}$ composable in the appropriate sense. We may represent (1.2) by

$$\begin{tikzcd}
\mathcal{E} \arrow[d, "p"'] & g \triangleright e \arrow[d, mapsto] & e \arrow[d, mapsto] \\
\mathcal{P} & Q \arrow[r, bend left, "g"] & P
\end{tikzcd}$$

Suppose $\mathcal{G}$ acts on $p : \mathcal{E} \to \mathcal{P}$ and on $p' : \mathcal{E}' \to \mathcal{P}$. A map $\psi : \mathcal{E} \to \mathcal{E}'$ *intertwines* the actions if

$$p'(\psi(e)) = p(e) \quad \text{and} \quad \psi(g \triangleright e) = g \triangleright \psi(e)\,. \tag{1.5}$$

There are two kinds of *trivial* actions. First, if $\mathcal{E}$ is of the form $\mathcal{E} = \mathcal{P} \times X$ for some set X and p is the projection onto the first coordinate, then an action on p of an arbitrary groupoid $\mathcal{G}$ with base $\mathcal{P}$ is called trivial if

$$g \triangleright (\mathfrak{e}(g), x) = (\mathfrak{s}(g), x) \tag{1.6}$$

for all $x \in X$, $g \in \mathcal{G}$. Second, if $\mathcal{N}$ is a group bundle over $\mathcal{P}$, an action of $\mathcal{N}$ on an arbitrary map $p : \mathcal{E} \to \mathcal{P}$ is called trivial if

$$n \triangleright e = e \tag{1.7}$$

for all $(n, e) \in \mathcal{N}\,{}_{\mathfrak{e}}\!\times_{p} \mathcal{E}$. If $\mathcal{N}$ is a group bundle and $\mathcal{E} = \mathcal{P} \times X$ then the two kinds of trivial actions agree.

Similarly, a *right action* of $\mathcal{G}$ on a map $q : \mathcal{E} \to \mathcal{P}$ is a map $\triangleleft : \mathcal{E}\,{}_{q}\!\times_{\mathfrak{s}} \mathcal{G} \to \mathcal{E}$ such that

(1.8) $q(e \triangleleft g) = \mathfrak{e}(g)$,

(1.9) $(e \triangleleft g) \triangleleft h = e \triangleleft gh$,

(1.10) $e \triangleleft \operatorname{id} q(e) = e$,

for all $g, h \in \mathcal{G}$, $e \in \mathcal{E}$ composable in the appropriate sense. Any left action gives rise to a right action on the same map by $e \triangleleft g = g^{-1} \triangleright e$, and vice versa.

Definition 1.1. [**11**, Definition 2.14]. A *matched pair of groupoids* is a pair of groupoids $(\mathcal{V}, \mathcal{H})$ with the same base $\mathcal{P}$ together with the following data. Let $t, b : \mathcal{V} \rightrightarrows \mathcal{P}$ be the source and end maps of $\mathcal{V}$, respectively, and $l, r : \mathcal{H} \rightrightarrows \mathcal{P}$ the source and end maps of $\mathcal{H}$, respectively. The data consists of a left action $\triangleright : \mathcal{H}\,{}_{r}\!\times_{t} \mathcal{V} \to \mathcal{V}$ of $\mathcal{H}$ on $t : \mathcal{V} \to \mathcal{P}$, and a right action $\triangleleft : \mathcal{H}\,{}_{r}\!\times_{t} \mathcal{V} \to \mathcal{H}$ of $\mathcal{V}$ on $r : \mathcal{H} \to \mathcal{P}$, satisfying

(1.11) $b(x \triangleright g) = l(x \triangleleft g)$,

(1.12) $x \triangleright fg = (x \triangleright f)((x \triangleleft f) \triangleright g)$,

(1.13) $xy \triangleleft g = (x \triangleleft (y \triangleright g))(y \triangleleft g)$,

for all $f, g \in \mathcal{V}$, $x, y \in \mathcal{H}$ for which the operations are defined.

We refer to $\mathcal{V}$ as the *vertical* groupoid, t and b stand for *top* and *bottom*; $\mathcal{H}$ is the *horizontal* groupoid, l and r stand for *left* and *right*. The following diagram illustrates the situation:

$$\begin{tikzcd} t(x \triangleright g) = l(x) \arrow[r, bend left, "x"] \arrow[d, bend right, "x \triangleright g"'] & r(x) = t(g) \arrow[d, bend left, "g"] \\ b(x \triangleright g) = l(x \triangleleft g) \arrow[r, bend right, "x \triangleleft g"'] & r(x \triangleleft g) = b(g) \end{tikzcd} \tag{1.14}$$

We refer to such diagrams as the *cells* of the matched pair. We will not need any notions from double category theory beyond this basic notation. For more in this direction, see [**11, 2**].

Given $P \in \mathcal{P}$, there are two identities: $\mathrm{id}_{\mathcal{H}} P \in \mathcal{H}$ and $\mathrm{id}_{\mathcal{V}} P \in \mathcal{V}$.

LEMMA 1.2. *For all $x, y \in \mathcal{H}$ and $f, g \in \mathcal{V}$ for which the operations are defined, we have*

$$x \triangleright \mathrm{id}_{\mathcal{V}}\, r(x) = \mathrm{id}_{\mathcal{V}}\, l(x)\,, \tag{1.15}$$

$$\mathrm{id}_{\mathcal{H}}\, t(g) \triangleleft g = \mathrm{id}_{\mathcal{H}}\, b(g)\,, \tag{1.16}$$

$$(x \triangleright g)^{-1} = (x \triangleleft g) \triangleright g^{-1}\,, \tag{1.17}$$

$$(x \triangleleft g)^{-1} = x^{-1} \triangleleft (x \triangleright g)\,, \tag{1.18}$$

$$(x \triangleleft g)^{-1} \triangleright (x \triangleright g)^{-1} = g^{-1}\,, \tag{1.19}$$

$$(x \triangleleft g)^{-1} \triangleleft (x \triangleright g)^{-1} = x^{-1}\,, \tag{1.20}$$

$$g^{-1}(x^{-1} \triangleright f) = (x \triangleleft g)^{-1} \triangleright \left((x \triangleright g)^{-1} f\right)\,, \tag{1.21}$$

$$(y \triangleleft g^{-1})x^{-1} = \left(y(x \triangleleft g)^{-1}\right) \triangleleft (x \triangleright g)^{-1}\,. \tag{1.22}$$

PROOF. First, by (1.10), $x \triangleleft \mathrm{id}_{\mathcal{V}}\, r(x) = x$; hence, by (1.12),

$$\begin{aligned} x \triangleright \mathrm{id}_{\mathcal{V}}\, r(x) &= x \triangleright \left(\mathrm{id}_{\mathcal{V}}\, r(x)\, \mathrm{id}_{\mathcal{V}}\, r(x)\right) \\ &= \left(x \triangleright \mathrm{id}_{\mathcal{V}}\, r(x)\right)\Big(\left(x \triangleleft \mathrm{id}_{\mathcal{V}}\, r(x)\right) \triangleright \mathrm{id}_{\mathcal{V}}\, r(x)\Big) = \left(x \triangleright \mathrm{id}_{\mathcal{V}}\, r(x)\right)\left(x \triangleright \mathrm{id}_{\mathcal{V}}\, r(x)\right). \end{aligned}$$

Cancelling we obtain (1.15). Now,

$$\mathrm{id}_{\mathcal{V}}\, l(x) \overset{(1.15)}{=} x \triangleright (gg^{-1}) \overset{(1.12)}{=} (x \triangleright g)\left((x \triangleleft g) \triangleright g^{-1}\right),$$

from which (1.17) follows. Furthermore,

$$(x \triangleleft g)^{-1} \triangleright (x \triangleright g)^{-1} = (x \triangleleft g)^{-1} \triangleright \left((x \triangleleft g) \triangleright g^{-1}\right) = g^{-1}\,,$$

which proves (1.19). Finally,

$$\begin{aligned} &(x \triangleleft g)^{-1} \triangleright \left((x \triangleright g)^{-1} f\right) \\ &\quad \overset{(1.12)}{=} \left((x \triangleleft g)^{-1} \triangleright (x \triangleright g)^{-1}\right)\left(\left((x \triangleleft g)^{-1} \triangleleft (x \triangleright g)^{-1}\right) \triangleright f\right) = g^{-1}(x^{-1} \triangleright f) \end{aligned}$$

which proves (1.21). The proofs of (1.16), (1.18), (1.20), and (1.22) are similar. $\square$

Conditions (1.19) and (1.20) may be succinctly expressed as follows: inverting all arrows in the cell (1.14) yields a new cell, as shown below:

$$
\begin{tikzcd}
r(x \triangleleft g) = b(g) \arrow[r, bend left, "(x \triangleleft g)^{-1}"] \arrow[d, bend right, "g^{-1}"'] & b(x \triangleright g) = l(x \triangleleft g) \arrow[d, bend left, "(x \triangleright g)^{-1}"] \\
r(x) = t(g) \arrow[r, bend right, "x^{-1}"'] & t(x \triangleright g) = l(x)
\end{tikzcd}
\tag{1.23}
$$

REMARK 1.3. As for Hopf algebras (or weak Hopf algebras), there is a number of basic constructions one may perform with matched pairs of groupoids. Let $(\mathcal{V}, \mathcal{H})$ be a matched pair. The *dual* or *transpose* matched pair is $(\mathcal{H}, \mathcal{V})$, with the following actions:

$$
f \rightharpoonup y := (y^{-1} \triangleleft f^{-1})^{-1} \quad \text{and} \quad f \leftharpoonup y := (y^{-1} \triangleright f^{-1})^{-1}
\tag{1.24}
$$

for $(f, y) \in \mathcal{V}\,{}_b\!\times_l \mathcal{H}$. Note that if $(x, g) \in \mathcal{H}\,{}_r\!\times_t \mathcal{V}$ then $(x \triangleright g, x \triangleleft g) \in \mathcal{V}\,{}_b\!\times_l \mathcal{H}$. According to (1.19) and (1.20), the corresponding cell of the matched pair $(\mathcal{H}, \mathcal{V})$ is

$$
\begin{tikzcd}
P \arrow[r, bend left, "x \triangleright g"] \arrow[d, bend right, "x"'] & Q \arrow[d, bend left, "x \triangleleft g"] \\
R \arrow[r, bend right, "g"'] & S
\end{tikzcd}
\tag{1.25}
$$

In other words, transposing a cell (1.14) of the matched pair $(\mathcal{V}, \mathcal{H})$ yields a cell of the dual matched pair $(\mathcal{H}, \mathcal{V})$.

The dual of the dual of $(\mathcal{V}, \mathcal{H})$ is the original matched pair $(\mathcal{V}, \mathcal{H})$.

The *opposite* of $(\mathcal{V}, \mathcal{H})$ is the matched pair $(\mathcal{V}, \mathcal{H}^{op})$ with the following actions:

$$
y \triangleright^{op} g := y^{-1} \triangleright g \quad \text{and} \quad y \triangleleft^{op} g := (y \triangleleft g)^{-1}
\tag{1.26}
$$

for $(y, g) \in (\mathcal{H}^{op})\,{}_{r^{op}}\!\times_t \mathcal{V} = \mathcal{H}\,{}_l\!\times_t \mathcal{V}$. Note that if $(x, g) \in \mathcal{H}\,{}_r\!\times_t \mathcal{V}$ then $(x^{-1}, g) \in \mathcal{H}\,{}_l\!\times_t \mathcal{V}$. The corresponding cell of the matched pair $(\mathcal{V}, \mathcal{H}^{op})$ is

$$
\begin{tikzcd}
Q \arrow[r, bend left, "x^{-1}"] \arrow[d, bend right, "x \triangleright g"'] & P \arrow[d, bend left, "g"] \\
S \arrow[r, bend right, "(x \triangleleft g)^{-1}"'] & R
\end{tikzcd}
\tag{1.27}
$$

In other words, inverting the horizontal arrows in a cell (1.14) of the matched pair $(\mathcal{V}, \mathcal{H})$ yields a cell of the opposite matched pair $(\mathcal{H}, \mathcal{V})$.

The *coopposite* matched pair $(\mathcal{V}^{op}, \mathcal{H})$ is defined similarly, by inverting the vertical arrows.

Example 1.4. Let $\mathcal{H} = \mathcal{P}$ be the discrete groupoid over $\mathcal{P}$ and $\mathcal{V} = \mathcal{P} \times \mathcal{P}$ be the coarse groupoid over $\mathcal{P}$ (Section 1.1). Then $(\mathcal{V}, \mathcal{H})$ is a matched pair of groupoids with the following actions:

$$P \triangleright (P, Q) = (P, Q) \quad \text{and} \quad P \triangleleft (P, Q) = Q\,.$$

The cells of this matched pair (1.14) are in this case simply

$$\begin{tikzcd} P \arrow[r, bend left, "P"] \arrow[d, bend right, "{(P,Q)}"'] & P \arrow[d, bend left, "{(P,Q)}"] \\ Q \arrow[r, bend right, "Q"'] & Q \end{tikzcd}$$

The dual matched pair has $\mathcal{H} = \mathcal{P} \times \mathcal{P}$, $\mathcal{V} = \mathcal{P}$, and actions

$$(P, Q) \triangleright Q = P \quad \text{and} \quad (P, Q) \triangleleft Q = (P, Q)\,.$$

We refer to $(\mathcal{P} \times \mathcal{P}, \mathcal{P})$ as the *initial* matched pair and to $(\mathcal{P}, \mathcal{P} \times \mathcal{P})$ as the *terminal* matched pair. This terminology is justified by Proposition 1.12.

Example 1.5. Let X and Y be arbitrary sets. Define $\mathcal{P} := X \times Y$, $\mathcal{H} := X \times X \times Y$, and $\mathcal{V} := X \times Y \times Y$. In order to define groupoid structures on $\mathcal{H}$ and $\mathcal{V}$ with base $\mathcal{P}$ and a matched pair structure on $(\mathcal{V}, \mathcal{H})$, we first specify the cells:

$$\begin{tikzcd} (x,y) \arrow[r, bend left, "{(x,x',y)}"] \arrow[d, bend right, "{(x,y,y')}"'] & (x',y) \arrow[d, bend left, "{(x',y,y')}"] \\ (x,y') \arrow[r, bend right, "{(x,x',y')}"'] & (x',y') \end{tikzcd}$$

From this diagram we can tell the source and end maps of $\mathcal{H}$ and $\mathcal{V}$, as well as the actions $\triangleright$ and $\triangleleft$. For instance,

$$l(x, x', y) := (x, y)\,,\ b(x, y, y') := (x, y')\,,\ \text{and}\ (x, x', y) \triangleright (x', y, y') := (x, y, y')\,.$$

It remains to describe the products on $\mathcal{H}$ and $\mathcal{V}$. They are

$$m_{\mathcal{H}}\big((x, x', y), (x', x'', y)\big) = (x, x'', y) \quad \text{and} \quad m_{\mathcal{V}}\big((x, y, y'), (x, y', y'')\big) = (x, y, y'')\,.$$

The groupoid and matched pair axioms are easily verified. We denote this matched pair by $M(X, Y)$. Note that if X is a singleton then $M(X, Y) = (\mathcal{P} \times \mathcal{P}, \mathcal{P})$, the initial matched pair, while if Y is a singleton then $M(X, Y) = (\mathcal{P}, \mathcal{P} \times \mathcal{P})$, the terminal matched pair. The dual of $M(X, Y)$ is $M(Y, X)$.

For a characterization of matched pairs of the form $M(X, Y)$ see [**2**, Proposition 2.14].

Example 1.6. Let $\mathcal{V}$ be any groupoid with base $\mathcal{P}$. There is a matched pair $(\mathcal{V}, \mathcal{P})$ with actions

$$t(f) \triangleright f = f \quad \text{and} \quad t(f) \triangleleft f = b(f)\,.$$

Similarly, for any groupoid $\mathcal{H}$ with base $\mathcal{P}$, there is a matched pair $(\mathcal{P}, \mathcal{H})$ with actions

$$x \triangleright r(x) = l(x) \quad \text{and} \quad x \triangleleft r(x) = x\,.$$

The dual of $(\mathcal{V}, \mathcal{P})$ is $(\mathcal{P}, \mathcal{V})$ (and the dual of $(\mathcal{P}, \mathcal{H})$ is $(\mathcal{H}, \mathcal{P})$).

Example 1.7. Let $\mathcal{V}$ be a groupoid with base $\mathcal{P}$, source t and end b. Let $\mathcal{N}$ be a group bundle over $\mathcal{P}$, viewed as a groupoid with source and end $l = r$. Consider the trivial left action of $\mathcal{N}$ on $t : \mathcal{V} \to \mathcal{P}$ as in (1.7): $n \triangleright g = g$ for all $(n, g) \in \mathcal{N}\,{}_r\!\times_t \mathcal{V}$.

Consider an arbitrary right action of $\mathcal{V}$ on $r : \mathcal{N} \to \mathcal{P}$. Then axioms (1.11) and (1.12) are satisfied, while axiom (1.13) is satisfied if and only if

$$nm \triangleleft g = (n \triangleleft g)(m \triangleleft g)$$

for all composable $n, m \in \mathcal{N}$, $g \in \mathcal{V}$. If this is the case, we say that the action of $\mathcal{V}$ on $\mathcal{N}$ is *by group bundle automorphisms*. Thus, an action of a groupoid $\mathcal{V}$ on a group bundle $\mathcal{N}$ by automorphisms together with the trivial action of the group bundle on the groupoid yield a matched pair of groupoids $(\mathcal{V}, \mathcal{N})$.

There is a similar notion of left action by group bundle automorphisms of a a groupoid $\mathcal{H}$ on a group bundle $\mathcal{N}$. Together with the trivial right action of $\mathcal{N}$ on $\mathcal{H}$, this gives rise to a matched pair $(\mathcal{N}, \mathcal{H})$.

There is a special case of particular interest. Let $\mathcal{H}$ be an arbitrary groupoid with base $\mathcal{P}$ and let $\mathcal{N} := \coprod_{P \in \mathcal{P}} \mathcal{H}(P, P)$. Then $\mathcal{H}$ acts on $\mathcal{N}$ by conjugation: $x \triangleright n := xnx^{-1}$. The matched pair $(\mathcal{N}, \mathcal{H})$ is one of the simplest instances of the double construction (see Example 1.20).

1.3. Two groupoids associated to a matched pair.

Definition 1.8. Let $(\mathcal{V}, \mathcal{H})$ be a matched pair of groupoids with base $\mathcal{P}$. The *diagonal* groupoid $\mathcal{V} \bowtie \mathcal{H}$ is defined as follows. The base is $\mathcal{P}$, the set of arrows is $\mathcal{V} \bowtie \mathcal{H} := \mathcal{V}\,{}_b\!\times_l \mathcal{H}$, the source map is $\mathfrak{s}(f, y) = t(f)$, the end map is $\mathfrak{e}(f, y) = r(y)$, the identities are $\operatorname{id} P = (\operatorname{id}_{\mathcal{V}} P, \operatorname{id}_{\mathcal{H}} P)$, and the composition is

$$(f, y)(h, z) = (f(y \triangleright h), (y \triangleleft h)z)\,. \tag{1.28}$$

The elements (f, y) of $\mathcal{V} \bowtie \mathcal{H}$ may be represented by *corner diagrams*

$$\begin{tikzcd} t(f) \arrow[d, bend right, "f"'] & \\ b(f) = l(y) \arrow[r, bend right, "y"'] & r(y) \end{tikzcd}$$

In this notation, the composition of $\mathcal{V} \bowtie \mathcal{H}$ results from the matched pair structure by stacking two such corners diagonally and filling in with a cell:

$$\begin{tikzcd}
t(f) \arrow[d, bend right, "f"'] & & \\
b(f) = l(y) \arrow[r, bend left, "y"] \arrow[d, dashed, "y \triangleright h"'] & r(y) = t(h) \arrow[d, bend left, "h"] & \\
b(y \triangleright h) = l(y \triangleleft h) \arrow[r, dashed, bend right, "y \triangleleft h"'] & b(h) = l(z) \arrow[r, bend right, "z"'] & r(z)
\end{tikzcd}$$

Note that, in view of (1.15) and (1.16), $\mathcal{V}$ and $\mathcal{H}$ may be viewed as subgroupoids of $\mathcal{V} \bowtie \mathcal{H}$ via $f \mapsto (f, \mathrm{id}_{\mathcal{H}}\, b(f))$ and $y \mapsto (\mathrm{id}_{\mathcal{V}}\, l(y), y)$.

Let $\mathcal{D}$ be a groupoid with base $\mathcal{P}$. An *exact factorization* of $\mathcal{D}$ is a pair of subgroupoids $\mathcal{V}$, $\mathcal{H}$ with the same base $\mathcal{P}$, such that for any $\alpha \in \mathcal{D}$, there exist unique $f \in \mathcal{V}$ and $y \in \mathcal{H}$ with $\alpha = fy$; in other words, the composition map $\mathcal{V}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}} \mathcal{H} \to \mathcal{D}$ is a bijection. A matched pair of groupoids $(\mathcal{V}, \mathcal{H})$ gives rise to an exact factorization of the diagonal groupoid $\mathcal{D} := \mathcal{V} \bowtie \mathcal{H}$ with factors $\mathcal{V}$ and $\mathcal{H}$.

Conversely, an exact factorization gives rise to a matched pair. Indeed, let t and b denote the restrictions of $\mathfrak{s}$ and $\mathfrak{e}$ to $\mathcal{V}$, and l and r the restrictions of $\mathfrak{s}$ and $\mathfrak{e}$ to $\mathcal{H}$. Then $\mathcal{H}\,{}_r\!\times_t \mathcal{V} \subseteq \mathcal{D}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}} \mathcal{D}$. Now, given $(x, g) \in \mathcal{H}\,{}_r\!\times_t \mathcal{V}$, the bijectivity of the composition allows to define elements $x \triangleright g \in \mathcal{V}$ and $x \triangleleft g \in \mathcal{H}$ by the equation

$$xg = (x \triangleright g)(x \triangleleft g)\,.$$

Then $\mathcal{V}$ and $\mathcal{H}$, together with these actions, form a matched pair.

The above constructions of exact factorizations and matched pairs are inverse.

For more details on these constructions, the reader may consult [**11**, Theorems 2.10 and 2.15] or [**2**, Prop 2.9].

Example 1.9. Consider a right action of a groupoid $\mathcal{V}$ on a group bundle $\mathcal{N}$ by automorphisms, and the corresponding matched pair $(\mathcal{V}, \mathcal{N})$ (Example 1.7). In this case, we denote the diagonal groupoid by $\mathcal{V} \ltimes \mathcal{N}$ and refer to it as the semidirect product of $\mathcal{V}$ by $\mathcal{N}$.

Note that the projection $\mathcal{V} \ltimes \mathcal{N} \to \mathcal{V}$, $(g, n) \mapsto g$ is a morphism of groupoids, and it has a section $\sigma : \mathcal{V} \to \mathcal{V} \ltimes \mathcal{N}$, $\sigma(g) = (g, \mathrm{id}\, b(g))$, which is a morphism of groupoids.

Conversely let $\alpha : \mathcal{G} \to \mathcal{K}$ be a morphism of groupoids over $\mathcal{P}$. If there is a section $\sigma : \mathcal{K} \to \mathcal{G}$ that is a morphism of groupoids, then $\mathcal{K}$ acts on the kernel $\mathcal{N}$ of α by $n \triangleleft k = \sigma(k)^{-1} n \sigma(k)$ and $\mathcal{G} \simeq \mathcal{K} \ltimes \mathcal{N}$. This may be seen as a special case of the characterization of matched pairs in terms of exact factorizations.

A simpler construction of groupoids is the following.

Definition 1.10. Let $\mathcal{G}$ and $\mathcal{K}$ be groupoids over $\mathcal{P}$. The *restricted product* of $\mathcal{G}$ and $\mathcal{K}$ is

$$\mathcal{G} \boxtimes \mathcal{K} := (\mathcal{G}\,{}_{\mathfrak{s}}\!\times_{\mathfrak{s}} \mathcal{K}) \cap (\mathcal{G}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{e}} \mathcal{K}) = \{(g, k) \in \mathcal{G} \times \mathcal{K} : \mathfrak{s}(g) = \mathfrak{s}(k),\ \mathfrak{e}(g) = \mathfrak{e}(k)\}.$$

Then $\mathcal{G} \boxtimes \mathcal{K}$ is a groupoid over the same base $\mathcal{P}$ with source and end maps defined by

$$\mathfrak{s}(g,k) = \mathfrak{s}(g) = \mathfrak{s}(k) \quad \text{and} \quad \mathfrak{e}(g,k) = \mathfrak{e}(g) = \mathfrak{e}(k)\,,$$

and composition

$$(g,k)(g',k') = (gg',kk')\,.$$

The restricted product is the product in the category of groupoids over $\mathcal{P}$: given a groupoid $\mathcal{F}$ with base $\mathcal{P}$ and morphisms $\alpha : \mathcal{F} \to \mathcal{G}$ and $\beta : \mathcal{F} \to \mathcal{K}$ of groupoids over $\mathcal{P}$, there is a unique morphism $\mathcal{F} \to \mathcal{G} \boxtimes \mathcal{K}$ of groupoids over $\mathcal{P}$ fitting in the commutative diagram

(1.29)
$$\begin{tikzcd}
 & \mathcal{F} \arrow[dl, "\alpha"'] \arrow[d, dashed] \arrow[dr, "\beta"] & \\
\mathcal{G} & \mathcal{G} \boxtimes \mathcal{K} \arrow[l] \arrow[r] & \mathcal{K}
\end{tikzcd}$$

Let $(\mathcal{V},\mathcal{H})$ be a matched pair. The diagonal groupoid, on the other hand, may be viewed as a sort of "restricted coproduct", in the sense that it satisfies the following universal property. Given a groupoid $\mathcal{F}$ with base $\mathcal{P}$ and morphisms $\alpha : \mathcal{H} \to \mathcal{F}$ and $\beta : \mathcal{V} \to \mathcal{F}$ of groupoids over $\mathcal{P}$ such that

$$\alpha(y)\beta(h) = \beta(y \triangleright h)\alpha(y \triangleleft h) \quad \text{for every} \quad (y,h) \in \mathcal{H}\,{}_r{\times}_t\,\mathcal{V}\,, \tag{1.30}$$

there is a unique morphism $\mathcal{V} \bowtie \mathcal{H} \to \mathcal{F}$ of groupoids over $\mathcal{P}$ fitting in the commutative diagram

(1.31)
$$\begin{tikzcd}
\mathcal{H} \arrow[r] \arrow[dr, "\alpha"'] & \mathcal{V} \bowtie \mathcal{H} \arrow[d, dashed] & \mathcal{V} \arrow[l] \arrow[dl, "\beta"] \\
 & \mathcal{F} &
\end{tikzcd}$$

Condition (1.30) may be illustrated as follows:

$$\begin{tikzcd}
P \arrow[r, bend left, "y"] \arrow[d, bend right, "y \triangleright h"'] & Q \arrow[d, bend left, "h"] \\
R \arrow[r, bend right, "y \triangleleft h"'] & S
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
 & Q \arrow[dr, "\beta(h)"] & \\
P \arrow[ur, "\alpha(y)"] \arrow[dr, "\beta(y \triangleright h)"'] & & S \\
 & R \arrow[ur, "\alpha(y \triangleleft h)"'] &
\end{tikzcd}$$

In other words, cells in the matched pair $(\mathcal{V},\mathcal{H})$ are transformed by (α,β) into *commutative* diagrams in the groupoid $\mathcal{F}$.

1.4. Morphisms of matched pairs. Duals and doubles.

Definition 1.11. Let $(\mathbb{V},\mathbb{H})$ and $(\mathcal{V},\mathcal{H})$ be two matched pairs of groupoids with the same base $\mathcal{P}$. A *morphism of matched pairs of groupoids* $(\mathbb{V},\mathbb{H}) \to (\mathcal{V},\mathcal{H})$ is a pair (α,β) of morphisms of groupoids over $\mathcal{P}$, $\alpha : \mathbb{H} \to \mathcal{H}$ and $\beta : \mathcal{V} \to \mathbb{V}$, such that

$$\beta(\alpha(h) \triangleright g) = h \triangleright \beta(g)\,, \tag{1.32}$$

$$\alpha(h \triangleleft \beta(g)) = \alpha(h) \triangleleft g\,, \tag{1.33}$$

for all $(h,g) \in \mathbb{H}\,{}_r\!\times_t \mathcal{V}$.

If $(\mathfrak{H},\mathfrak{V})$ is another matched pair of groupoids with base $\mathcal{P}$ and $(\delta,\omega) : (\mathcal{V},\mathcal{H}) \to (\mathfrak{V},\mathfrak{H})$ is a morphism of matched pairs, then $(\alpha\delta,\omega\beta)$ is a morphism of matched pairs $(\mathbb{V},\mathbb{H}) \to (\mathfrak{V},\mathfrak{H})$. Thus, matched pairs of groupoids with the same base $\mathcal{P}$ form a category.

There are two distinguished objects in this category.

Proposition 1.12. *The matched pairs $(\mathcal{P}\times\mathcal{P},\mathcal{P})$ and $(\mathcal{P},\mathcal{P}\times\mathcal{P})$ of Example 1.4 are respectively the initial and the terminal objects in the category of matched pair of groupoids with base $\mathcal{P}$.*

Proof. Given an arbitrary matched pair $(\mathcal{V},\mathcal{H})$, consider the unique morphisms of groupoids over $\mathcal{P}$, $\alpha_0 : \mathcal{P} \to \mathcal{H}$ and $\beta_0 : \mathcal{V} \to \mathcal{P}\times\mathcal{P}$ (Section 1.1):

$$\alpha_0(P) = \mathrm{id}_{\mathcal{H}}\, P\,, \quad \beta_0(g) = (t(g), b(g))\,.$$

Then (α_0,β_0) is (the unique) morphism of matched pairs $(\mathcal{P}\times\mathcal{P},\mathcal{P}) \to (\mathcal{V},\mathcal{H})$: condition (1.32) holds by (1.4) and condition (1.33) holds by (1.16). Similarly, the unique morphisms of groupoids over $\mathcal{P}$, $\alpha_1 : \mathcal{H} \to \mathcal{P}\times\mathcal{P}$ and $\beta_1 : \mathcal{P} \to \mathcal{V}$ combine to give the unique morphism of matched pairs $(\alpha_1,\beta_1) : (\mathcal{V},\mathcal{H}) \to (\mathcal{P},\mathcal{P}\times\mathcal{P})$. $\square$

Remark 1.13. Any matched pair $(\mathcal{V},\mathcal{H})$ is isomorphic to its opposite $(\mathcal{V},\mathcal{H}^{op})$ (Remark 1.3). The isomorphism consist of the pair $\alpha : \mathcal{H} \to \mathcal{H}^{op}$, $\alpha(x) = x^{-1}$, and $\beta : \mathcal{V} \to \mathcal{V}$, $\beta(g) = g$. Similarly, any matched pair is isomorphic to its coopposite.

We will construct a new matched pair from a morphism of matched pairs. The following lemma is a preliminary step towards this goal. We use t and b to denote the source and end maps of both $\mathbb{V}$ and $\mathcal{V}$, and similarly l and r for the source and end maps of both $\mathbb{H}$ and $\mathcal{H}$.

Lemma 1.14. *Let $(\alpha,\beta) : (\mathbb{V},\mathbb{H}) \to (\mathcal{V},\mathcal{H})$ be a morphism of matched pairs of groupoids. Define maps $\rightharpoonup : \mathbb{H}\,{}_r\!\times_t \mathcal{V} \to \mathcal{V}$ and $\leftharpoonup : \mathbb{H}\,{}_r\!\times_t \mathcal{V} \to \mathbb{H}$ by*

$$h \rightharpoonup g := \alpha(h) \triangleright g\,, \qquad h \leftharpoonup g := h \triangleleft \beta(g)\,. \tag{1.34}$$

Then $(\mathcal{V},\mathbb{H},\rightharpoonup,\leftharpoonup)$ is a matched pair of groupoids.

Proof. The maps $\rightharpoonup$ and $\leftharpoonup$ are well-defined and are actions since α and β are morphisms of groupoids. We check (1.11): $b(h \rightharpoonup g) = b(\alpha(h)\triangleright g) = l(\alpha(h)\triangleleft g) = l(\alpha(h\triangleleft\beta(g))) = l(h\triangleleft\beta(g)) = l(h \leftharpoonup g)$, where we have used (1.33) in the third equality. We check (1.12): $h \rightharpoonup fg = \alpha(h)\triangleright fg = (\alpha(h)\triangleright f)((\alpha(h)\triangleleft f)\triangleright g) = (h \rightharpoonup f)(\alpha(h \leftharpoonup f)\triangleright g) = (h \rightharpoonup f)((h \leftharpoonup f) \rightharpoonup g)$, where we have used (1.33) in the third equality. We check (1.13): $hk \leftharpoonup g = hk\triangleleft\beta(g) = (h\triangleleft(k\triangleright\beta(g)))(k\triangleleft\beta(g)) = (h\triangleleft\beta(k \rightharpoonup g))(k \leftharpoonup g) = (h \leftharpoonup (k \rightharpoonup g))(k \leftharpoonup g)$, where we have used (1.32) in the third equality. $\square$

We may now consider the diagonal groupoid $\mathcal{V} \bowtie \mathbb{H}$ (Definition 1.8) associated to the matched pair of Lemma 1.14. Let l and r be its source and end maps. According to Definition 1.8,

$$l(g,h) = t(g) \quad \text{and} \quad r(g,h) = r(h)$$

for all $(g,h) \in \mathcal{V} \bowtie \mathbb{H}$; and the product is given by

$$(f,u)(g,h) = \Big(f\big(\alpha(u) \triangleright g\big), \big(u \triangleleft \beta(g)\big)h \Big). \tag{1.35}$$

The groupoid $\mathcal{V} \bowtie \mathbb{H}$ will be the horizontal groupoid of a matched pair to be defined shortly. The vertical groupoid will be $\mathbb{V} \boxtimes \mathcal{H}^{op}$. Let t and b be the source and end maps of $\mathbb{V} \boxtimes \mathcal{H}^{op}$. According to Definition 1.10,

$$t(\gamma, x) = t(\gamma) = r(x) \quad \text{and} \quad b(\gamma, x) = b(\gamma) = l(x)$$

for all $(\gamma, x) \in \mathbb{V} \boxtimes \mathcal{H}^{op}$.

Theorem 1.15. *Let $(\alpha, \beta) : (\mathbb{V}, \mathbb{H}) \to (\mathcal{V}, \mathcal{H})$ be a morphism of matched pairs of groupoids. Define maps $\rightharpoonup : (\mathcal{V} \bowtie \mathbb{H})\, {}_r\!\times_t (\mathbb{V} \boxtimes \mathcal{H}^{op}) \to \mathbb{V} \boxtimes \mathcal{H}^{op}$ and $\leftharpoonup : (\mathcal{V} \bowtie \mathbb{H})\, {}_r\!\times_t (\mathbb{V} \boxtimes \mathcal{H}^{op}) \to \mathcal{V} \bowtie \mathbb{H}$ by*

(1.36)

$$(g,h) \rightharpoonup (\gamma, x) := \left(\beta(g)(h \triangleright \gamma)\beta\Big(\big(\alpha(h \triangleleft \gamma) x \alpha(h)^{-1}\big) \triangleright g^{-1}\Big), \Big(\alpha(h \triangleleft \gamma) x \alpha(h)^{-1}\Big) \triangleleft g^{-1} \right),$$

(1.37)

$$(g,h) \leftharpoonup (\gamma, x) := \left(\Big(\big(\alpha(h \triangleleft \gamma) x \alpha(h)^{-1}\big) \triangleleft g^{-1}\Big) \triangleright g\,,\ h \triangleleft \gamma \right),$$

for all $(g,h) \in \mathcal{V} \bowtie \mathbb{H}$, $(\gamma, x) \in \mathbb{V} \boxtimes \mathcal{H}^{op}$ with $r(h) = t(\gamma)$.

Then $(\mathbb{V} \boxtimes \mathcal{H}^{op}, \mathcal{V} \bowtie \mathbb{H}, \rightharpoonup, \leftharpoonup)$ is a matched pair of groupoids.

The proof of this theorem is lengthy but for the most part straightforward and is given in the Appendix. Here we content ourselves with an illustration of formulas (1.36) and (1.37), in the special case when $(\mathbb{V}, \mathbb{H}) = (\mathcal{V}, \mathcal{H})$ and α and β are identities. We start from $(g,h) \in \mathcal{V} \bowtie \mathcal{H}$ and $(\gamma, x) \in \mathcal{V} \boxtimes \mathcal{H}^{op}$ with $r(h) = t(\gamma)$ (so $b(g) = l(h)$, $t(\gamma) = r(x)$, and $b(\gamma) = l(x)$), and fill in with cells:

(1.38)

$$\begin{tikzcd}
P \arrow[d, bend right, "g"'] & & & \\
Q \arrow[r, bend left, "h"] \arrow[d, bend right, "h \triangleright \gamma"'] & R \arrow[d, bend left, "\gamma"] & & \\
R' \arrow[r, bend right, "h \triangleleft \gamma"'] \arrow[d, bend right, "\Omega \triangleright g^{-1}"'] & S \arrow[r, bend right, "x"'] & R \arrow[r, bend right, "h^{-1}"'] & Q \arrow[d, bend left, "g^{-1}"] \\
Q' \arrow[rrr, bend right, "\Omega \triangleleft g^{-1}"'] & & & P
\end{tikzcd}$$

where $\Omega := (h \triangleleft \gamma) x h^{-1} \in \mathcal{H}$. Formulas (1.36) and (1.37) can be written as

$(g,h) \rightharpoonup (\gamma, x) = \big(g(h \triangleright \gamma)(\Omega \triangleright g^{-1}), \Omega \triangleleft g^{-1}\big)$ and $(g,h) \leftharpoonup (\gamma, x) = \big((\Omega \triangleright g^{-1})^{-1}, h \triangleleft \gamma\big)$.

Definition 1.16. The matched pair associated to the identity morphism $(\mathcal{V},\mathcal{H}) \to (\mathcal{V},\mathcal{H})$ by means of Theorem 1.15 is called the *double* of $(\mathcal{V},\mathcal{H})$ and denoted $D(\mathcal{V},\mathcal{H})$. The matched pair associated to a general morphism (α,β) : $(\mathbb{V},\mathbb{H}) \to (\mathcal{V},\mathcal{H})$ is called a *generalized double* and denoted $D(\alpha,\beta)$.

This terminology is justified by Theorem 3.1, Corollary 3.4, and Theorem 5.10 below.

Remark 1.17. Consider the unique morphism $(\alpha_0,\beta_0) : (\mathcal{P}\times\mathcal{P},\mathcal{P}) \to (\mathcal{V},\mathcal{H})$ (Proposition 1.12). We describe its generalized double. We have

$$(\mathcal{P}\times\mathcal{P})\boxtimes\mathcal{H}^{op} \cong \mathcal{H}^{op} \quad\text{and}\quad \mathcal{V}\bowtie\mathcal{P}\cong\mathcal{V}$$

via $\big(r(x),l(x),x\big) \leftrightarrow x$ and $\big(g,b(g)\big) \leftrightarrow g$. After these identifications, (1.36) and (1.37) boil down to

$$g \rightharpoonup x = x \triangleleft g^{-1} \quad\text{and}\quad g \leftharpoonup x = (x\triangleleft g^{-1})\triangleright g = (x\triangleright g^{-1})^{-1}$$

for $g\in\mathcal{V}$ and $x\in\mathcal{H}$ with $b(g)=r(x)$. Comparing these expressions with (1.24) and (1.26) we see that the generalized double $D(\alpha_0,\beta_0)$ is the dual of the opposite of $(\mathcal{V},\mathcal{H})$. Since any matched pair is isomorphic to its opposite (Remark 1.13), this generalized double is isomorphic to the dual of $(\mathcal{V},\mathcal{H})$: $D(\alpha_0,\beta_0)\cong(\mathcal{H},\mathcal{V})$.

Consider the generalized double of the (unique) morphism $(\alpha_1,\beta_1) : (\mathbb{V},\mathbb{H}) \to (\mathcal{P},\mathcal{P}\times\mathcal{P})$ of Proposition 1.12. We have

$$\mathbb{V}\boxtimes(\mathcal{P}\times\mathcal{P})^{op}\cong\mathbb{V} \quad\text{and}\quad \mathcal{P}\bowtie\mathbb{H}\cong\mathbb{H}\,,$$

and after these identifications, (1.36) and (1.37) boil down to the original actions of the matched pair $(\mathbb{V},\mathbb{H})$. Thus $D(\alpha_1,\beta_1)\cong(\mathbb{V},\mathbb{H})$.

Example 1.18. Consider the initial matched pair $(\mathcal{P}\times\mathcal{P},\mathcal{P})$ (Example 1.4). The only endomorphism of $(\mathcal{P}\times\mathcal{P},\mathcal{P})$ is the identity, so the dual and the double of this matched pair coincide. Therefore, the double of this matched pair is $(\mathcal{P},\mathcal{P}\times\mathcal{P})$, the terminal matched pair (Example 1.4).

The unique morphism $(\alpha_1,\beta_1) : (\mathcal{P},\mathcal{P}\times\mathcal{P}) \to (\mathcal{P},\mathcal{P}\times\mathcal{P})$ is the identity. Therefore, by Remark 1.17, $(\mathcal{P},\mathcal{P}\times\mathcal{P})$ coincides with its own double.

Example 1.19. Let X and Y be sets and consider the matched pair $M(X,Y)$ of Example 1.5. We have $\mathcal{P}=X\times Y$, $\mathcal{H}=X\times X\times Y$, $\mathcal{V}=X\times Y\times Y$, and

$$\mathcal{V}\boxtimes\mathcal{H}^{op}\cong X\times Y \quad\text{and}\quad \mathcal{V}\bowtie\mathcal{H}\cong(X\times Y)\times(X\times Y)$$

via $\big((x,y,y),(x,x,y)\big) \leftrightarrow (x,y)$ and $\big((x,y,y''),(x,x'',y'')\big) \leftrightarrow \big((x,y),(x'',y'')\big)$. Thus, the double of this matched pair is the terminal matched pair $M(X\times Y,\{*\}) = (\mathcal{P},\mathcal{P}\times\mathcal{P})$.

Example 1.20. Let $(\mathcal{V},\mathcal{P})$ and $(\mathcal{P},\mathcal{H})$ be the matched pairs of Example 1.6. For $(\mathcal{P},\mathcal{H})$, we have $\mathcal{P}\boxtimes\mathcal{H}^{op}=\coprod_{P\in\mathcal{P}}\mathcal{H}(P,P)^{op}$ (a group bundle) and $\mathcal{P}\bowtie\mathcal{H}\cong\mathcal{H}$.

Actions (1.36) and (1.37) reduce to

$$h \rightharpoonup x = hxh^{-1} \quad \text{and} \quad h \leftharpoonup x = h$$

for $h \in \mathcal{H}(P,Q)$, $x \in \mathcal{H}(Q,Q)$. Thus, the double of $(\mathcal{P},\mathcal{H})$ is the coopposite of (and hence isomorphic to) the matched pair $(\coprod_{P\in\mathcal{P}} \mathcal{H}(P,P),\mathcal{H})$ of Example 1.7.

Similarly, the double of $(\mathcal{V},\mathcal{P})$ is the matched pair $(\coprod_{P\in\mathcal{P}} \mathcal{V}(P,P),\mathcal{V})$ of Example 1.7. In particular, $D(\mathcal{V},\mathcal{P}) \cong D(\mathcal{P},\mathcal{V})$.

Next we discuss the functoriality of generalized doubles.

Proposition 1.21. *Consider a commutative diagram of morphisms of matched pairs:*

$$\begin{tikzcd} (\mathbb{V},\mathbb{H}) \arrow[rr, "(\alpha,\beta)"] \arrow[dr, "(\varphi,\psi)"'] & & (\mathcal{V},\mathcal{H}) \arrow[dl, "(\chi,\omega)"] \\ & (\mathfrak{V},\mathfrak{H}) & \end{tikzcd}$$

There are morphisms of matched pairs

$$(\alpha_{\#},\beta_{\#}) : D(\varphi,\psi) \to D(\chi,\omega) \quad \textit{and} \quad (\chi^{\#},\omega^{\#}) : D(\varphi,\psi) \to D(\alpha,\beta)$$

given by

$$\alpha_{\#} := \mathfrak{V} \bowtie \alpha\,, \;\; \beta_{\#} := \beta \boxtimes \mathfrak{H}^{op}\,, \;\; \chi^{\#} := \chi \bowtie \mathbb{H}\,, \;\; \omega^{\#} := \mathbb{V} \boxtimes \omega\,.$$

Proof. We only note that $D(\varphi,\psi) = (\mathbb{V}\boxtimes\mathfrak{H}^{op}, \mathfrak{V}\bowtie\mathbb{H})$, $D(\chi,\omega) = (\mathcal{V}\boxtimes\mathfrak{H}^{op}, \mathfrak{V}\bowtie\mathcal{H})$, and $D(\alpha,\beta) = (\mathbb{V}\boxtimes\mathcal{H}^{op}, \mathcal{V}\bowtie\mathbb{H})$, so the above maps are well-defined, and omit the remaining verifications. $\square$

Remark 1.22. Let us apply Proposition 1.21 to the commutative diagram

$$\begin{tikzcd} (\mathbb{V},\mathbb{H}) \arrow[rr, "(\alpha,\beta)"] \arrow[dr] & & (\mathcal{V},\mathcal{H}) \arrow[dl] \\ & (\mathcal{P},\mathcal{P}\times\mathcal{P}) & \end{tikzcd}$$

where the diagonal maps are the unique morphisms to the terminal matched pair. According to Remark 1.17, the generalized double of the left diagonal map can be identified with $(\mathbb{V},\mathbb{H})$. Therefore, the right diagonal map induces a morphism of matched pairs

$$(\mathbb{V},\mathbb{H}) \to D(\alpha,\beta)\,.$$

Explicitly, this consists of the pair of morphisms of groupoids $\iota : \mathbb{H} \to \mathcal{V}\bowtie\mathbb{H}$ and $\pi : \mathbb{V}\boxtimes\mathcal{H}^{op} \to \mathbb{V}$ given by

$$\iota(h) = (\mathrm{id}_{\mathcal{V}}\, l(h), h) \quad \text{and} \quad \pi(\gamma, x) = \gamma\,.$$

This morphism plays a crucial role in Section 4.

Finally, let us apply Proposition 1.21 to the commutative diagram

$$\begin{tikzcd} (\mathbb{V},\mathbb{H}) \arrow[rr, "(\alpha,\beta)"] & & (\mathcal{V},\mathcal{H}) \\ & (\mathcal{P}\times\mathcal{P},\mathcal{P}) \arrow[ul] \arrow[ur] & \end{tikzcd}$$

where the diagonal maps are the unique morphisms from the initial matched pair. By Remark 1.17, we obtain a morphism of matched pairs $(\alpha^{\#},\beta^{\#})$ from the dual of $(\mathcal{V},\mathcal{H})$ to the dual of $(\mathbb{V},\mathbb{H})$. Explicitly, $\alpha^{\#}=\beta:\mathcal{V}\to\mathbb{V}$ and $\beta^{\#}=\alpha:\mathbb{H}\to\mathcal{H}$. We refer to the morphism (β,α) as the dual of the morphism (α,β).

The generalized double of a morphism and that of its dual coincide:

Proposition 1.23. *Let $(\alpha,\beta):(\mathbb{V},\mathbb{H})\to(\mathcal{V},\mathcal{H})$ be a morphism of matched pairs and consider its dual $(\beta,\alpha):(\mathcal{H},\mathcal{V})\to(\mathbb{H},\mathbb{V})$. There is an isomorphism of matched pairs*

$$D(\alpha,\beta)\cong D(\beta,\alpha)\,.$$

Proof. We define the isomorphism and omit the verifications. We have

$$D(\alpha,\beta)=(\mathbb{V}\boxtimes\mathcal{H}^{op},\mathcal{V}\bowtie\mathbb{H})\ \text{ and }\ D(\beta,\alpha)=(\mathcal{H}\boxtimes\mathbb{V}^{op},\mathbb{H}\bowtie\mathcal{V})\,.$$

The isomorphism $D(\alpha,\beta)\to D(\beta,\alpha)$ is given by the following pair of morphisms of groupoids:

$$\mathcal{V}\bowtie\mathbb{H}\to\mathbb{H}\bowtie\mathcal{V}\,,\qquad (g,h)\mapsto\Big(\big(h^{-1}\triangleleft\beta(g)^{-1}\big)^{-1},\big(\alpha(h)^{-1}\triangleright g^{-1}\big)^{-1}\Big)\,;$$
$$\mathcal{H}\boxtimes\mathbb{V}^{op}\to\mathbb{V}\boxtimes\mathcal{H}^{op}\,,\qquad (x,\gamma)\mapsto(\gamma^{-1},x^{-1})\,.$$

□

We deduce that the constructions of doubles and duals commute:

Corollary 1.24. *Let $(\mathcal{V},\mathcal{H})$ be a matched pair and consider its dual $(\mathcal{H},\mathcal{V})$. We have*

$$D(\mathcal{V},\mathcal{H})\cong D(\mathcal{H},\mathcal{V})\,.$$

Proof. Take α and β identities in Proposition 1.23. □

2. Representations of matched pairs of groupoids

2.1. Definition and monoidal structure.

Let $\mathcal{P}$ be a set. We denote by $\mathrm{Quiv}(\mathcal{P})$ the category of *quivers* over $\mathcal{P}$. The objects of $\mathrm{Quiv}(\mathcal{P})$ are sets X equipped with two maps $p:X\to\mathcal{P}$ and $q:X\to\mathcal{P}$, and the morphisms are maps $X\to Y$ for which $\begin{tikzcd}[sep=small] X \arrow[rr] \arrow[dr, "p"'] & & Y \arrow[dl, "p"] \\ & \mathcal{P} & \end{tikzcd}$ and $\begin{tikzcd}[sep=small] X \arrow[rr] \arrow[dr, "q"'] & & Y \arrow[dl, "q"] \\ & \mathcal{P} & \end{tikzcd}$ commute. This is a monoidal category: the tensor product of two objects X and Y is the set $X\,{}_q\!\times_p Y$ (1.1) with $p(x,y)=p(x)$ and $q(x,y)=q(y)$. The unit object is the set $\mathcal{P}$ with $p=q=\mathrm{id}:\mathcal{P}\to\mathcal{P}$.

Definition 2.1. Let $(\mathcal{V},\mathcal{H})$ be a matched pair of groupoids with base $\mathcal{P}$. A (set-theoretic) *representation* of $(\mathcal{V},\mathcal{H})$ is an object $(\mathcal{E},p,q)$ of $\mathrm{Quiv}(\mathcal{P})$ together with

- A left action $\triangleright : \mathcal{H}\,{}_r\!\times_p \mathcal{E} \to \mathcal{E}$ of $\mathcal{H}$ on p.
- A decomposition $\mathcal{E} = \coprod_{g\in\mathcal{V}} \mathcal{E}(g)$ called the *grading* of $\mathcal{E}$ over $\mathcal{V}$. If $e \in \mathcal{E}(g)$, we set $|e| := g$.

These are subject to the following conditions:

$$p(e) = t(|e|) \text{ and } q(e) = b(|e|) \text{ for all } e \in \mathcal{E}\,, \tag{2.1}$$

$$|x \triangleright e| = x \triangleright |e| \text{ for all } (x, e) \in \mathcal{H}\,{}_r\!\times_p \mathcal{E}\,. \tag{2.2}$$

Note that p and q are determined by the rest of the structure, in view of (2.1). The following diagram displays the above conditions:

$$\begin{tikzcd}
t(x\triangleright|e|) = l(x) \arrow[r, bend left, "x"] \arrow[d, bend right, "|x\triangleright e| = x\triangleright|e|"'] & r(x) = p(e) = t(|e|) \arrow[d, bend left, "|e|"] \\
b(x\triangleright|e|) = l(x\triangleleft|e|) \arrow[r, bend right, "x\triangleleft|e|"'] & r(x\triangleleft|e|) = q(e) = b(|e|)
\end{tikzcd}$$

A morphism between representations $\mathcal{E}$ and $\mathcal{F}$ of $(\mathcal{V}, \mathcal{H})$ is a map $\psi : \mathcal{E} \to \mathcal{F}$ which intertwines the actions of $\mathcal{H}$ (1.5) and preserves the gradings in the sense that $|\psi(e)| = |e|$. It follows that ψ preserves p and q.

The resulting category of representations of the matched pair $(\mathcal{V}, \mathcal{H})$ is denoted $\mathrm{Rep}(\mathcal{V}, \mathcal{H})$. It comes endowed with a functor $f\!\ell : \mathrm{Rep}(\mathcal{V}, \mathcal{H}) \to \mathrm{Quiv}(\mathcal{P})$ which forgets the action and the grading.

Example 2.2. The category of quivers over $\mathcal{P}$ may be identified with the category of representations of the initial matched pair $(\mathcal{P} \times \mathcal{P}, \mathcal{P})$ (Example 1.4). The action of $\mathcal{P}$ on a quiver $(\mathcal{E}, p, q)$ is $P \triangleright e = e$, for $e \in \mathcal{E}$ with $p(e) = P$, and the grading over $\mathcal{P} \times \mathcal{P}$ is $|e| = \big(p(e), q(e)\big)$.

The category of representations of the terminal matched pair $(\mathcal{P}, \mathcal{P} \times \mathcal{P})$ is less familiar. Its objects are sets $\mathcal{E}$ equipped with a decomposition $\mathcal{E} = \coprod_{P\in\mathcal{P}} \mathcal{E}_P$ and a family of maps $\pi_P : \mathcal{E} \to \mathcal{E}_P$, one for each $P \in \mathcal{P}$, with the following properties:

$$\pi_P(e) = e \text{ if } e \in \mathcal{E}_P \text{ and } \pi_P\big(\pi_Q(e)\big) = \pi_P(e) \text{ for every } e \in \mathcal{E},\ P, Q \in \mathcal{P}.$$

In particular, each π_P is a projection $\mathcal{E} \twoheadrightarrow \mathcal{E}_P$. The structure of representation is as follows. The quiver structure maps and the grading are $p(e) = q(e) = |e| = P$, for $e \in \mathcal{E}_P$, while the action of $\mathcal{P} \times \mathcal{P}$ is $(P, Q) \triangleright e = \pi_P(e)$, for $e \in \mathcal{E}_Q$.

More generally, a representation of a matched pair of the form $(\mathcal{V}, \mathcal{P})$ (Example 1.6) is just a set $\mathcal{E}$ with a grading over $\mathcal{V}$, and a representation of a matched pair of the form $(\mathcal{P}, \mathcal{H})$ is just a left action of $\mathcal{H}$ on a map $p : \mathcal{E} \to \mathcal{P}$.

We describe the representations of the matched pair $M(X, Y)$ (Example 1.5). These objects are quivers $\mathcal{E}$ with base Y, together with a decomposition $\mathcal{E} = \coprod_{x\in X} \mathcal{E}_x$ and a family of maps $\pi_x : \mathcal{E} \to \mathcal{E}_x$, one for each $x \in X$, which are morphisms of quivers over Y and have the following properties:

$$\pi_x(e) = e \text{ if } e \in \mathcal{E}_x \text{ and } \pi_x\big(\pi_{x'}(e)\big) = \pi_x(e) \text{ for every } e \in \mathcal{E},\ x, x' \in X.$$

Choosing X or Y as singletons we recover the above descriptions of the representations of the initial and terminal matched pairs respectively.

REMARK 2.3. We mention a construction due to the referee. Given a quiver $\mathcal{E}$ over $\mathcal{P}$, there is a matched pair $\big(\mathcal{V}(\mathcal{E}),\mathcal{H}(\mathcal{E})\big)$ with the property that representations of an arbitrary matched pair $(\mathcal{V},\mathcal{H})$ on $\mathcal{E}$ are in one-to-one correspondence with morphisms $(\mathcal{V},\mathcal{H})\to\big(\mathcal{V}(\mathcal{E}),\mathcal{H}(\mathcal{E})\big)$. See [**1**] for the details.

The notion of representation of $(\mathcal{V},\mathcal{H})$ makes use of the groupoid structure of $\mathcal{H}$ and of the action of $\mathcal{H}$ on $\mathcal{V}$ only. Below, we introduce a monoidal structure on the category of representations which brings into play the groupoid structure of $\mathcal{V}$ and the action of $\mathcal{V}$ on $\mathcal{H}$.

PROPOSITION 2.4. *Let $\mathcal{E}$ and $\mathcal{F}$ be representations of a matched pair $(\mathcal{V},\mathcal{H})$. Consider the set*

$$\mathcal{E}\otimes\mathcal{F} \ := \ \mathcal{E}\,{}_q\!\times_p\mathcal{F}=\{(e,f)\in\mathcal{E}\times\mathcal{F}: b(|e|)=t(|f|)\}.$$

This set can be made into a representation of $(\mathcal{V},\mathcal{H})$ by defining

$$p,q:\mathcal{E}\otimes\mathcal{F}\to\mathcal{P}\,,\quad p(e,f)=p(e)\,,\quad q(e,f)=q(f)\,, \tag{2.3}$$

$$x\triangleright(e,f)=\big(x\triangleright e,(x\triangleleft|e|)\triangleright f\big)\,, \tag{2.4}$$

$$|(e,f)|=|e||f|\,. \tag{2.5}$$

PROOF. We check that this structure is well-defined and that $\mathcal{E}\otimes\mathcal{F}$ is indeed a representation of $(\mathcal{V},\mathcal{H})$.

Since $b(|e|)=t(|f|)$, for $(e,f)\in\mathcal{E}\otimes\mathcal{F}$, the grading is well-defined. Since $p(f)=b(|e|)=r(x\triangleleft|e|)$, the second component in (2.4) is well-defined. Finally, $b(|x\triangleright e|)=b(x\triangleright|e|)=l(x\triangleleft|e|)=p((x\triangleleft|e|)\triangleright f)=t(|(x\triangleleft|e|)\triangleright f)|)$; hence $(x\triangleright e,(x\triangleleft|e|)\triangleright f)\in\mathcal{E}\otimes\mathcal{F}$.

A straightforward computation shows that (1.2), (1.3), and (1.4) hold for (2.4).

We check condition (2.1) for $\mathcal{E}\otimes\mathcal{F}$:

$$p(e,f)=p(e)=t(|e|)=t(|e||f|)=t\big(|(e,f)|\big)\,,$$

by (2.1) for $\mathcal{E}$. Similarly, $q(e,f)=b\big(|(e,f)|\big)$. We check condition (2.2) for $\mathcal{E}\otimes\mathcal{F}$:

$$\begin{aligned}|x\triangleright(e,f)|=|(x\triangleright e,(x\triangleleft|e|)\triangleright f)|&=|x\triangleright e||(x\triangleleft|e|)\triangleright f|\\&=(x\triangleright|e|)\big((x\triangleleft|e|)\triangleright|f|\big)=x\triangleright(|e||f|)\,,\end{aligned}$$

by (2.2) for $\mathcal{F}$ and by (1.12). $\square$

The set $\mathcal{P}$ can be made into a representation of any matched pair $(\mathcal{V},\mathcal{H})$ of groupoids with base $\mathcal{P}$. We set $p=q=\mathrm{id}:\mathcal{P}\to\mathcal{P}$, $|P|=\mathrm{id}_{\mathcal{V}}\,P\in\mathcal{V}$, and we let $\mathcal{H}$ act trivially on p, as in (1.6), which in this case boils down to $x\triangleright r(x)=l(x)$ for all $x\in\mathcal{H}$.

PROPOSITION 2.5. *The operation $\mathcal{E}\otimes\mathcal{F}$ of Proposition 2.4 turns* $\mathrm{Rep}(\mathcal{V},\mathcal{H})$ *into a monoidal category. The unit object is $\mathcal{P}$, as above. The forgetful functor* $f\ell:\mathrm{Rep}(\mathcal{V},\mathcal{H})\to\mathrm{Quiv}(\mathcal{P})$ *is monoidal.*

Proof. If $\mathcal{E}$, $\mathcal{F}$, and $\mathcal{K}$ are in $\mathrm{Rep}(\mathcal{V},\mathcal{H})$, then there are canonical identifications

$$(\mathcal{E}\otimes\mathcal{F})\otimes\mathcal{K}\simeq\{(e,f,k)\in\mathcal{E}\times\mathcal{F}\times\mathcal{K}:q(e)=p(f)\,,\ q(f)=p(k)\}\simeq\mathcal{E}\otimes(\mathcal{F}\otimes\mathcal{K})\,.$$

The grading is preserved under these identifications by associativity of the product of $\mathcal{V}$. As for the action of $\mathcal{H}$, we have

$$\begin{aligned} x\triangleright\big(e,(f,k)\big) &= \big(x\triangleright e,(x\triangleleft|e|)\triangleright(f,k)\big)=\Big(x\triangleright e,(x\triangleleft|e|)\triangleright f,\big((x\triangleleft|e|)\triangleleft|f|\big)\triangleright k\Big)\\ &=\Big(x\triangleright e,(x\triangleleft|e|)\triangleright f,\big(x\triangleleft(|e|\,|f|)\big)\triangleright k\Big)=\big(x\triangleright(e,f),(x\triangleleft|(e,f)|)\triangleright k\big)=x\triangleright\big((e,f),k\big)\end{aligned}$$

as needed.

The verification of the remaining axioms offers no difficulty. □

2.2. Some distinguished representations.

We discuss certain canonical representations of a matched pair $(\mathcal{V},\mathcal{H})$ which play an important role in our proofs.

First, $\mathcal{V}$ becomes a representation of $(\mathcal{V},\mathcal{H})$ by means of the maps $t,b:\mathcal{V}\to\mathcal{P}$ (for p and q respectively), the identity grading $|g|=g$, and the action $\mathcal{H}\,{}_r\!\times_t\mathcal{V}\xrightarrow{\triangleright}\mathcal{V}$ from the definition of matched pair. Conditions (2.1) and (2.2) are trivially satisfied. This is in fact the final object in the category $\mathrm{Rep}(\mathcal{V},\mathcal{H})$, as one may easily check:

Lemma 2.6. *Given any representation $\mathcal{E}\to\mathcal{P}$ of $(\mathcal{V},\mathcal{H})$, there exists a unique morphism of representations $\psi:\mathcal{E}\to\mathcal{V}$. It is $\psi(e)=|e|$.* □

Another distinguished representation of $(\mathcal{V},\mathcal{H})$ is $\mathcal{H}\,{}_r\!\times_t\mathcal{V}$, with action and grading given by

$$x\triangleright(y,g)=(xy,g),\qquad |(x,g)|=x\triangleright g\,. \tag{2.6}$$

Condition (2.2) holds by (1.3). Condition (2.1) says that $p(x,g)=l(x)$ and $q(x,g)=l(x\triangleleft g)$.

This representation has several interesting subrepresentations. Let $\sigma:\mathcal{P}\to\mathcal{V}$ be a section of $t:\mathcal{V}\to\mathcal{P}$ and let

$$\mathcal{H}_\sigma:=\{(x,\sigma r(x)):x\in\mathcal{H}\}\subseteq\mathcal{H}\,{}_r\!\times_t\mathcal{V}\,.$$

Then $\mathcal{H}_\sigma$ is a subrepresentation of $\mathcal{H}\,{}_r\!\times_t\mathcal{V}$.

In particular, consider the section σ_0 defined by $\sigma_0(P)=\mathrm{id}_{\mathcal{V}}\,P$, $P\in\mathcal{P}$. We may identify $\mathcal{H}_{\sigma_0}$ with $\mathcal{H}$; then $p(x)=q(x)=l(x)$, the action is $x\triangleright y=xy$, and the grading is $|x|=\mathrm{id}_{\mathcal{V}}\,l(x)$, by (1.15).

It is natural to wonder if there is a relation between the representations $\mathcal{V}$, $\mathcal{H}$, and $\mathcal{H}\,{}_r\!\times_t\mathcal{V}$ just defined. One readily sees that $\mathcal{H}\,{}_r\!\times_t\mathcal{V}$ is *not* the tensor product of the representations $\mathcal{H}$ and $\mathcal{V}$. However, one has the following result.

Lemma 2.7. *The map $\psi:\mathcal{H}\,{}_r\!\times_t\mathcal{V}\to\mathcal{V}\otimes\mathcal{H}=\mathcal{V}\,{}_b\!\times_l\mathcal{H}$ given by*

$$\psi(x,g)=(x\triangleright g,x\triangleleft g) \tag{2.7}$$

is an isomorphism of representations.

PROOF. From (1.14) we see that ψ is well-defined and preserves maps to $\mathcal{P}$:

$$p(x,g) = l(x) = t(x \triangleright g) = p(x \triangleright g, x \triangleleft g)\,.$$

The map $\varphi(g,x) = \big((x^{-1} \triangleleft g^{-1})^{-1}, (x^{-1} \triangleright g^{-1})^{-1}\big)$ is the inverse of ψ, by (1.19) and (1.20), so ψ is a bijection.

We check that ψ preserves actions. If $x, y \in \mathcal{H}$ and $g \in \mathcal{V}$, with $r(x) = l(y)$, $r(y) = t(g)$, then

$$\begin{aligned} x \triangleright \psi(y,g) = x \triangleright (y \triangleright g, y \triangleleft g) &\overset{(2.4)}{=} (x \triangleright (y \triangleright g), (x \triangleleft (y \triangleright g))(y \triangleleft g)) \\ &= (xy \triangleright g, xy \triangleleft g) = \psi(xy, g) \overset{(2.6)}{=} \psi(x \triangleright (y,g)). \end{aligned}$$

We check that ψ preserves gradings. If $x \in \mathcal{H}$ and $g \in \mathcal{V}$, with $r(x) = t(g)$, then

$$|\psi(x,g)| = |(x \triangleright g, x \triangleleft g)| \overset{(2.5)}{=} (x \triangleright g)\, \mathrm{id}_{\mathcal{V}}\, l(x \triangleleft g) = (x \triangleright g) \overset{(2.6)}{=} |(x,g)|\,.$$

$\square$

There are additional important subrepresentations of $\mathcal{H}\,{}_r{\times}_t\,\mathcal{V}$. Recall that $\mathcal{H}_P = \{x \in \mathcal{H} : r(x) = P\}$, for $P \in \mathcal{P}$. If $g \in \mathcal{V}$ is such that $t(g) = P$, then $\mathcal{H}_P \times \{g\}$ is a subrepresentation of $\mathcal{H}\,{}_r{\times}_t\,\mathcal{V}$. This representation satisfies the following universal property.

LEMMA 2.8. *Let $\mathcal{E}$ be a representation of $(\mathcal{V}, \mathcal{H})$. Let $e \in \mathcal{E}$, $g = |e|$, and $P = t(g)$. There exists a unique morphism of representations $\rho_e : \mathcal{H}_P \times \{g\} \to \mathcal{E}$ such that*

$$\rho_e(\mathrm{id}\, P, g) = e. \tag{2.8}$$

We refer to the morphism ρ_e as the *expansion* of e.

PROOF. Define $\rho_e : \mathcal{H}_P \times \{g\} \to \mathcal{E}$ by $\rho_e(x,g) = x \triangleright e$. This clearly satisfies (2.8), and preserves action and grading. Conversely, assume that $\rho : \mathcal{H}_P \times \{g\} \to \mathcal{E}$ is a morphism satisfying (2.8). Then $\rho(x,g) = \rho(x\, \mathrm{id}\, P, g) = x \triangleright \rho(\mathrm{id}\, P, g) = x \triangleright e = \rho_e(x,g)$, and uniqueness follows. $\square$

Finally, we mention that $\mathcal{H}_P = \mathcal{H}_P \times \{\mathrm{id}\, P\}$ is a subrepresentation of $\mathcal{H} = \mathcal{H}_{\sigma_0}$, and for any $y \in \mathcal{H}(P,Q)$, the right multiplication map $R_y : \mathcal{H}_P \to \mathcal{H}_Q$, $R_y(x) = xy$, is an isomorphism of representations, with inverse $R_{y^{-1}}$.

2.3. Restriction of representations.
Let $(\alpha, \beta) : (\mathbb{V}, \mathbb{H}) \to (\mathcal{V}, \mathcal{H})$ be a morphism of matched pairs of groupoids with base $\mathcal{P}$. We construct a functor

$$\mathrm{Res}_\alpha^\beta : \mathrm{Rep}(\mathcal{V}, \mathcal{H}) \to \mathrm{Rep}(\mathbb{V}, \mathbb{H})$$

called the *restriction along* (α, β). Let $\mathcal{E}$ be a representation of $(\mathcal{V}, \mathcal{H})$ with structure maps $p, q : \mathcal{E} \to \mathcal{P}$, action $\triangleright$ and grading $|\ |$. Consider the action of $\mathbb{H}$ on the map $p : \mathcal{E} \to \mathcal{P}$ given by

$$h \triangleright e := \alpha(h) \triangleright e\,, \ \text{ for all } (h,e) \in \mathbb{H}\,{}_r{\times}_p\,\mathcal{E}, \tag{2.9}$$

and the grading $\| \ \| : \mathcal{E} \to \mathbb{V}$ given by

$$\|e\| := \beta(|e|)\,. \tag{2.10}$$

We let $\operatorname{Res}_{\alpha}^{\beta}(\mathcal{E})$ denote the set $\mathcal{E}$, equipped with the same maps $p, q : \mathcal{E} \to \mathcal{P}$ and these new action and grading.

Given a morphism of representations $\psi : \mathcal{E} \to \mathcal{F}$, we let $\operatorname{Res}_{\alpha}^{\beta}(\psi) = \psi$.

Proposition 2.9. *Let $(\alpha, \beta) : (\mathbb{V}, \mathbb{H}) \to (\mathcal{V}, \mathcal{H})$ be a morphism of matched pairs of groupoids.*

(a) *For any representation $\mathcal{E}$ of $(\mathcal{V}, \mathcal{H})$, $\operatorname{Res}_{\alpha}^{\beta}(\mathcal{E})$ is a representation of $(\mathbb{V}, \mathbb{H})$.*
(b) $\operatorname{Res}_{\alpha}^{\beta} : \operatorname{Rep}(\mathcal{V}, \mathcal{H}) \to \operatorname{Rep}(\mathbb{V}, \mathbb{H})$ *is a monoidal functor and the diagram*

$$\begin{tikzcd} \operatorname{Rep}(\mathcal{V}, \mathcal{H}) \arrow[rr, "\operatorname{Res}_{\alpha}^{\beta}"] \arrow[dr, "f\ell"'] & & \operatorname{Rep}(\mathbb{V}, \mathbb{H}) \arrow[dl, "f\ell"] \\ & \operatorname{Quiv}(\mathcal{P}) & \end{tikzcd}$$

commutes.
(c) *If $(\chi, \omega) : (\mathcal{V}, \mathcal{H}) \to (\mathfrak{V}, \mathfrak{H})$ is another morphism of matched pairs then diagram*

$$\begin{tikzcd} \operatorname{Rep}(\mathfrak{V}, \mathfrak{H}) \arrow[rr, "\operatorname{Res}_{\chi\circ\alpha}^{\beta\circ\omega}"] \arrow[dr, "\operatorname{Res}_{\chi}^{\omega}"'] & & \operatorname{Rep}(\mathbb{V}, \mathbb{H}) \\ & \operatorname{Rep}(\mathcal{V}, \mathcal{H}) \arrow[ur, "\operatorname{Res}_{\alpha}^{\beta}"'] & \end{tikzcd}$$

commutes.

Proof. (a) We first check that the new action and grading satisfy (2.1) and (2.2). Given $e \in \mathcal{E}$ we have

$$p(e) = t(|e|) = t\big(\beta(|e|)\big) = t(\|e\|)\,,$$

by (2.1) for $\mathcal{E}$ as a representation of $(\mathcal{V}, \mathcal{H})$. Similarly, $q(e) = b(\|e\|)$. If $(h, e) \in \mathbb{H}\,{}_r\!\times_p \mathcal{E}$ then

$$\|h \triangleright e\| = \|\alpha(h) \triangleright e\| = \beta(|\alpha(h) \triangleright e|) = \beta(\alpha(h) \triangleright |e|) \overset{(1.32)}{=} h \triangleright \beta(|e|) = h \triangleright \|e\|\,;$$

by (2.2) for $\mathcal{E}$ as a representation of $(\mathcal{V}, \mathcal{H})$. Thus, $\operatorname{Res}_{\alpha}^{\beta}$ is a well-defined functor, the rest of the verifications being straightfoward.

We next check that $\operatorname{Res}_{\alpha}^{\beta}$ is monoidal. Since the functor does not change the maps p and q, $\operatorname{Res}_{\alpha}^{\beta}(\mathcal{E} \otimes \mathcal{F}) = \operatorname{Res}_{\alpha}^{\beta}(\mathcal{E}) \otimes \operatorname{Res}_{\alpha}^{\beta}(\mathcal{F})$ as sets. Let $e \in \mathcal{E}$, $f \in \mathcal{F}$. The degree of (e, f) in $\operatorname{Res}_{\alpha}^{\beta}(\mathcal{E} \otimes \mathcal{F})$ is

$$\|(e, f)\| = \beta(|(e, f)|) = \beta(|e||f|) = \beta(|e|)\beta(|f|) = \|e\|\|f\|,$$

which agrees with the degree of (e, f) in $\operatorname{Res}_\alpha^\beta(\mathcal{E}) \otimes \operatorname{Res}_\alpha^\beta(\mathcal{F})$. Let $h \in \mathbb{H}$. The action of h on (e, f) in $\operatorname{Res}_\alpha^\beta(\mathcal{E} \otimes \mathcal{F})$ is

$$h \triangleright (e, f) = \alpha(h) \triangleright (e, f) = \Big(\alpha(h) \triangleright e, \big(\alpha(h) \triangleleft |e|\big) \triangleright f\Big)$$
$$\overset{(1.33)}{=} \Big(\alpha(h) \triangleright e, \alpha\big(h \triangleleft \beta(|e|)\big) \triangleright f\Big) = \Big(\alpha(h) \triangleright e, \alpha\big(h \triangleleft \|e\|\big) \triangleright f\Big),$$

which agrees with the action of h on (e, f) in $\operatorname{Res}_\alpha^\beta(\mathcal{E}) \otimes \operatorname{Res}_\alpha^\beta(\mathcal{F})$.

The proof of (c) is left to the reader. □

THEOREM 2.10. *Let $F : \operatorname{Rep}(\mathcal{V}, \mathcal{H}) \to \operatorname{Rep}(\mathbb{V}, \mathbb{H})$ be a monoidal functor such that diagram*

$$\begin{tikzcd} \operatorname{Rep}(\mathcal{V}, \mathcal{H}) \arrow[rr, "F"] \arrow[dr, "f\ell"'] & & \operatorname{Rep}(\mathbb{V}, \mathbb{H}) \arrow[dl, "f\ell"] \\ & \operatorname{Quiv}(\mathcal{P}) & \end{tikzcd}$$

commutes. There exists a unique morphism of matched pairs of groupoids $(\alpha, \beta) : (\mathbb{V}, \mathbb{H}) \to (\mathcal{V}, \mathcal{H})$ such that

$$F = \operatorname{Res}_\alpha^\beta .$$

PROOF. We make use of the representation $\mathcal{V}$ of $(\mathcal{V}, \mathcal{H})$ of Lemma 2.6. Since $F(\mathcal{V})$ is an object in $\operatorname{Rep}(\mathbb{V}, \mathbb{H})$ with underlying set $\mathcal{V}$, we may define a map $\beta : \mathcal{V} \to \mathbb{V}$ by

$$\beta(v) = \|v\| .$$

Let $\mathcal{E}$ be an arbitrary representation of $(\mathcal{V}, \mathcal{H})$. By Lemma 2.6, the map $\psi : \mathcal{E} \to \mathcal{V}$ defined by $\psi(e) = |e|$ is a morphism in $\operatorname{Rep}(\mathcal{V}, \mathcal{H})$. Since F preserves forgetful functors, the same map ψ is a morphism $F(\mathcal{E}) \to F(\mathcal{V})$ in $\operatorname{Rep}(\mathbb{V}, \mathbb{H})$; in particular, it must preserve the gradings. We deduce that

$$\beta(|e|) = \beta(\psi(e)) = \|\psi(e)\| = \|e\| .$$

We have obtained condition (2.10) in the definition of restriction functor.

From $F(\mathcal{V} \otimes \mathcal{V}) = F(\mathcal{V}) \otimes F(\mathcal{V})$ we deduce that β is a morphism of groupoids.

Let $h \in \mathbb{H}$, $P = r(h)$, and $g \in \mathcal{V}$ be such that $t(g) = P$. Consider the representation $\mathcal{H}_P \times \{g\}$ of $(\mathcal{V}, \mathcal{H})$. Using the action of $\mathbb{H}$ on $F(\mathcal{H}_P \times \{g\})$, we define an element $\alpha_g(h) \in \mathcal{H}_P$ by

$$h \triangleright (\operatorname{id}_{\mathcal{H}} P, g) = (\alpha_g(h), g).$$

Let $\mathcal{E}$ be a representation of $(\mathcal{V}, \mathcal{H})$. Let $e \in \mathcal{E}$ be such that $g = |e|$. We claim that the action of $\mathbb{H}$ on $F(\mathcal{E})$ is given by $h \triangleright e = \alpha_g(h) \triangleright e$. To see this, consider the expansion ρ_e of e, that is the map $\rho_e : \mathcal{H}_P \times \{g\} \to \mathcal{E}$, $\rho_e(x) = x \triangleright e$. By Lemma 2.8, ρ_e is a morphism in $\operatorname{Rep}(\mathcal{V}, \mathcal{H})$. Since F preserves forgetful functors, ρ_e is also a morphism in $\operatorname{Rep}(\mathbb{V}, \mathbb{H})$; in particular, it preserves the actions of $\mathbb{H}$. Therefore,

$$(*) \qquad h \triangleright e = h \triangleright \big(\rho_e(\operatorname{id} P, g)\big) = \rho_e\big(h \triangleright (\operatorname{id} P, g)\big) = \rho_e(\alpha_g(h), g) = \alpha_g(h) \triangleright e .$$

We next show that $\alpha_g(h)$ is independent of g. Let $f \in \mathcal{V}$ be another element such that $t(f) = P$. Let $X = \mathcal{H}_P \times \{f\}$ and $Y = \mathcal{H}_P \times \{g\}$. We compute the action of $\mathbb{H}$ on $F(X \otimes Y) = F(X) \otimes F(Y)$ in two different ways. Let $\big((x, f), (y, g)\big) \in X \otimes Y$. According to (2.5) and (2.6), the degree of this element is

$$|((x, f), (y, g))| = |(x, f)||(y, g)| = (x \triangleright f)(y \triangleright g)\,.$$

The top of $(x \triangleright f)(y \triangleright g) \in \mathcal{V}$ is, by definition of $\mathcal{H}_P$, $t(f) = P$. Therefore, we may apply $(*)$ to this element and conclude that the action of h on $F(X \otimes Y)$ is

$$\begin{aligned} h \triangleright \big((x, f), (y, g)\big) &= \alpha_{(x \triangleright f)(y \triangleright g)}(h) \triangleright \big((x, f), (y, g)\big) \\ &= \Big(\big(\alpha_{(x \triangleright f)(y \triangleright g)}(h)x, f\big), \big(\big(\alpha_{(x \triangleright f)(y \triangleright g)}(h) \triangleleft (x \triangleright f)\big)y, g\big)\Big)\,. \end{aligned}$$

On the other hand, the action of h on $F(X) \otimes F(Y)$ is

$$\begin{aligned} h \triangleright \big((x, f), (y, g)\big) &= \Big(h \triangleright (x, f), \big(h \triangleleft \|(x, f)\|\big) \triangleright (y, g)\Big) \\ &= \Big(\big(\alpha_{x \triangleright f}(h)x, f\big), \big(\alpha_{y \triangleright g}\big(h \triangleleft \beta(x \triangleright f)\big)y, g\big)\Big)\,. \end{aligned}$$

Comparing these two expressions we obtain

$$\alpha_{(x \triangleright f)(y \triangleright g)}(h) = \alpha_{x \triangleright f}(h)\,, \tag{2.11}$$

$$\alpha_{(x \triangleright f)(y \triangleright g)}(h) \triangleleft (x \triangleright f) = \alpha_{y \triangleright g}\big(h \triangleleft \beta(x \triangleright f)\big)\,. \tag{2.12}$$

Choose $x = y = \mathrm{id}_{\mathcal{H}} P \in \mathcal{H}_P$ and $f = \mathrm{id}_{\mathcal{V}} P$. Since $q((x, f)) = P = p((y, g))$, we have $\big((x, f), (y, g)\big) \in X \otimes Y$. Hence, (2.11) applies and we obtain that $\alpha_g(h) = \alpha_{\mathrm{id}_{\mathcal{V}} P}(h)$. Thus, we may define a map $\alpha : \mathbb{H} \to \mathcal{H}$ by $\alpha(h) = \alpha_g(h)$ where g is any element of $\mathcal{V}$ such that $t(g) = r(h)$. Equation $(*)$ becomes condition (2.9) in the definition of restriction functor.

It follows easily now that α is a morphism of groupoids: if $h, k \in \mathbb{H}$, $r(k) = l(h)$ then $(*)$ implies

$$\begin{aligned} (\alpha(kh), \mathrm{id}_{\mathcal{V}} P) = kh \triangleright (\mathrm{id}_{\mathcal{H}}, \mathrm{id}_{\mathcal{V}} P) &= k \triangleright (h \triangleright (\mathrm{id}_{\mathcal{H}}, \mathrm{id}_{\mathcal{V}} P)) \\ &= k \triangleright (\alpha(h), \mathrm{id}_{\mathcal{V}} P) = (\alpha(k)\alpha(h), \mathrm{id}_{\mathcal{V}} P)\,. \end{aligned}$$

Also, (2.12) yields $\alpha(h) \triangleleft (x \triangleright f) = \alpha\big(h \triangleleft \beta(x \triangleright f)\big)$. Choosing $x = \mathrm{id}_{\mathcal{H}}(t(f))$ we deduce (1.33) in the definition of morphism of matched pairs of groupoids. To complete the proof, it only remains to check that (1.32) holds. Let $h \in \mathbb{H}$, $g \in \mathcal{V}$. Then

$$\beta(\alpha(h) \triangleright g) = \|\alpha(h) \triangleright g\| = \|h \triangleright g\| = h \triangleright \|g\| = h \triangleright \beta(g)\,.$$

Thus, $(\alpha, \beta) : (\mathbb{V}, \mathbb{H}) \to (\mathcal{V}, \mathcal{H})$ is a morphism of matched pairs and $\mathrm{Res}_\alpha^\beta = F$.

Uniqueness can be traced back through the above argument. □

Example 2.11. Consider the forgetful functor $f\ell : \mathrm{Rep}(\mathcal{V}, \mathcal{H}) \to \mathrm{Quiv}(\mathcal{P})$. As discussed in Example 2.2, $\mathrm{Quiv}(\mathcal{P}) = \mathrm{Rep}(\mathcal{P} \times \mathcal{P}, \mathcal{P})$. According to Theorem 2.10, $f\ell$ must be the restriction functor along a morphism of matched pairs

$(\mathcal{P} \times \mathcal{P}, \mathcal{P}) \to (\mathcal{V}, \mathcal{H})$. In fact, one has $f\ell = \mathrm{Res}_{\alpha_0}^{\beta_0}$ where (α_0, β_0) is the morphism of Proposition 1.12.

Dually, consider the unique morphism of matched pairs from $(\mathcal{V}, \mathcal{H})$ to $(\mathcal{P}, \mathcal{P} \times \mathcal{P})$. The corresponding restriction functor allows to view any representation of $(\mathcal{P}, \mathcal{P} \times \mathcal{P})$ (see Example 2.2) as a representation of $(\mathcal{V}, \mathcal{H})$. Such representations are called *trivial*. None of the representations of Section 2.2 are trivial.

3. Centers and centralizers

3.1. Generalities on centralizers, centers, and braidings.

We review the connection between centers and braidings for monoidal categories, and discuss the related notion of centralizers. We restrict our considerations to the case of strict monoidal functors between strict monoidal categories, as this is the case of present interest. For more information, see [**6**] or [**8**, XIII.4].

Let $F : \mathcal{C} \to \mathcal{D}$ be a (strict) monoidal functor between (strict) monoidal categories. There is a (strict) monoidal category $Z(F)$, the *centralizer* of F, defined as follows.

- The objects of $Z(F)$ are pairs (D, ζ), where D is an object of $\mathcal{D}$ and $\zeta_{(-)} : F(-) \otimes D \to D \otimes F(-)$ is a natural isomorphism such that the following diagrams commute:

(3.1)
$$\begin{tikzcd}
F(X \otimes Y) \otimes D \arrow[rr, "\zeta_{X\otimes Y}"] \arrow[d, equal] & & D \otimes F(X \otimes Y) \arrow[d, equal] \\
F(X) \otimes F(Y) \otimes D \arrow[dr, "\mathrm{id} \otimes \zeta_Y"'] & & D \otimes F(X) \otimes F(Y) \\
& F(X) \otimes D \otimes F(Y) \arrow[ur, "\zeta_X \otimes \mathrm{id}"'] &
\end{tikzcd}$$

(3.2)
$$\begin{tikzcd}
F(\mathbf{1}) \otimes D \arrow[rr, "\zeta_{\mathbf{1}}"] \arrow[d, equal] & & D \otimes F(\mathbf{1}) \arrow[d, equal] \\
\mathbf{1} \otimes D \arrow[r, equal] & D \arrow[r, equal] & D \otimes \mathbf{1}
\end{tikzcd}$$

- A morphism $f : (D, \zeta) \to (E, \vartheta)$ in $Z(F)$ is a map $f : D \to E$ in $\mathcal{D}$ such that for all objects X of $\mathcal{C}$ the following diagram commutes

$$\begin{tikzcd}
F(X) \otimes D \arrow[r, "\zeta_X"] \arrow[d, "\mathrm{id} \otimes f"'] & D \otimes F(X) \arrow[d, "f \otimes \mathrm{id}"] \\
F(X) \otimes E \arrow[r, "\vartheta_X"'] & E \otimes F(X)
\end{tikzcd}$$

- The tensor product of two objects (D, ζ) and (E, ϑ) in $Z(F)$ is the object $(D \otimes E, \zeta \otimes \vartheta)$ where $(\zeta \otimes \vartheta)_{(-)} : D \otimes E \otimes F(-) \to F(-) \otimes D \otimes E$ is defined for each object X of $\mathcal{C}$ by the diagram

(3.3)
$$\begin{tikzcd}
D \otimes E \otimes F(X) \arrow[rr, dashed, "(\zeta\otimes\vartheta)_X"] \arrow[dr, "\mathrm{id} \otimes \vartheta_X"'] & & F(X) \otimes D \otimes E \\
& D \otimes F(X) \otimes E \arrow[ur, "\zeta_X \otimes \mathrm{id}"'] &
\end{tikzcd}$$

- The tensor product of morphisms is the tensor product of morphisms in $\mathcal{D}$.
- The unit object of $Z(F)$ is $(\mathbf{1}, \mathrm{id})$ where $\mathrm{id}_{(-)} : \mathbf{1} \times F(-) \to F(-) \times \mathbf{1}$ is the identity natural transformation.

There is a strict monoidal functor $U : Z(F) \to \mathcal{D}$ given by $U(D, \zeta) = D$.

The *center* of $\mathcal{C}$ is the centralizer of the identity functor $\mathcal{C} \to \mathcal{C}$. It is denoted $Z(\mathcal{C})$. It is a *braided* category with braiding

$$c_{(E,\vartheta),(D,\zeta)} : (E, \vartheta) \otimes (D, \zeta) \to (D, \zeta) \otimes (E, \vartheta)\,, \qquad c_{(E,\vartheta),(D,\zeta)} = \zeta_E\,. \tag{3.4}$$

Let $U : Z(\mathcal{C}) \to \mathcal{C}$ be as before. Braidings in $\mathcal{C}$ are in one-to-one correspondance with monoidal sections of U. More precisely, a family of maps $c_{E,D} : E \otimes D \to D \otimes E$ is a braiding in the monoidal category $\mathcal{C}$ if and only if $D \mapsto (D, c_{-,D})$ defines a monoidal functor $\mathcal{C} \to Z(\mathcal{C})$.

3.2. Centralizers of restriction functors.

Let $(\alpha, \beta) : (\mathbb{V}, \mathbb{H}) \to (\mathcal{V}, \mathcal{H})$ be a morphism of matched pairs of groupoids. Consider the restriction functor $\mathrm{Res}_\alpha^\beta : \mathrm{Rep}(\mathcal{V}, \mathcal{H}) \to \mathrm{Rep}(\mathbb{V}, \mathbb{H})$ (Section 2.3). We show that the centralizer of this monoidal functor is the category of representations of a matched pair; namely, the generalized double of (α, β) (Definition 1.16).

Theorem 3.1. *Let $(\alpha, \beta) : (\mathbb{V}, \mathbb{H}) \to (\mathcal{V}, \mathcal{H})$ be a morphism of matched pairs of groupoids with base $\mathcal{P}$. There is a strict monoidal equivalence*

$$\mathrm{Rep}\, D(\alpha, \beta) \cong Z(\mathrm{Res}_\alpha^\beta)\,,$$

and the following diagram is commutative

$$\begin{tikzcd} \mathrm{Rep}\, D(\alpha,\beta) \arrow[r, "\cong"] \arrow[dr, "f\ell"'] & Z(\mathrm{Res}_\alpha^\beta) \arrow[r, "U"] & \mathrm{Rep}(\mathbb{V}, \mathbb{H}) \arrow[dl, "f\ell"] \\ & \mathrm{Quiv}(\mathcal{P}) & \end{tikzcd}$$

We approach the proof of this theorem in several steps.

Let (D, ζ) be an object in $Z(\mathrm{Res}_\alpha^\beta)$. In particular, D is a representation of $(\mathbb{V}, \mathbb{H})$ and a quiver over $\mathcal{P}$. Let $p, q : D \to \mathcal{P}$ be the quiver structure maps and $\|d\| \in \mathbb{V}$ denote the degree of an element $d \in D$.

For any object $\mathcal{E}$ in $\mathrm{Rep}(\mathcal{V}, \mathcal{H})$, we have the natural isomorphism $\zeta_{\mathcal{E}} : \mathrm{Res}_\alpha^\beta(\mathcal{E}) \otimes D \to D \otimes \mathrm{Res}_\alpha^\beta(\mathcal{E})$.

For any sets X and Y, we let pr_1 and pr_2 denote the projections from $X \times Y$ onto X and Y, respectively.

Claim 1. There is a map $\rightharpoonup : \mathcal{V}\, {}_b\!\times_p D \to D$ such that

$$\mathrm{pr}_1\, \zeta_{\mathcal{E}}(e, d) = |e| \rightharpoonup d\,, \tag{3.5}$$

for all $e \in \mathcal{E}$ and $d \in D$ with $b(|e|) = p(d)$. Moreover, $\rightharpoonup$ is a left action of $\mathcal{V}$ on $p : D \to \mathcal{P}$.

Proof. Consider the object $\mathcal{V}$ of $\operatorname{Rep}(\mathcal{V},\mathcal{H})$ of Lemma 2.6. Define

$$g \rightharpoonup d := \operatorname{pr}_1 \zeta_{\mathcal{V}}(g,d) \in D\,,$$

for $g \in \mathcal{V}$ and $d \in D$ with $b(g) = p(d)$. Applying the naturality condition of ζ to the unique morphism of representations $\mathcal{E} \to \mathcal{V}$ (the degree) we deduce (3.5).

We verify the conditions for $\rightharpoonup\colon \mathcal{V}\,{}_b\!\times_p D \to D$ to be a left action of $\mathcal{V}$ on $p : D \to \mathcal{P}$: (1.2) holds since $\zeta_{\mathcal{V}}$ commutes with maps to $\mathcal{P}$ (being a map of quivers over $\mathcal{P}$), (1.3) follows from (3.1) applied to $X = Y = \mathcal{V}$, and (1.4) follows from (3.2). $\square$

Claim 2. There is a map $[\ ] : D \to \mathcal{H}$ such that

$$\zeta_{\mathcal{E}}(e,d) = \big(|e| \rightharpoonup d, ([d] \triangleleft |e|^{-1}) \triangleright e\big)\,, \tag{3.6}$$

for all $e \in \mathcal{E}$ and $d \in D$ with $b(|e|) = p(d)$. Moreover, the map $d \mapsto (\|d\|, [d])$ maps D to $\mathbb{V} \boxtimes \mathcal{H}^{op}$.

Proof. Consider the object $\mathcal{H}$ of $\operatorname{Rep}(\mathcal{V},\mathcal{H})$ of Section 2.2. Define, for $d \in D$,

$$[d] := \operatorname{pr}_2 \zeta_{\mathcal{H}}\big(\mathrm{id}_{\mathcal{H}}\, p(d), d\big) \in \mathcal{H}\,.$$

Since $\zeta_{\mathcal{H}}\big(\mathrm{id}_{\mathcal{H}}\, p(d), d\big) \in D \otimes \operatorname{Res}_{\alpha}^{\beta}(\mathcal{H}) = D\,{}_q\!\times_l \mathcal{H}$, we have

$$\begin{aligned} l([d]) = q\Big(\operatorname{pr}_1 \zeta_{\mathcal{H}}\big(\mathrm{id}_{\mathcal{H}}\, p(d), d\big)\Big) &\overset{(3.5)}{=} q\big(|\mathrm{id}_{\mathcal{H}}\, p(d)| \rightharpoonup d\big) \\ &= q\big(\mathrm{id}_{\mathcal{V}}\, p(d) \rightharpoonup d\big) \overset{(1.4)}{=} q(d) \overset{(2.1)}{=} b(\|d\|)\,. \end{aligned}$$

(we used (2.1) for D as a representation of $(\mathbb{V},\mathbb{H})$).

Let $P := p(d) \in \mathcal{P}$. Consider the subobject $\mathcal{H}_P$ of $\mathcal{H}$ in $\operatorname{Rep}(\mathcal{V},\mathcal{H})$ (Section 2.2). By naturality of ζ,

$$[d] = \operatorname{pr}_2 \zeta_{\mathcal{H}_P}\big(\mathrm{id}_{\mathcal{H}}\, P, d\big) \in \mathcal{H}_P\,.$$

Hence, by definition of $\mathcal{H}_P$,

$$r([d]) = P = p(d) \overset{(2.1)}{=} t(\|d\|)\,.$$

Thus $(\|d\|, [d])$ belongs to $\mathbb{V} \boxtimes \mathcal{H}^{op} = (\mathbb{V}\,{}_b\!\times_l \mathcal{H}) \cap (\mathbb{V}\,{}_t\!\times_r \mathcal{H})$.

Let e be as above, $g = |e| \in \mathcal{V}$, and $Q = t(g)$. Consider the object $\mathcal{H}_Q \times \{g\}$ in $\operatorname{Rep}(\mathcal{V},\mathcal{H})$ (Section 2.2). Since $|(\mathrm{id}_{\mathcal{H}}\, Q, g)| = g$, (3.5) says that

$$\zeta_{\mathcal{H}_Q \times \{g\}}(\mathrm{id}_{\mathcal{H}}\, Q, g, d) = (g \rightharpoonup d, [d]_g, g)$$

for some element $[d]_g \in \mathcal{H}_Q$.

Let $\rho_e : \mathcal{H}_Q \times \{g\} \to \mathcal{E}$ be the expansion of e (Lemma 2.8). Applying the naturality of ζ to the morphism ρ_e we deduce that

$$\zeta_{\mathcal{E}}(e,d) = (g \rightharpoonup d, [d]_g \triangleright e)\,. \tag{$*$}$$

Consider now the isomorphism ψ of Lemma 2.7. Combining the naturality of ζ with (3.1) we obtain the commutativity of

$$\begin{tikzcd}[column sep=small]
\mathrm{Res}^{\beta}_{\alpha}(\mathcal{H}\,{}_r\!\times_t \mathcal{V}) \otimes D \arrow[rr, "\zeta_{\mathcal{H}\,{}_r\times_t\,\mathcal{V}}"] \arrow[d, "\mathrm{Res}^{\beta}_{\alpha}(\psi)\otimes \mathrm{id}"'] & & D \otimes \mathrm{Res}^{\beta}_{\alpha}(\mathcal{H}\,{}_r\!\times_t \mathcal{V}) \arrow[d, "\mathrm{id}\otimes \mathrm{Res}^{\beta}_{\alpha}(\psi)"] \\
\mathrm{Res}^{\beta}_{\alpha}(\mathcal{V}) \otimes \mathrm{Res}^{\beta}_{\alpha}(\mathcal{H}) \otimes D \arrow[dr, "\mathrm{id}\otimes\zeta_{\mathcal{H}}"'] & & D \otimes \mathrm{Res}^{\beta}_{\alpha}(\mathcal{V}) \otimes \mathrm{Res}^{\beta}_{\alpha}(\mathcal{H}) \\
& \mathrm{Res}^{\beta}_{\alpha}(\mathcal{V}) \otimes D \otimes \mathrm{Res}^{\beta}_{\alpha}(\mathcal{H}) \arrow[ur, "\zeta_{\mathcal{V}}\otimes\mathrm{id}"'] &
\end{tikzcd}$$

On one hand,

$$\begin{aligned}
&\Big(\mathrm{id}\otimes \mathrm{Res}^{\beta}_{\alpha}(\psi)\Big)\,\zeta_{\mathcal{H}\,{}_r\times_t\,\mathcal{V}}(\mathrm{id}_{\mathcal{H}}\,Q, g, d)\\
&\qquad\overset{(3.6)}{=}\Big(\mathrm{id}\otimes \mathrm{Res}^{\beta}_{\alpha}(\psi)\Big)\,(g \rightharpoonup d, [d]_g, g) \overset{(2.7)}{=} (g \rightharpoonup d, [d]_g \triangleright g, [d]_g \triangleleft g)\,.
\end{aligned}$$

On the other, by (2.7) and (1.16),

$$\begin{aligned}
(\zeta_{\mathcal{V}}\otimes \mathrm{id})(\mathrm{id}\otimes\zeta_{\mathcal{H}})(\mathrm{Res}^{\beta}_{\alpha}(\psi)\otimes\mathrm{id})(\mathrm{id}_{\mathcal{H}}\,Q, g, d) &= (\zeta_{\mathcal{V}}\otimes \mathrm{id})(\mathrm{id}\otimes\zeta_{\mathcal{H}})(g, \mathrm{id}_{\mathcal{H}}\,P, d)\\
&= (\zeta_{\mathcal{V}}\otimes \mathrm{id})(g, d, [d]) \overset{(*)}{=} (g \rightharpoonup d, [d]_g \triangleright g, [d]).
\end{aligned}$$

Hence $[d] = [d]_g \triangleleft g$, or $[d]_g = [d] \triangleleft g^{-1}$. Together with $(*)$, this proves (3.6). $\square$

Claim 3. The action $\rightharpoonup$ of $\mathcal{V}$ on D and the action of $\mathbb{H}$ on D extend to an action of $\mathcal{V} \bowtie \mathbb{H}$ on D. Together with the map $(\|\ \|, [\ ])$, they turn D into a representation of the matched pair $D(\alpha, \beta) = (\mathbb{V} \boxtimes \mathcal{H}^{op}, \mathcal{V} \bowtie \mathbb{H})$.

Proof. We make use of the fact that $\zeta_{\mathcal{E}}$ preserves the action of $\mathbb{H}$. Let $h \in \mathbb{H}$. On one hand,

$$\begin{aligned}
&\zeta_{\mathcal{E}}\big(h \triangleright (e,d)\big) \overset{(2.4)}{=} \zeta_{\mathcal{E}}\big(h \triangleright e, (h \triangleleft \|e\|) \triangleright d\big) \overset{(2.9),\,(2.10)}{=} \zeta_{\mathcal{E}}\Big(\alpha(h) \triangleright e, \big(h \triangleleft \beta(|e|)\big) \triangleright d\Big)\\
&\overset{(3.6)}{=}\Bigg(|\alpha(h) \triangleright e| \rightharpoonup \Big(\big(h \triangleleft \beta(|e|)\big) \triangleright d\Big), \Big(\Big[\big(h \triangleleft \beta(|e|)\big) \triangleright d\Big] \triangleleft |\alpha(h) \triangleright e|^{-1}\Big) \triangleright \big(\alpha(h) \triangleright e\big)\Bigg)\,.
\end{aligned}$$

On the other,

$$\begin{aligned}
h \triangleright \zeta_{\mathcal{E}}(e,d) &\overset{(3.6)}{=} h \triangleright \big(|e| \rightharpoonup d, ([d] \triangleleft |e|^{-1}) \triangleright e\big)\\
&\overset{(2.4)}{=} \Big(h \triangleright (|e| \rightharpoonup d), \big(h \triangleleft \big\|\,|e| \rightharpoonup d\,\big\|\big) \triangleright \big([d] \triangleleft |e|^{-1}\big) \triangleright e\Big)\\
&\overset{(2.9)}{=} \Big(h \triangleright (|e| \rightharpoonup d), \alpha\big(h \triangleleft \big\|\,|e| \rightharpoonup d\,\big\|\big) \triangleright \big([d] \triangleleft |e|^{-1}\big) \triangleright e\Big)\,.
\end{aligned}$$

The equality between the first components of the above expressions and formula (1.35) for the product of the groupoid $\mathcal{V} \bowtie \mathbb{H}$ imply that the actions $\rightharpoonup$ and $\triangleright$ combine to give a left action of $\mathcal{V} \bowtie \mathbb{H}$ on D.

Consider the equality between the second components, in the special case when $\mathcal{E} = \mathcal{H}$ and e is an identity. Recalling the structure of $\mathcal{H}$ as a representation of $(\mathcal{V}, \mathcal{H})$ (Section 2.2), we obtain

$$[h \triangleright d]\alpha(h) = \alpha\big(h \triangleleft \|d\|\big)[d]\,. \tag{3.7}$$

We need to derive additional identities. Applying (3.1) to the product $\mathrm{Res}_\alpha^\beta(\mathcal{H}) \otimes \mathrm{Res}_\alpha^\beta(\mathcal{V}) \otimes D$ and the element $(\mathrm{id}_{\mathcal{H}}\, Q, g, d)$ we obtain

$$[g \rightharpoonup d] = [d] \triangleleft g^{-1}\,. \tag{3.8}$$

Finally, we make use of the fact that $\zeta_{\mathcal{E}}$ preserves degrees (in $\mathbb{V}$). Calculating degrees on both sides of (3.6) (and letting $g = |e|$) we obtain

$$\beta(g)\|d\| = \|g \rightharpoonup d\|\beta\big(([d] \triangleleft g^{-1}) \triangleright g\big)\,, \tag{3.9}$$

for any $d \in$ and $g \in \mathcal{V}$ with $b(g) = q(d)$.

We have already seen that D, with its given structure of quiver over $\mathcal{P}$, carries a grading over $\mathbb{V} \boxtimes \mathcal{H}^{op}$ (Claim 2) and a left action of $\mathcal{V} \bowtie \mathbb{H}$. To conclude that D is a representation of the matched pair $D(\alpha, \beta) = (\mathbb{V} \boxtimes \mathcal{H}^{op}, \mathcal{V} \bowtie \mathbb{H})$ we only need to check conditions (2.1) and (2.2). The former boils down to the corresponding condition for D as a representation of $(\mathbb{V}, \mathbb{H})$. The latter requires the verification of the following two identities (in view of (1.36)):

$$\big[g \rightharpoonup (h \triangleright d)\big] = \Big(\alpha\big(h \triangleleft \|d\|\big)[d]\alpha(h)^{-1}\Big) \triangleleft g^{-1}\,, \tag{3.10}$$

$$\|g \rightharpoonup (h \triangleright d)\| = \beta(g)\big(h \triangleleft \|d\|\big)\beta\Big(\big(\alpha(h \triangleleft \|d\|)[d]\alpha(h)^{-1}\big) \triangleleft g^{-1}\Big)\,. \tag{3.11}$$

Now, (3.10) is a consequence of (3.7) and (3.8), while (3.11) follows from (1.17), (3.9), (3.10), and (2.2) for D as a representation of $(\mathbb{V}, \mathbb{H})$. □

We have shown that any object D in the centralizer $Z(\mathrm{Res}_\alpha^\beta)$ can be endowed with the structure of a representation of the matched pair $D(\alpha, \beta)$. Conversely, any such representation D gives rise to an object in $Z(\mathrm{Res}_\alpha^\beta)$. First, D is a representation of $(\mathbb{V}, \mathbb{H})$ by restriction via the morphism of matched pairs $(\mathbb{V}, \mathbb{H}) \to D(\alpha, \beta)$ of Remark 1.22. Then, one defines a transformation $\zeta_{\mathcal{E}} : \mathrm{Res}_\alpha^\beta(\mathcal{E}) \otimes D \to D \otimes \mathrm{Res}_\alpha^\beta(\mathcal{E})$ by means of (3.6). Reversing some of the arguments in the above proofs, one verifies that (D, ζ) is an object in $Z(\mathrm{Res}_\alpha^\beta)$.

It is clear that these constructions yield an equivalence of categories

$$\mathrm{Rep}\, D(\alpha, \beta) \cong Z(\mathrm{Res}_\alpha^\beta)\,,$$

whose composition with the functor $U : Z(\mathrm{Res}_\alpha^\beta) \to \mathrm{Rep}(\mathbb{V}, \mathbb{H})$ preserves the forgetful functors to $\mathrm{Quiv}(\mathcal{P})$. To complete the proof of Theorem 3.1, we need to verify that the monoidal structures are preserved under this equivalence.

Claim 4. Tensor products of objects in the categories $Z(\mathrm{Res}_\alpha^\beta)$ and $\mathrm{Rep}\, D(\alpha, \beta)$ agree.

Proof. Let D and D' be objects in $\mathrm{Rep}\, D(\alpha, \beta)$. They give rise to objects (D, ζ) and (D', ζ') of $Z(\mathrm{Res}_\alpha^\beta)$, by means of the above construction. On the other hand, since $D(\alpha,\beta)$ is a matched pair, $D\otimes D'$ has a natural structure of representation of $D(\alpha,\beta)$ (Proposition 2.4), which gives rise to an object $(D\otimes D',\zeta'')$ of $Z(\mathrm{Res}^\beta_\alpha)$. According to (3.6), the transformation $\zeta''_{\mathcal{E}}:\mathrm{Res}^\beta_\alpha(\mathcal{E})\otimes D\otimes D'\to D\otimes D'\otimes\mathrm{Res}^\beta_\alpha(\mathcal{E})$ is given by

$$\begin{aligned}\zeta''_{\mathcal{E}}(e,d,d') &= \Big(|e|\rightharpoonup(d,d'),\big([d,d']\triangleleft|e|^{-1}\big)\triangleright e\Big)\\ &= \Big(|e|\rightharpoonup d,\ \big(|e|\leftharpoonup(\|d\|,[d])\big)\rightharpoonup d',\ \big([d'][d]\triangleleft|e|^{-1}\big)\triangleright e\Big)\\ &= \Big(|e|\rightharpoonup d,\ \big(([d]\triangleleft|e|^{-1})\triangleright|e|\big)\rightharpoonup d',\ \big([d'][d]\triangleleft|e|^{-1}\big)\triangleright e\Big).\end{aligned}$$

In the second equality we made use of (2.4), (2.5), and the definition of the product in $\mathbb{V}\boxtimes\mathcal{H}^{op}$; in the last equality we used (1.37) (in the special case when the element of $\mathbb{H}$ is an identity).

We need to check that $\zeta''=\zeta\otimes\zeta'$, as prescribed by the definition of tensor product of objects in $Z(\mathrm{Res}^\beta_\alpha)$ (3.3). We calculate

$$\begin{aligned}(\zeta\otimes\zeta')_{\mathcal{E}}(e,d,d') &\overset{(3.3)}{=}(\mathrm{id}\otimes\zeta'_{\mathcal{E}})(\zeta_{\mathcal{E}}\otimes\mathrm{id})(e,d,d')\\ &\overset{(3.6)}{=}(\mathrm{id}\otimes\zeta'_{\mathcal{E}})\big(|e|\rightharpoonup d,\ ([d]\triangleleft|e|^{-1})\triangleright e,\ d'\big)\\ &\overset{(3.6)}{=}\Big(|e|\rightharpoonup d,\ \big|([d]\triangleleft|e|^{-1})\triangleright e\big|\rightharpoonup d',\ \big([d']\triangleleft\big|([d]\triangleleft|e|^{-1})\triangleright e\big|^{-1}\big)\triangleright\big(([d]\triangleleft|e|^{-1})\triangleright e\big)\Big)\\ &\overset{(2.2),(1.17)}{=}\Big(|e|\rightharpoonup d,\ \big(([d]\triangleleft|e|^{-1})\triangleright|e|\big)\rightharpoonup d',\ \big([d']\triangleleft([d]\triangleright|e|^{-1})\big)\triangleright\big(([d]\triangleleft|e|^{-1})\triangleright e\big)\Big)\\ &\overset{(1.3),(1.13)}{=}\Big(|e|\rightharpoonup d,\ \big(([d]\triangleleft|e|^{-1})\triangleright|e|\big)\rightharpoonup d',\ \big([d'][d]\triangleleft|e|^{-1}\big)\triangleright e\Big).\end{aligned}$$

Thus $\zeta''=\zeta\otimes\zeta'$ and the proof is complete. □

Similarly, one may verify that the rest of the structure (unit object, morphisms) is preserved by the equivalence. This completes the proof of Theorem 3.1.

Remark 3.2. According to Theorem 3.1, the monoidal functor $\mathrm{Rep}\,D(\alpha,\beta)\to\mathrm{Rep}(\mathbb{V},\mathbb{H})$ preserves the forgetful functors to $\mathrm{Quiv}(\mathcal{P})$. Therefore, by Theorem 2.10, this functor must be the restriction along a certain morphism of matched pairs $(\mathbb{V},\mathbb{H})\to D(\alpha,\beta)$. It is easy to see that the latter is the morphism of matched pairs (ι,π) of Remark 1.22.

Corollary 3.3. *Let $(\mathcal{V},\mathcal{H})$ be a matched pair of groupoids with base $\mathcal{P}$. There is a strict monoidal equivalence*

$$\mathrm{Rep}(\mathcal{H},\mathcal{V})\cong Z(f\ell)$$

between the category of representations of the dual and the centralizer of the forgetful functor. Moreover, the following diagram is commutative

$$\begin{tikzcd} \mathrm{Rep}(\mathcal{H},\mathcal{V}) \arrow[rr, "\cong"] \arrow[dr, "f\ell"'] & & Z\big(f\ell\big) \arrow[dl, "U"] \\ & \mathrm{Quiv}(\mathcal{P}) & \end{tikzcd}$$

Proof. This is a special case of Theorem 3.1, since $f\ell = \mathrm{Res}_{\alpha_0}^{\beta_0}$ where $(\alpha_0,\beta_0) : (\mathcal{P}\times\mathcal{P},\mathcal{P}) \to (\mathcal{V},\mathcal{H})$ is the unique morphism (Example 2.11), and the dual of $(\mathcal{V},\mathcal{H})$ is $D(\alpha_0,\beta_0)$ (Remark 1.17). $\square$

Corollary 3.4. *Let $(\mathcal{V},\mathcal{H})$ be a matched pair of groupoids with base $\mathcal{P}$. There is a strict monoidal equivalence*

$$\mathrm{Rep}\, D(\mathcal{V},\mathcal{H}) \cong Z\big(\mathrm{Rep}(\mathcal{V},\mathcal{H})\big)$$

between the category of representations of the double and the center of the category of representations. Moreover, the following diagram is commutative

$$\begin{tikzcd} \mathrm{Rep}\, D(\mathcal{V},\mathcal{H}) \arrow[r, "\cong"] \arrow[dr, "f\ell"'] & Z\big(\mathrm{Rep}(\mathcal{V},\mathcal{H})\big) \arrow[r, "U"] & \mathrm{Rep}(\mathcal{V},\mathcal{H}) \arrow[dl, "f\ell"] \\ & \mathrm{Quiv}(\mathcal{P}) & \end{tikzcd}$$

Proof. This is the special case of Theorem 3.1 when $\mathcal{V} = \mathbb{V}$, $\mathcal{H} = \mathbb{H}$, and α and β are identities. $\square$

Let $(\mathcal{V},\mathcal{H})$ be a matched pair of groupoids and D a representation of the double $D(\mathcal{V},\mathcal{H}) = (\mathcal{V}\boxtimes\mathcal{H}^{op}, \mathcal{V}\bowtie\mathcal{H})$. Denote the degree of an element $d \in D$ by $|d| := (|d|_{\mathcal{V}}, |d|_{\mathcal{H}}) \in \mathcal{V}\boxtimes\mathcal{H}^{op}$ and the action of $(g,h) \in \mathcal{V}\bowtie\mathcal{H}$ on d by $(g,h)\triangleright e$. By restriction, this yields actions of $\mathcal{V}$ and $\mathcal{H}$ on D that we denote by the same symbol $\triangleright$. We make use of this notation in the following result.

Corollary 3.5. *Let $(\mathcal{V},\mathcal{H})$ be a matched pair of groupoids. The category of representations of the double is braided. If D and E are representations of $D(\mathcal{V},\mathcal{H})$, the braiding $c_{E,D} : E\otimes D \to D\otimes E$ is given by*

$$c_{E,D}(e,d) = \big(|e|_{\mathcal{V}}\triangleright d, (|d|_{\mathcal{H}}\triangleleft|e|_{\mathcal{V}}^{-1})\triangleright e\big)\,. \tag{3.12}$$

Proof. This follows from (3.4) and (3.6). $\square$

4. Braidings on the representations of a matched pair

We are now in position to classify braidings on the category of representations of a matched pair in intrinsic terms (that is, in terms of the given groupoid-theoretic structure).

4.1. Matched pairs of rotations.

Definition 4.1. A *rotation* for a matched pair $(\mathcal{V}, \mathcal{H})$ is a morphism of groupoids $\kappa : \mathcal{V} \to \mathcal{H}$ satisfying

$$y\kappa(g) = \kappa(y \triangleright g)(y \triangleleft g) \tag{4.1}$$

for all $(y, g) \in \mathcal{H}\,{}_r\!\times_t \mathcal{V}$.

We may picture a rotation κ as follows:

$$\begin{tikzcd}
 & P \arrow[r, bend left, "y"] \arrow[d, bend right, "y \triangleright g"'] & Q \arrow[r, bend left, "\kappa(g)"] \arrow[d, bend left, "g"] & S \\
P \arrow[r, bend right, "\kappa(y \triangleright g)"'] & R \arrow[r, bend right, "y \triangleleft g"'] & S &
\end{tikzcd}$$

Lemma 4.2. *Let $(\mathcal{V}, \mathcal{H})$ be a matched pair of groupoids and $\kappa : \mathcal{V} \to \mathcal{H}$ a map. Let $\hat{\kappa} : \mathcal{V} \bowtie \mathcal{H} \to \mathcal{H}$ be given by $\hat{\kappa}(g, x) = \kappa(g)x$. Then $\hat{\kappa}$ is a morphism of groupoids if and only if κ is a rotation.*

Proof. According to (1.28), $\hat{\kappa}$ is a morphism of groupoids if and only if $\kappa(f)y\kappa(g)x = \kappa(f(y \triangleright g))(y \triangleleft g)x$ for all composable pairs (f, y) and (g, x) in $\mathcal{V} \bowtie \mathcal{H}$. This implies that $\kappa : \mathcal{V} \to \mathcal{H}$ is a morphism of groupoids (letting y, x be identities) and (4.1) (letting f, x be identities). The converse is a special case of the universal property of the diagonal product (1.31). $\square$

Definition 4.3. A *matched pair of rotations* for $(\mathcal{V}, \mathcal{H})$ is a pair of rotations (ξ, η) such that

$$b\big(\eta(g) \triangleright f\big) = b\big(\xi(f)^{-1} \triangleright g^{-1}\big)\,, \tag{4.2}$$

$$\big(\eta(g) \triangleright f\big)\big(\xi(f)^{-1} \triangleright g^{-1}\big)^{-1} = gf\,. \tag{4.3}$$

for every f and g in $\mathcal{V}$ with $b(g) = t(f)$.

The situation may be illustrated as follows:

$$\begin{tikzcd}
 & P \arrow[d, bend right, "g"'] & R \arrow[l, bend right, "\xi(f)^{-1} \triangleleft g^{-1}"'] \\
P \arrow[r, bend right, "\eta(g)"] \arrow[d, bend right, "\eta(g) \triangleright f"'] & Q \arrow[u, bend right, "g^{-1}"'] \arrow[d, bend right, "f"'] & S \arrow[l, bend right, "\xi(f)^{-1}"] \arrow[u, bend right, "\xi(f)^{-1} \triangleright g^{-1}"'] \\
R \arrow[r, bend right, "\eta(g) \triangleleft f"'] & S \arrow[u, bend right, "f^{-1}"'] &
\end{tikzcd}$$

This diagram also helps visualize the identities that follow.

Lemma 4.4. *Any matched pair of rotations (ξ,η) satisfies the following conditions:*

$$\xi\big(\eta(g)\triangleright f\big)=\big(\xi(f)^{-1}\triangleleft g^{-1}\big)^{-1}\,, \tag{4.4}$$

$$\eta\big(\xi(f)^{-1}\triangleright g^{-1}\big)^{-1}=\eta(g)\triangleleft f\,, \tag{4.5}$$

$$\eta(g)\xi(f)=\big(\xi(f)^{-1}\triangleleft g^{-1}\big)^{-1}\big(\eta(g)\triangleleft f\big)\,, \tag{4.6}$$

for every f and g in $\mathcal{V}$ with $b(g)=t(f)$.

Proof. Applying the morphism ξ to (4.3) we obtain

$$\xi\big(\eta(g)\triangleright f\big)\xi\big(\xi(f)^{-1}\triangleright g^{-1}\big)^{-1}=\xi(g)\xi(f)\,.$$

From (4.1) for ξ we deduce

$$\xi\big(\xi(f)^{-1}\triangleright g^{-1}\big)=\xi(f)^{-1}\xi(g)^{-1}\big(\xi(f)^{-1}\triangleleft g^{-1}\big)^{-1}\,.$$

These two identities combine to give (4.4).

Similarly, applying the morphism η to (4.3) we obtain

$$\eta\big(\eta(g)\triangleright f\big)\eta\big(\xi(f)^{-1}\triangleright g^{-1}\big)^{-1}=\eta(g)\eta(f)\,.$$

From (4.1) for η we deduce

$$\eta\big(\eta(g)\triangleright f\big)=\eta(g)\eta(f)\big(\eta(g)\triangleleft f\big)^{-1}\,.$$

These two identities combine to give (4.5).

Finally, from (4.1) for ξ we may also deduce

$$\eta(g)\xi(f)=\xi\big(\eta(g)\triangleright f\big)\big(\eta(g)\triangleleft f\big)\,.$$

Together with (4.4) this gives (4.6). $\square$

4.2. Classification of braidings.

Braidings on the category of representations of a matched pair are parametrized by matched pairs of rotations.

Theorem 4.5. *Let $(\mathcal{V},\mathcal{H})$ be a matched pair of groupoids. There is a bijective correspondence between matched pairs of rotations for $(\mathcal{V},\mathcal{H})$ and braidings on $\operatorname{Rep}(\mathcal{V},\mathcal{H})$. Let (ξ,η) be a matched pair of rotations. For any representations D and E, the corresponding braiding $c_{E,D}:E\otimes D\to D\otimes E$ is given by*

$$c_{E,D}(e,d)=\Big(\eta(|e|)\triangleright d,\,\big(\xi(|d|)^{-1}\triangleleft|e|^{-1}\big)\triangleright e\Big)\,. \tag{4.7}$$

Proof. As recalled in Section 3.1, braidings on $\operatorname{Rep}(\mathcal{V},\mathcal{H})$ are in bijective correspondence with monoidal sections of the functor $U:Z\big(\operatorname{Rep}(\mathcal{V},\mathcal{H})\big)\to\operatorname{Rep}(\mathcal{V},\mathcal{H})$. These are in bijection with sections of

$$\operatorname{Rep}D(\mathcal{V},\mathcal{H})\xrightarrow{\cong}Z\big(\operatorname{Rep}(\mathcal{V},\mathcal{H})\big)\xrightarrow{U}\operatorname{Rep}(\mathcal{V},\mathcal{H})\,.$$

As noted in Remark 3.2, this composite is $\operatorname{Res}^{\pi}_{\iota}$, where $\pi:\mathcal{V}\boxtimes\mathcal{H}^{op}\to\mathcal{V}$ and $\iota:\mathcal{H}\to\mathcal{V}\bowtie\mathcal{H}$ are given by

$$\pi(f,y)=f\quad\text{and}\quad\iota(x)=(\mathrm{id}_{\mathcal{V}}\,l(x),x)\,.$$

Note that if a functor F preserves forgetful functors, then so does any section G of F:

$$\mathit{f\ell} \circ F = \mathit{f\ell} \quad \Rightarrow \quad \mathit{f\ell} \circ G = \mathit{f\ell} \circ F \circ G = \mathit{f\ell}\,.$$

Therefore, monoidal sections G of Res_ι^π preserve the forgetful functors and by Theorem 2.10 they are in bijective correspondence with morphisms of matched pairs $(\alpha,\beta) : D(\mathcal{V},\mathcal{H}) \to (\mathcal{V},\mathcal{H})$ ($\alpha : \mathcal{V} \bowtie \mathcal{H} \to \mathcal{H}$, $\beta : \mathcal{V} \to \mathcal{V} \boxtimes \mathcal{H}^{op}$), such that $(\alpha,\beta)\circ(\iota,\pi) = \mathrm{id}_{(\mathcal{V},\mathcal{H})}$, via $G = \mathrm{Res}_\alpha^\beta$.

By Proposition 2.9 the above condition is equivalent to

$$\alpha \circ \iota = \mathrm{id}_{\mathcal{H}} \quad \text{and} \quad \pi \circ \beta = \mathrm{id}_{\mathcal{V}}\ .$$

In turn, for these conditions to be met, α and β must be of the form

$$\alpha(g,x) = \eta(g)x, \qquad \beta(f) = \big(f, \mu(f)\big)\,,$$

for certain maps $\mu, \eta : \mathcal{V} \to \mathcal{H}$ (to obtain the expression for α, note that $(g,x) = \big(g, \mathrm{id}_{\mathcal{H}}\, b(g)\big)\iota(x)$ in $\mathcal{V} \bowtie \mathcal{H}$).

We will show that (α,β) is a morphism of matched pairs if and only if (ξ,η) is a matched pair of rotations. This will prove that there is a bijective correspondence between braidings and matched pairs of rotations. Expression (4.7) for the braiding in $\mathrm{Rep}(\mathcal{V},\mathcal{H})$ follows from expression (3.12) for the braiding in $\mathrm{Rep}\, D(\mathcal{V},\mathcal{H})$, taking into account (2.9) and (2.10) and the above expressions for α and β in terms of η and ξ.

By Lemma 4.2, α is a morphism of groupoids if and only if η is a rotation. Also, by (1.29), β is a morphism of groupoids if and only if $\mu : \mathcal{V} \to \mathcal{H}^{op}$ is a morphism of groupoids; or equivalently, if $\xi : \mathcal{V} \to \mathcal{H}$ given by $\xi(f) = \mu(f)^{-1}$ is a morphism of groupoids.

Assume then that α, β, η, and ξ are morphisms of groupoids. To finish the proof, we will show that conditions (1.32) and (1.33) for (α,β) are equivalent to condition (4.1) for ξ and conditions (4.2) and (4.3) for (ξ,η) (this will complete the proof of the fact that (α,β) is a morphism of matched pairs if and only if (ξ,η) is a matched pair of rotations). It will turn out that, in this particular case, (1.33) is a consequence of (1.32).

Let us spell out (1.32) and (1.33). They are, respectively,

$$\beta\big(\alpha(g,x) \triangleright f\big) = (g,x) \rightharpoonup \beta(f) \quad \text{and} \quad \alpha\big((g,x) \leftharpoonup (f,\mu(f)\big) = \alpha(g,x) \triangleleft f\,,$$

for every $(g,x) \in \mathcal{V} \bowtie \mathcal{H}$ and $f \in \mathcal{V}$ for which the products are defined. The actions $\leftharpoonup$ and $\rightharpoonup$ of the matched pair $D(\mathcal{V},\mathcal{H})$ are defined by (1.36) and (1.37) (the morphisms α and β appearing in those formulas are in this case identities). Combining these formulas with the expressions for α and β in terms of η and μ we obtain that (1.32) is equivalent to (A) and (B) below, while (1.33) is equivalent to (C).

$$\big(\eta(g)x\big) \triangleright f = g(x \triangleright f)\Big(\big((x \triangleleft f)\mu(f)x^{-1}\big) \triangleright g^{-1}\Big)\,, \tag{A}$$

$$\mu\Big(\big(\eta(g)x\big) \triangleright f\Big) = \big((x \triangleleft f)\mu(f)x^{-1}\big) \triangleleft g^{-1}\,, \tag{B}$$

$$\big(\eta(g)x\big) \triangleleft f = \eta\left(\Big(\big((x \triangleleft f)\mu(f)x^{-1}\big) \triangleleft g^{-1}\Big) \triangleright g\right)(x \triangleleft f)\,, \tag{C}$$

for every $f, g \in \mathcal{V}$ and $x \in \mathcal{H}$ with $b(g) = l(x)$ and $r(x) = t(f)$.

Letting x be an identity in (A), we get (4.2) and (4.3). Letting g be an identity in (B), we get that ξ satisfies (4.1) (recall $\xi(f) = \mu(f)^{-1}$). This proves one implication.

Conversely, assume that ξ satisfies (4.1) and conditions (4.2) and (4.3) hold. By Lemma 4.4, conditions (4.4) and (4.5) hold as well. We then have

$$\begin{aligned}\big(\eta(g)x\big) \triangleright f &\overset{(1.3)}{=} \eta(g) \triangleright (x \triangleright f) \overset{(4.3)}{=} g(x \triangleright f)\big(\mu(x \triangleright f) \triangleright g^{-1}\big)\\ &\overset{(4.1)}{=} g(x \triangleright f)\Big(\big((x \triangleleft f)\mu(f)x^{-1}\big) \triangleright g^{-1}\Big)\,,\end{aligned}$$

which is (A). Also,

$$\mu\Big(\big(\eta(g)x\big) \triangleright f\Big) \overset{(1.3)}{=} \mu\big(\eta(g) \triangleright (x \triangleright f)\big) \overset{(4.4)}{=} \mu(x \triangleright f) \triangleleft g^{-1} \overset{(4.1)}{=} \big((x \triangleleft f)\mu(f)x^{-1}\big) \triangleleft g^{-1}\,,$$

which is (B). Finally,

$$\begin{aligned}\big(\eta(g)x\big) \triangleleft f &\overset{(1.13)}{=} \big(\eta(g) \triangleleft (x \triangleright f)\big)(x \triangleleft f) \overset{(4.5)}{=} \eta\Big(\big(\mu(x \triangleright f) \triangleleft g^{-1}\big) \triangleright g\Big)(x \triangleleft f)\\ &\overset{(4.1)}{=} \eta\left(\Big(\big((x \triangleleft f)\mu(f)x^{-1}\big) \triangleleft g^{-1}\Big) \triangleright g\right)(x \triangleleft f)\,,\end{aligned}$$

which is (C). The proof is complete. $\square$

EXAMPLE 4.6. Let $(\mathcal{V}, \mathcal{H})$ be a matched pair. By Corollary 3.5, there is a braiding in the category of representations of $D(\mathcal{V}, \mathcal{H})$. According to Theorem 4.5, this braiding must come from a matched pair of rotations for $D(\mathcal{V}, \mathcal{H})$. It follows from (3.12) and (4.7) that this consists of the rotations $\xi, \eta : \mathcal{V} \boxtimes \mathcal{H}^{op} \to \mathcal{V} \bowtie \mathcal{H}$ given by

$$\xi(\gamma, x) = \big(\mathrm{id}_{\mathcal{V}}\, l(x), x^{-1}\big) \quad \text{and} \quad \eta(\gamma, x) = \big(\gamma, \mathrm{id}_{\mathcal{H}}\, t(\gamma)\big)\,.$$

REMARK 4.7. Suppose $\mathcal{E}$ is a quiver and $c : \mathcal{E} \otimes \mathcal{E} \to \mathcal{E} \otimes \mathcal{E}$ is a map satisfying the Yang-Baxter equation. If c is *non-degenerate*, it is possible to construct a matched pair of groupoids $\big(\mathcal{V}(c), \mathcal{H}(c)\big)$ and a matched pair of rotations for $\big(\mathcal{V}(c), \mathcal{H}(c)\big)$ such that $\mathcal{E}$ is a representation of $\big(\mathcal{V}(c), \mathcal{H}(c)\big)$ and the corresponding braiding satisfies $c_{\mathcal{E},\mathcal{E}} = c$. This version of the FRT-construction is given in [**1**].

5. Applications to weak Hopf algebras

5.1. Weak Hopf algebras from matched pairs of groupoids.

We first recall the definition of weak Hopf algebras; more information can be found in [**15**].

Definition 5.1. [**4, 5**] A *weak bialgebra* is a collection $(H, m, 1, \Delta, \varepsilon)$, where $(H, m, 1)$ is a unital associative algebra, (H, Δ, ε) is a counital coassociative coalgebra, and the following conditions hold, for all $a, b, c \in H$:

$$\Delta(ab) = \Delta(a)\Delta(b)\,,$$
$$\big(\Delta(1)\otimes 1\big)\big(1\otimes\Delta(1)\big) = \Delta^{(2)}(1) = \big(1\otimes\Delta(1)\big)\big(\Delta(1)\otimes 1\big)\,,$$
$$\varepsilon(ab_1)\varepsilon(b_2c) = \varepsilon(abc) = \varepsilon(ab_2)\varepsilon(b_1c)\,.$$

Consider the maps $\varepsilon_{\mathfrak{s}} : H \to H$ and $\varepsilon_{\mathfrak{t}}(h) : H \to H$ defined by

$$\varepsilon_{\mathfrak{s}}(h) = (\mathrm{id}\otimes\varepsilon)\big((1\otimes h)\Delta(1)\big) \text{ and } \varepsilon_{\mathfrak{t}}(h) = (\varepsilon\otimes\mathrm{id})\big(\Delta(1)(h\otimes 1)\big)\,.$$

They are not morphisms of algebras in general, but their images are subalgebras of H. They are called the source and target subalgebras, respectively. Elements of one commute with elements of the other.

A weak bialgebra H is called a *weak Hopf algebra* or a *quantum groupoid* if there exists a linear map $\mathcal{S} : H \to H$ such that

$$m(\mathcal{S}\otimes\mathrm{id})\Delta = \varepsilon_{\mathfrak{s}}\,,\quad m(\mathrm{id}\otimes\mathcal{S})\Delta = \varepsilon_{\mathfrak{t}}\,,\quad\text{and}\quad m^{(2)}(\mathcal{S}\otimes\mathrm{id}\otimes\mathcal{S})\Delta^{(2)} = \mathcal{S}\,.$$

In this case, the map $\mathcal{S}$ is unique and it is an antimorphism of algebras and coalgebras. It is called the antipode of H. If H is finite dimensional, $\mathcal{S}$ is invertible and restricts to an anti-isomorphism between the source and target subalgebras.

A construction of a weak Hopf algebra from a matched pair of groupoids was introduced in [**2**]. It generalizes a construction of Hopf algebras from matched pairs of groups due to G. I. Kac and M. Takeuchi. In order to recall it, we need some terminology from the theory of double categories.

Let $(\mathcal{V}, \mathcal{H})$ be a matched pair of groupoids over $\mathcal{P}$. The set of cells of $(\mathcal{V}, \mathcal{H})$ is

$$\mathfrak{B} := \mathcal{H}\,{}_r\!\times_t \mathcal{V} = \{(x, g) : x\in\mathcal{H},\ g\in\mathcal{V},\ \text{and } r(x) = t(g)\}\,.$$

We use the letters A, B, C to denote cells. A cell may at times be represented by a diagram (1.14), which displays the pair (x, g) together with the actions $x \triangleright g$ and $x \triangleleft g$. Two cells $A = (x, g)$ and $B = (y, f)$ are horizontally composable if $g = y \triangleright f$; in this case, their horizontal composition is the cell $AB := (xy, f)$. Given an element $g \in \mathcal{V}$, we let $I_{\mathcal{H}}(g) := \big(\mathrm{id}_{\mathcal{H}}\, t(g), g\big)$. This is an identity for the horizontal composition. Vertical composition is defined similarly, and given $x \in \mathcal{H}$ we let $I_{\mathcal{V}}(x) := \big(x, \mathrm{id}_{\mathcal{V}}\, r(x)\big)$. The inverse of a cell $A = (x, g)$ is $A^{-1} := \big((x\triangleleft g)^{-1}, (x\triangleright g)^{-1}\big)$, as illustrated in (1.23).

We assume from now on that $\mathcal{H}$, $\mathcal{V}$ (and hence also $\mathcal{P}$) are finite sets. The weak Hopf algebra $\Bbbk(\mathcal{V}, \mathcal{H})$ is the $\Bbbk$-vector space with basis $\mathfrak{B}$ and the following structure, which we specify on basis elements A, B:

- The product is $A\cdot B = \begin{cases} AB & \text{if } A \text{ and } B \text{ are horizontally composable,}\\ 0 & \text{otherwise.}\end{cases}$

- The unit is $\mathbf{1} := \sum_{g\in\mathcal{V}} I_{\mathcal{H}}(g)$.
- The coproduct is $\Delta(A) = \sum B\otimes C$, where the sum is over all pairs (B, C) of vertically composable cells whose vertical composition is A.
- The counit is $\varepsilon(A) = \begin{cases} 1 & \text{if } A = I_{\mathcal{V}}(x) \text{ for some } x\in\mathcal{V}, \\ 0 & \text{otherwise.}\end{cases}$
- The antipode is $\mathcal{S}(A) = A^{-1}$.

For each $P\in\mathcal{P}$, let

$$\mathbf{1}_P := \sum_{g\in\mathcal{V},\, b(g)=P} I_{\mathcal{H}}(g) \quad\text{and}\quad \mathbf{1}^P := \sum_{g\in\mathcal{V},\, t(g)=P} I_{\mathcal{H}}(g)\,.$$

The source subalgebra is the subspace generated by $\{\mathbf{1}_P\}_{P\in\mathcal{P}}$, and the target subalgebra is the subspace generated by $\{\mathbf{1}^P\}_{P\in\mathcal{P}}$. Each of these is a complete set of orthogonal idempotents. Thus, both subalgebras are semisimple commutative of dimension $|\mathcal{P}|$, and hence isomorphic to the algebra of functions $\Bbbk^{\mathcal{P}}$.

Remark 5.2. Two points regarding notation. In the terminology of [**2**], $\Bbbk(\mathcal{V},\mathcal{H})$ is the weak Hopf algebra corresponding to the *vacant double groupoid* associated to the transpose of the matched pair $(\mathcal{V},\mathcal{H})$. Also, our convention for writing products in a groupoid from left to right is the opposite of that in [**4, 15**]; for this reason, the source and target subalgebras of $\Bbbk(\mathcal{V},\mathcal{H})$ are respectively described by the target b and the source t of $\mathcal{V}$.

Note that the set $\mathfrak{B}$ is essentially the same as the set of arrows of the diagonal groupoid $\mathcal{V}\bowtie\mathcal{H}$. The algebra $\Bbbk(\mathcal{V},\mathcal{H})$, on the other hand, is *not* the groupoid algebra of $\mathcal{V}\bowtie\mathcal{H}$, in general. As an algebra, $\Bbbk(\mathcal{V},\mathcal{H})$ is the groupoid algebra of a groupoid whose arrows are the cells of $(\mathcal{V},\mathcal{H})$ and whose objects are the arrows of $\mathcal{V}$. As a coalgebra, it is the dual of the groupoid algebra of a groupoid whose arrows are the cells of $(\mathcal{V},\mathcal{H})$ and whose objects are the arrows of $\mathcal{H}$. For more in this direction, see [**2**].

Example 5.3. The weak Hopf algebra corresponding to the initial matched pair is the algebra of functions on $\mathcal{P}\times\mathcal{P}$:

$$\Bbbk(\mathcal{P}\times\mathcal{P},\mathcal{P}) \cong \Bbbk^{\mathcal{P}\times\mathcal{P}}\,.$$

The isomorphism identifies a cell $\big(P,(P,Q)\big)$ with the function

$$(R,S)\mapsto \begin{cases} 1 & \text{if } (R,S) = (P,Q), \\ 0 & \text{if not.}\end{cases}$$

On the other hand, the weak Hopf algebra corresponding to the terminal matched pair is the algebra of linear endomorphisms of the $\Bbbk$-space with basis $\mathcal{P}$ (a matrix algebra):

$$\Bbbk(\mathcal{P},\mathcal{P}\times\mathcal{P}) \cong \operatorname{End}_{\Bbbk}(\Bbbk\mathcal{P})\,.$$

The isomorphism identifies a cell $\big((P,Q),Q)\big)$ with the endomorphism

$$R \mapsto \begin{cases} P & \text{if } R = Q, \\ 0 & \text{if not.} \end{cases}$$

The weak Hopf algebras corresponding to the matched pair $(\mathcal{P}, \mathcal{H})$ is the groupoid algebra of $\mathcal{H}$, while the weak Hopf algebra corresponding to $(\mathcal{V}, \mathcal{P})$ is the algebra of functions $\Bbbk^{\mathcal{V}}$.

The construction is functorial with respect to morphisms of matched pairs (Definition 1.11).

Proposition 5.4. *Let $(\alpha, \beta) : (\mathbb{V}, \mathbb{H}) \to (\mathcal{V}, \mathcal{H})$ be a morphism of matched pairs of finite groupoids. There is a morphism of weak Hopf algebras $\Bbbk(\mathbb{V}, \mathbb{H}) \to \Bbbk(\mathcal{V}, \mathcal{H})$ given by*

$$(h, \gamma) \mapsto \sum_{g \in \mathcal{V},\, \beta(g) = \gamma} \big(\alpha(h), g\big)\,. \tag{5.1}$$

This defines a functor from the category of matched pairs to that of weak Hopf algebras.

Proof. First note that if (h, γ) is a cell in $(\mathbb{V}, \mathbb{H})$, then $\big(\alpha(h), g\big)$ is a cell in $(\mathcal{V}, \mathcal{H})$ for any $g \in \mathcal{V}$ with $\beta(g) = \gamma$, since $\beta : \mathcal{V} \to \mathbb{V}$ is a morphism of groupoids. Thus, (5.1) is well-defined.

Let (h, γ) and (h', γ') be two horizontally composable cells in $(\mathbb{V}, \mathbb{H})$. There is a bijection between the set of cells in $(\mathcal{V}, \mathcal{H})$ of the form $\big(\alpha(hh'), g'\big)$ with $\beta(g') = \gamma'$ and the set of pairs of cells in $(\mathcal{V}, \mathcal{H})$ of the form $\big(\alpha(h), g\big)$, $\big(\alpha(h'), g'\big)$, with $\beta(g) = \gamma$ and $\beta(g') = \gamma'$. These pairs are necessarily horizontally composable. In one direction, the bijection is given by horizontal composition; in the other, one sets $g := \alpha(h') \triangleright g'$ and appeals to (1.32). This shows that (5.1) preserves multiplications. The remaining conditions can be verified similarly. $\square$

We denote the morphism of matched pairs and the corresponding morphism of weak Hopf algebras by the same symbol (α, β).

In view of Example 5.3, the morphisms from the initial and to the terminal matched pair (Proposition 1.12) yield morphisms of weak Hopf algebras

$$\Bbbk^{\mathcal{P} \times \mathcal{P}} \to \Bbbk(\mathcal{V}, \mathcal{H}) \to \mathrm{End}_{\Bbbk}(\Bbbk\mathcal{P})\,.$$

The image of the first map is the subalgebra of $\Bbbk(\mathcal{V}, \mathcal{H})$ generated by the source and target subalgebras (which commute elementwise).

Remark 5.5. The weak Hopf algebra associated to the opposite of a matched pair $(\mathcal{V}, \mathcal{H})$ (Remark 1.3) is the opposite of the weak Hopf algebra associated to $(\mathcal{V}, \mathcal{H})$. Copposites are similarly preserved. Recall that any matched pair is isomorphic to its opposite and to its coopposite (Remark 1.13). In view of Proposition 5.4, the same is true of the weak Hopf algebra of a matched pair.

5.2. The linearization functor.

Let H be a weak bialgebra. The category $\mathrm{Rep}_{\Bbbk}(H)$ of left H-modules is a monoidal category with tensor product

$$M \bigotimes N := \Delta(1)\cdot M \otimes_{\Bbbk} N\,,$$

where we let $H\otimes H$ act on $M\otimes_{\Bbbk} N$ by $(a,b)\cdot(m,n) := a\cdot m\otimes b\cdot n$. The action of H on $M\bigotimes N$ is by restriction of this action via Δ.

Let $(\mathcal{V},\mathcal{H})$ be a matched pair of groupoids. Given a representation $\mathcal{E}$ of $(\mathcal{V},\mathcal{H})$, consider the vector space $\Bbbk\mathcal{E}$ with basis $\mathcal{E}$. Define a map $\Bbbk(\mathcal{V},\mathcal{H})\otimes\Bbbk\mathcal{E}\to\Bbbk\mathcal{E}$ on the basis $\mathfrak{B}\times\mathcal{E}$ by

$$(x,g)\cdot e = \begin{cases} x\triangleright e & \text{if } |e| = g,\\ 0 & \text{otherwise.}\end{cases} \tag{5.2}$$

Proposition 5.6. *In the situation above, $\Bbbk\mathcal{E}$ is a left module over $\Bbbk(\mathcal{V},\mathcal{H})$. Moreover, this defines a monoidal functor*

$$\mathrm{Lin} : \mathrm{Rep}(\mathcal{V},\mathcal{H}) \to \mathrm{Rep}_{\Bbbk}\big(\Bbbk(\mathcal{V},\mathcal{H})\big)\,.$$

Proof. Note that if $|e| = g$ and $(x,g)\in\mathfrak{B}$ then $p(e) = t(g) = r(x)$ so $x\triangleright e$ is defined. Associativity and unitality of (5.2) follow from (1.2), (1.3), and (1.4). A morphism of representations $\psi:\mathcal{E}\to\mathcal{F}$ preserves the actions of $\mathcal{H}$ and gradings by $\mathcal{V}$, so its linearization $\psi:\Bbbk\mathcal{E}\to\Bbbk\mathcal{F}$ preserves the actions of $\Bbbk(\mathcal{V},\mathcal{H})$. Thus, Lin is a functor.

Let $\mathcal{E}$ and $\mathcal{F}$ be representations of $(\mathcal{V},\mathcal{H})$. We identify $\Bbbk\mathcal{E}\otimes_{\Bbbk}\Bbbk\mathcal{F}$ with $\Bbbk(\mathcal{E}\times\mathcal{F})$. An element $I_{\mathcal{H}}(g)\otimes I_{\mathcal{H}}(g')$ acts on a basis element (e,f) of $\Bbbk\mathcal{E}\otimes_{\Bbbk}\Bbbk\mathcal{F}$ as zero unless $g = |e|$ and $g' = |f|$, in which case it acts as the identity, by (1.4). Therefore,

$$(\mathbf{1}_P\otimes\mathbf{1}^P)\cdot(e,f) = \begin{cases}(e,f) & \text{if } b(|e|) = P = t(|f|),\\ 0 & \text{otherwise;}\end{cases}$$

$$\Delta(\mathbf{1})\cdot(e,f) = \begin{cases}(e,f) & \text{if } b(|e|) = t(|f|),\\ 0 & \text{otherwise,}\end{cases}$$

since $\Delta(\mathbf{1}) = \sum_{P\in\mathcal{P}}\mathbf{1}_P\otimes\mathbf{1}^P$. Recalling that $\mathcal{E}\otimes\mathcal{F} = \{(e,f)\in\mathcal{E}\times\mathcal{F} : b(|e|) = t(|f|)\}$, we conclude that

$$\Delta(\mathbf{1})\cdot\Bbbk\mathcal{E}\otimes_{\Bbbk}\Bbbk\mathcal{F} = \Bbbk(\mathcal{E}\otimes\mathcal{F})\,,$$

or in other words, $\mathrm{Lin}(\mathcal{E})\bigotimes\mathrm{Lin}(\mathcal{F}) = \mathrm{Lin}(\mathcal{E}\otimes\mathcal{F})$ as vector spaces. It remains to verify that the actions of $\Bbbk(\mathcal{V},\mathcal{H})$ on both sides agree. Choose a cell $(x,g)\in\mathfrak{B}$. Consider first its action on a basis element (e,f) of $\Bbbk\mathcal{E}\bigotimes\Bbbk\mathcal{F}$. For a tensor $B\otimes C$ to appear in $\Delta(x,g)$ we must have $B = (x,g_1)$ and $C = (x\triangleleft g_1, g_2)$ with $g_1g_2 = g$. The action of $B\otimes C$ on (e,f) is zero unless $g_1 = |e|$ and $g_2 = |f|$, in which case it is equal to $\big(x\triangleright e,\,(x\triangleleft g_1)\triangleright f\big)$. Therefore,

$$\Delta(x,g)\cdot(e,f) = \begin{cases}\big(x\triangleright e,\,(x\triangleleft|e|)\triangleright f\big) & \text{if } |e||f| = g\,,\\ 0 & \text{otherwise.}\end{cases}$$

This agrees with the action of (x, g) on $\Bbbk(\mathcal{E} \otimes \mathcal{F})$, according to (2.4), (2.5), and (5.2). □

EXAMPLE 5.7. Consider the distinguished representation $\mathcal{H}\,{}_r{\times}_t\,\mathcal{V}$ of a matched pair $(\mathcal{V}, \mathcal{H})$ (Section 2.2). It follows readily from (2.6) and (5.2) that the module $\Bbbk(\mathcal{H}\,{}_r{\times}_t\,\mathcal{V})$ affords the regular representation of the weak Hopf algebra $\Bbbk(\mathcal{V}, \mathcal{H})$.

The linearization functor satisfies the following naturality condition. Given a morphism of matched pairs $(\alpha, \beta) : (\mathbb{V}, \mathbb{H}) \to (\mathcal{V}, \mathcal{H})$, the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Rep}(\mathcal{V}, \mathcal{H}) \arrow[r, "\mathrm{Lin}"] \arrow[d, "\mathrm{Res}_\alpha^\beta"'] & \mathrm{Rep}_{\Bbbk}\big(\Bbbk(\mathcal{V}, \mathcal{H})\big) \arrow[d, "\mathrm{Res}_\alpha^\beta"] \\
\mathrm{Rep}(\mathbb{V}, \mathbb{H}) \arrow[r, "\mathrm{Lin}"'] & \mathrm{Rep}_{\Bbbk}\big(\Bbbk(\mathbb{V}, \mathbb{H})\big)
\end{tikzcd}$$

where the functor on the right is the restriction along the morphism of algebras $\Bbbk(\mathbb{V}, \mathbb{H}) \to \Bbbk(\mathcal{V}, \mathcal{H})$ of Proposition 5.4.

Consider the case of the unique morphism $(\mathcal{P} \times \mathcal{P}, \mathcal{P}) \to (\mathcal{V}, \mathcal{H})$. The category of representations of the initial matched pair is $\mathrm{Quiv}(\mathcal{P})$, the category of quivers over $\mathcal{P}$ (Example 2.2). On the other hand, since $\Bbbk(\mathcal{P} \times \mathcal{P}, \mathcal{P}) = \Bbbk^{\mathcal{P} \times \mathcal{P}}$ (Example 5.3), the category of representations of this weak Hopf algebra can be identified with $\mathrm{Bimod}(\Bbbk^{\mathcal{P}})$, the category of bimodules over $\Bbbk^{\mathcal{P}}$. This may also be described as the category of $(\mathcal{P} \times \mathcal{P})$-graded vector spaces (with homogeneous maps of degree 0 as morphisms). We obtain a commutative diagram

$$\begin{tikzcd}
\mathrm{Rep}(\mathcal{V}, \mathcal{H}) \arrow[r, "\mathrm{Lin}"] \arrow[d, "f\ell"'] & \mathrm{Rep}_{\Bbbk}\big(\Bbbk(\mathcal{V}, \mathcal{H})\big) \arrow[d, "f\ell"] \\
\mathrm{Quiv}(\mathcal{P}) \arrow[r, "\mathrm{Lin}"'] & \mathrm{Bimod}(\Bbbk^{\mathcal{P}})
\end{tikzcd}$$

The functor on the right is the restriction along the canonical morphism $\Bbbk^{\mathcal{P} \times \mathcal{P}} \to \Bbbk(\mathcal{V}, \mathcal{H})$. Explicitly, if M is a left $\Bbbk(\mathcal{V}, \mathcal{H})$-module, the $(\mathcal{P} \times \mathcal{P})$-grading on M is defined by $M_{P,Q} := \mathbf{1}^Q \cdot (\mathbf{1}_P \cdot M)$.

5.3. Quasitriangular weak Hopf algebras.

DEFINITION 5.8. [**5, 14, 15**]. A *quasitriangular weak Hopf algebra* is a pair $(H, \mathcal{R})$ where H is a weak Hopf algebra, $\mathcal{R} \in \Delta^{cop}(1)(H \otimes_k H)\Delta(1)$ satisfies

$$\Delta^{cop}(h)\mathcal{R} = \mathcal{R}\Delta(h)\,, \quad \text{for all} \quad h \in H\,, \tag{5.3}$$

$$(\mathrm{id} \otimes \Delta)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{12}\,, \tag{5.4}$$

$$(\Delta \otimes \mathrm{id})(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{23}, \tag{5.5}$$

and there exists an element $\overline{\mathcal{R}} \in \Delta(1)(H \otimes_{\Bbbk} H)\Delta^{cop}(1)$ with

$$\mathcal{R}\overline{\mathcal{R}} = \Delta^{cop}(1) \quad \text{and} \quad \overline{\mathcal{R}}\mathcal{R} = \Delta(1)\,. \tag{5.6}$$

In this case, the element $\overline{\mathcal{R}}$ unique, and we say that $\mathcal{R}$ is a quasitriangular structure on H.

A quasitriangular structure $\mathcal{R}$ on H gives rise to a braiding c in the monoidal category $\operatorname{Rep}_{\Bbbk}(H)$ [**15**, Proposition 5.2.2]. If $\mathcal{R} = \sum_i R_i \otimes R_i'$, and M and N are H-modules, the transformation $c_{M,N} : M \otimes N \to N \otimes M$ is given by

$$c_{M,N}(m \otimes n) = \sum_i R_i' \cdot n \otimes R_i \cdot m\,. \tag{5.7}$$

Conversely, suppose that H is a weak Hopf algebra, $(\mathcal{C}, c)$ is a braided category, $F : \mathcal{C} \to \operatorname{Rep}_{\Bbbk}(H)$ is a monoidal functor, and there are an object X in $\mathcal{C}$ such that $F(X) = H$ and an element $\mathcal{R} \in H \otimes H$ such that $F(c_{E,D}) : F(E) \otimes F(D) \to F(D) \otimes F(E)$ is given by (5.7), for all objects E and D of $\mathcal{C}$. Then $\mathcal{R}$ is a quasitriangular structure on H. The proof of [**8**, Proposition XIII.1.4] can be easily adapted to derive this result. Briefly, (5.3) follows from the H-linearity of $F(c_{X,X})$, (5.4) and (5.5) follow from the braiding axioms, and (5.6) from the invertibility of $F(c_{X,X})$.

We make use of this fact to deduce that matched pairs of rotations for a matched pair yield quasitriangular structures for the corresponding weak Hopf algebra.

THEOREM 5.9. *Let (ξ, η) be a matched pair of rotations for $(\mathcal{V}, \mathcal{H})$. Define $\mathcal{R}_{\xi,\eta} \in \Bbbk(\mathcal{V}, \mathcal{H}) \otimes_{\Bbbk} \Bbbk(\mathcal{V}, \mathcal{H})$ by*

$$\mathcal{R}_{\xi,\eta} := \sum_{(f,g) \in \mathcal{V}\,{}_b\!\times_t \mathcal{V}} \big(\xi(f)^{-1} \triangleleft g^{-1}, g\big) \otimes \big(\eta(g), f\big)\,. \tag{5.8}$$

Then $\big(\Bbbk(\mathcal{V}, \mathcal{H}), \mathcal{R}_{\xi,\eta}\big)$ is a quasitriangular weak Hopf algebra.

PROOF. Let $H = \Bbbk(\mathcal{V}, \mathcal{H})$ and $X = \mathcal{H}\,{}_r\!\times_t \mathcal{V}$. Consider the linearization functor $\operatorname{Lin} : \operatorname{Rep}(\mathcal{V}, \mathcal{H}) \to \operatorname{Rep}_{\Bbbk}(H)$ (Proposition 5.6). As mentioned in Example 5.7, we have $\operatorname{Lin}(X) = H$. According to the previous remark, it suffices to check that for any representations $\mathcal{E}$ and $\mathcal{D}$ of $(\mathcal{V}, \mathcal{H})$ we have

$$c_{\mathcal{E},\mathcal{D}}(e, d) = \sum_{(f,g) \in \mathcal{V}\,{}_b\!\times_t \mathcal{V}} \big(\eta(g), f\big) \cdot d \otimes \big(\xi(f)^{-1} \triangleleft g^{-1}, g\big) \cdot e\,.$$

But this is immediate from (4.7) and (5.2) (recall that if $(e, d) \in \mathcal{E} \otimes \mathcal{D}$ then $b(|e|) = t(|d|)$). □

The *Drinfeld element* of a quasitriangular weak Hopf algebra $(H, \mathcal{R})$ is $u := \sum_i \mathcal{S}(R_i')R_i$. It satisfies many remarkable properties [**15**, Proposition 5.2.6]. In addition, when H is involutory ($\mathcal{S}^2 = \mathrm{id}$) then u is central and $\mathcal{S}(u) = u$.

Let (ξ, η) be a matched pair of rotations for a matched pair $(\mathcal{V}, \mathcal{H})$, and $\mathcal{R}$ the quasitriangular structure on $\Bbbk(\mathcal{V}, \mathcal{H})$ of Theorem 5.9. The Drinfeld element turns out to be

$$u = \sum_{f \in \mathcal{V}} \big(\varphi(f), f\big)$$

where $\varphi(f) := \xi(f)\eta(f)^{-1}$. Note that $\varphi(f) \in \mathcal{H}(P, P)$ where $P = t(f)$. This map φ is not a morphism of groupoids but satisfies the following properties (from which the properties of u may be deduced):

$$\varphi(x \triangleright f) = x\varphi(f)x^{-1}\,, \quad \varphi(f) \triangleright f = f\,, \quad \text{and} \quad \varphi(f) \triangleleft f = \varphi(f^{-1})^{-1}\,,$$

for every $f \in \mathcal{V}$ and $x \in \mathcal{H}$ with $r(x) = t(f)$. One also finds that

$$u^n = \sum_{f \in \mathcal{V}} \big(\varphi(f)^n, f\big)$$

for every $n \in \mathbb{Z}$.

5.4. Duals and doubles of weak Hopf algebras of matched pairs. Let $(\mathcal{V}, \mathcal{H})$ be a matched pair and $(\mathcal{H}, \mathcal{V})$ its dual. As noted in Remark 1.3, there is a bijection between the set of cells of these matched pairs, given by transposition:

$$\mathcal{H}\,{}_r{\times}_t\,\mathcal{V} \xrightarrow{\cong} \mathcal{V}\,{}_b{\times}_l\,\mathcal{H}\,, \quad (x, g) \mapsto (x \triangleright g, x \triangleleft g)\,.$$

Define a pairing on $\Bbbk(\mathcal{V}, \mathcal{H}) \otimes \Bbbk(\mathcal{H}, \mathcal{V})$ by

$$\langle (x, g),\, (f, y) \rangle := \begin{cases} 1 & \text{if } f = x \triangleright g \text{ and } y = x \triangleleft g, \\ 0 & \text{otherwise.} \end{cases}$$

This pairing is non-degenerate and it identifies the weak Hopf algebra $\Bbbk(\mathcal{H}, \mathcal{V})$ with the dual of the weak Hopf algebra $\Bbbk(\mathcal{V}, \mathcal{H})$ [**2**, Proposition 3.11].

We briefly review the construction of the *Drinfeld double* of a finite dimensional weak Hopf algebra H [**15**, Section 5.3]. The space $H^* \otimes_{\Bbbk} H$ is an algebra with multiplication

$$(\phi \otimes h) \cdot (\phi' \otimes h') := \langle \mathcal{S}(h_1),\, \phi'_1 \rangle \phi'_2 \phi \otimes h_2 h' \langle h_3,\, \phi'_3 \rangle\,. \tag{5.9}$$

The linear span of the elements

$$\phi \otimes zh - \langle z, \epsilon_1 \rangle \epsilon_2 \phi \otimes h\,, \quad z \text{ in the target subalgebra of } H, \tag{5.10}$$

$$\phi \otimes zh - \langle z, \epsilon_2 \rangle \epsilon_1 \phi \otimes h\,, \quad z \text{ in the source subalgebra of } H, \tag{5.11}$$

is an ideal for this multiplication. The quotient of $H^* \otimes_{\Bbbk} H$ by this ideal is the Drinfeld double of H. It is a quasitriangular weak Hopf algebra. For more details, see [**15**] or [**14**].

Theorem 5.10. *There is an isomorphism of quasitriangular weak Hopf algebras between the Drinfeld double of $\Bbbk(\mathcal{V}, \mathcal{H})$ and the weak Hopf algebra of the matched pair $(\mathcal{V} \boxtimes \mathcal{H}^{op}, \mathcal{V} \bowtie \mathcal{H})$. For basis elements $(f, y) \in \Bbbk(\mathcal{H}, \mathcal{V})$ and $(x, g) \in \Bbbk(\mathcal{V}, \mathcal{H})$, the isomorphism is given by*

$$(f, y) \otimes (x, g) \mapsto \Big((f^{-1}, x),\, \big(g, (x \triangleleft g)^{-1} (y^{-1} \triangleleft f^{-1}) x\big) \Big)\,. \tag{5.12}$$

Proof. Let $H = \Bbbk(\mathcal{V}, \mathcal{H})$. Let (x, g) be a cell in $(\mathcal{V}, \mathcal{H})$ (a basis element of H) and (f, y) a cell in $(\mathcal{H}, \mathcal{V})$ (a basis element of H^*). Conditions (5.10) and (5.11) impose the following relations on $H^* \otimes_{\Bbbk} H$:

$$(f, y) \otimes (x, g) \equiv 0 \;\text{ whenever } t(f) \neq l(x) \text{ or } b(x \triangleright g) \neq l(y^{-1} \triangleleft f^{-1}).$$

Therefore, the quotient of $H^* \otimes_{\Bbbk} H$ by these relations can be identified with the space with basis elements $(f, y) \otimes (x, g)$, where $t(f) = l(x)$ and $b(x \triangleright g) = l(y^{-1} \triangleleft f^{-1})$.

These conditions can be depicted as follows:

$$\begin{tikzcd} P \arrow[r, bend left, "x"] \arrow[d, bend right, "x\triangleright g"'] & Q \arrow[d, bend left, "g"] \\ R \arrow[r, bend right, "x\triangleleft g"'] & S \end{tikzcd} \qquad \begin{tikzcd} P \arrow[r, bend left, "f"] \arrow[d, bend right, "(y^{-1}\triangleleft f^{-1})^{-1}"'] & Q' \arrow[d, bend left, "y"] \\ R \arrow[r, bend right, "(y^{-1}\triangleright f^{-1})^{-1}"'] & S' \end{tikzcd}$$

We see that $b(f^{-1}) = P = l(x)$, so $(f^{-1}, x) \in \mathcal{V} \bowtie \mathcal{H}$. Also,

$$t(g) = Q = r\big((x\triangleleft g)^{-1}(y^{-1}\triangleleft f^{-1})x\big) \quad \text{and} \quad b(g) = S = l\big((x\triangleleft g)^{-1}(y^{-1}\triangleleft f^{-1})x\big)\,,$$

so $\big(g, (x\triangleleft g)^{-1}(y^{-1}\triangleleft f^{-1})x\big) \in \mathcal{V} \boxtimes \mathcal{H}^{op}$. This shows that the map (5.12) is well-defined. Let us denote it by Ψ. Since it takes basis elements bijectively onto basis elements, Ψ is a linear isomorphism.

Let $\phi = (f, y)$, $h = (x, g)$, $\phi' = (f', y')$, and $h' = (x', g')$. Computing with (5.9) one finds that the product $(\phi \otimes h) \cdot (\phi' \otimes h')$ is zero unless

$$f'g = (x'\triangleright g')({y'}^{-1}\triangleleft {f'}^{-1})^{-1} \quad \text{and} \quad xy' = (y^{-1}\triangleleft f^{-1})^{-1}(x\triangleleft g)\,.$$

These conditions are equivalent to

$$\big(g, (x\triangleleft g)^{-1}(y^{-1}\triangleleft f^{-1})x\big) = ({f'}^{-1}, x') \rightharpoonup \big(g', (x'\triangleleft g')^{-1}({y'}^{-1}\triangleleft {f'}^{-1})x'\big)\,,$$

as one may see from (1.36). This is precisely when the product $\Psi(\phi\otimes h)\cdot\Psi(\phi'\otimes h')$ is not zero (when these cells are horizontally composable in $D(\mathcal{V}, \mathcal{H})$). Using these conditions one may verify that $\Psi\big((\phi\otimes h)\cdot(\phi'\otimes h')\big)$ agrees with $\Psi(\phi\otimes h)\cdot\Psi(\phi'\otimes h')$ when these products are non-zero. We omit the details. $\square$

6. Appendix. Proof of theorem 1.15

We proceed in several steps.

Claim 1. The map $\rightharpoonup$ given by (1.36) is a left action of $\mathcal{V} \bowtie \mathbb{H}$ on $\mathbb{V} \boxtimes \mathcal{H}^{op}$.

Proof. Let $(g, h) \in \mathcal{V} \bowtie \mathbb{H}$, $(\gamma, x) \in \mathbb{V} \boxtimes \mathcal{H}^{op}$ with $r(h) = t(\gamma)$. For notational convenience, we set (A, B) for the right-hand side of (1.36). Then we also have

$$A = \beta(g)\left(h \triangleright \left(\gamma\beta\left(\left(x\alpha(h)^{-1}\right) \triangleright g^{-1}\right)\right)\right).$$

One may see from (1.38) that A, B, and the action given by (1.36) are well-defined. In detail, $b(A) = b(\big(\alpha(h\triangleleft\gamma)x\alpha(h)^{-1}\big) \triangleright g^{-1}) = l(B)$ by (1.11); and $t(A) = t(g) = r(g^{-1}) = l(B)$ by (1.8); hence $(A, B) \in \mathbb{V} \boxtimes \mathcal{H}^{op}$, and also (1.2) holds. We check now (1.3). We have

$$\begin{aligned}(f, u) \rightharpoonup ((g, h) \rightharpoonup (\gamma, x)) &= (f, u) \rightharpoonup (A, B) \\ &= \left(\beta(f)\left(u \triangleright \left(A\beta\left(\left(B\alpha(u)^{-1}\right) \triangleright f^{-1}\right)\right)\right), \left(\alpha(u\triangleleft A)B\alpha(u)^{-1}\right) \triangleleft f^{-1}\right).\end{aligned}$$

The first component in this expression equals

$$\beta(f)\left(u \triangleright \left(\beta(g)\left(h \triangleright \left(\gamma\beta\left(\left(x\alpha(h)^{-1}\right) \triangleright g^{-1}\right)\right)\right)\beta\left(\left(B\alpha(u)^{-1}\right) \triangleright f^{-1}\right)\right)\right), \tag{6.1}$$

while the second is

$$\left(\alpha(u \triangleleft \left(\beta(g)\left(h \triangleright \left(\gamma\beta\left(\left(x\alpha(h)^{-1}\right) \triangleright g^{-1}\right)\right)\right)\right)\right)B\alpha(u)^{-1}\right) \triangleleft f^{-1}. \tag{6.2}$$

On the other hand, the second component of

$$(f,u)(g,h) \rightharpoonup (\gamma,x) = (f(\alpha(u) \triangleright g), (u \triangleleft \beta(g))h) \rightharpoonup (\gamma,x)$$

equals

$$\left(\alpha(((u \triangleleft \beta(g))h) \triangleleft \gamma)x\alpha((u \triangleleft \beta(g))h)^{-1}\right) \triangleleft (f(\alpha(u) \triangleright g))^{-1}, \tag{6.3}$$

while the first is

$$\begin{aligned}
&\beta(f)\beta((\alpha(u) \triangleright g)\left((u \triangleleft \beta(g))h) \triangleright \left(\gamma\beta\left(\left(x\alpha(u \triangleleft \beta(g))h)^{-1}\right) \triangleright (f(\alpha(u) \triangleright g))^{-1}\right)\right)\right) \\
&= \beta(f)(u \triangleright \beta(g))\left((u \triangleleft \beta(g)) \triangleright \left(h \triangleright \left(\gamma\beta\left(\left(x\alpha(u \triangleleft \beta(g))h)^{-1}\right) \triangleright (f(\alpha(u) \triangleright g))^{-1}\right)\right)\right)\right) \\
&= \beta(f)\left(u \triangleright \left(\beta(g)\left(h \triangleright \left(\gamma\beta\left(\left(x\alpha(u \triangleleft \beta(g))h)^{-1}\right) \triangleright (f(\alpha(u) \triangleright g))^{-1}\right)\right)\right)\right)\right);
\end{aligned}$$

thus, to see that this equals (6.1), it is enough to verify the equality

$$\begin{aligned}
&\left(h \triangleright \left(\gamma\beta\left(\left(x\alpha(h)^{-1}\right) \triangleright g^{-1}\right)\right)\right)\beta\left(\left(B\alpha(u)^{-1}\right) \triangleright f^{-1}\right) \\
&\qquad \stackrel{?}{=} h \triangleright \left(\gamma\beta\left((x\alpha(u \triangleleft \beta(g))h)^{-1} \triangleright (f(\alpha(u) \triangleright g))^{-1}\right)\right).
\end{aligned} \tag{6.4}$$

The left-hand side of (6.4) is

$$\begin{aligned}
&(h \triangleright \gamma)\left((h \triangleleft \gamma) \triangleright \left(\beta\left(\left(x\alpha(h)^{-1}\right) \triangleright g^{-1}\right)\right)\right) \\
&\qquad \beta\left(\left(\left(\left(\alpha(h \triangleleft \gamma)x\alpha(h)^{-1}\right) \triangleleft g^{-1}\right)\alpha(u)^{-1}\right) \triangleright f^{-1}\right) \\
&= (h \triangleright \gamma)\beta\left(\left(\alpha(h \triangleleft \gamma)x\alpha(h)^{-1}\right) \triangleright g^{-1}\right) \\
&\qquad \beta\left(\left(\left(\left(\alpha(h \triangleleft \gamma)x\alpha(h)^{-1}\right) \triangleleft g^{-1}\right)\alpha(u)^{-1}\right) \triangleright f^{-1}\right) \\
&= (h \triangleright \gamma)\beta\left(\left(\alpha(h \triangleleft \gamma)x\alpha(h)^{-1}\right) \triangleright \left(g^{-1}\left(\alpha(u)^{-1} \triangleright f^{-1}\right)\right)\right) \\
&\qquad = (h \triangleright \gamma)\beta\left(\alpha(h \triangleleft \gamma) \triangleright \left(\left(x\alpha(h)^{-1}\right) \triangleright \left(g^{-1}\left(\alpha(u)^{-1} \triangleright f^{-1}\right)\right)\right)\right) \\
&\qquad\qquad = h \triangleright \left(\gamma\beta\left(\left(x\alpha(h)^{-1}\right) \triangleright \left(g^{-1}\left(\alpha(u)^{-1} \triangleright f^{-1}\right)\right)\right)\right).
\end{aligned}$$

Thus, to verify (6.4), it is enough to check

$$\alpha(h)^{-1} \triangleright \left(g^{-1}\left(\alpha(u)^{-1} \triangleright f^{-1}\right)\right) \stackrel{?}{=} (\alpha(u \triangleleft \beta(g))h)^{-1} \triangleright \left((\alpha(u) \triangleright g)^{-1}f^{-1}\right). \tag{6.5}$$

Now, the right-hand side of (6.5) is

$$\begin{aligned}
&\left(\alpha(h)^{-1}\alpha(u \triangleleft \beta(g))^{-1}\right) \triangleright \left((\alpha(u) \triangleright g)^{-1}f^{-1}\right) \\
&\qquad = \alpha(h)^{-1} \triangleright \left((\alpha(u) \triangleleft g)^{-1} \triangleright \left((\alpha(u) \triangleright g)^{-1}f^{-1}\right)\right)
\end{aligned}$$

and this equals the left-hand side of (6.5) by (1.21). Thus, we have checked the equality of the first components in (1.3). We next verify the equality of the second components, that is, the equality of (6.2) and (6.3). Acting on (6.2) by $\triangleleft f$ we have

$$\begin{aligned}
&\alpha\left(u \triangleleft \left(\beta(g)\left(h \triangleright \left(\gamma\beta\left(\left(x\alpha(h)^{-1}\right) \triangleright g^{-1}\right)\right)\right)\right)\right) \\
&\qquad \left(\left(\alpha(h \triangleleft \gamma)x\alpha(h)^{-1}\right) \triangleleft g^{-1}\right)\alpha(u)^{-1}
\end{aligned}$$

$$
\begin{aligned}
&= \alpha\left(u \triangleleft \left(\beta(g)(h \triangleright \gamma)\left((h \triangleleft \gamma) \triangleright \beta\left(\left(x\alpha(h)^{-1}\right) \triangleright g^{-1}\right)\right)\right)\right) \\
&\qquad \left(\left(\alpha(h \triangleleft \gamma)x\alpha(h)^{-1}\right) \triangleleft g^{-1}\right)\alpha(u)^{-1} \\
&= \alpha\left(u \triangleleft \left(\beta(g)(h \triangleright \gamma)\beta\left(\left(\alpha((h \triangleleft \gamma))x\alpha(h)^{-1}\right) \triangleright g^{-1}\right)\right)\right) \\
&\qquad \left(\left(\alpha(h \triangleleft \gamma)x\alpha(h)^{-1}\right) \triangleleft g^{-1}\right)\alpha(u)^{-1} \\
&= \left(\left(\alpha\left(u \triangleleft \left(\beta(g)(h \triangleright \gamma)\right)\right) \triangleleft \left(\alpha(h \triangleleft \gamma)x\alpha(h)^{-1}\right) \triangleright g^{-1}\right)\right) \\
&\qquad \left(\left(\alpha(h \triangleleft \gamma)x\alpha(h)^{-1}\right) \triangleleft g^{-1}\right)\alpha(u)^{-1} \\
&= \left(\left(\alpha\left(u \triangleleft \left(\beta(g)(h \triangleright \gamma)\right)\right)\alpha(h \triangleleft \gamma)x\alpha(h)^{-1}\right) \triangleleft g^{-1}\right)\alpha(u)^{-1} \\
&\qquad = \left(\left(\alpha\left(u \triangleleft \left(\beta(g)(h \triangleright \gamma)\right)(h \triangleleft \gamma)\right)x\alpha(h)^{-1}\right) \triangleleft g^{-1}\right)\alpha(u)^{-1} \\
&= \left(\left(\alpha\left(\left(u \triangleleft \left(\beta(g)(h \triangleright \gamma)\right)\right)(h \triangleleft \gamma)\right)x\alpha(h)^{-1}\right) \triangleleft g^{-1}\right)\alpha(u)^{-1} \\
&\qquad = \left(\left(\left(\alpha\left(\left(u \triangleleft \beta(g)\right)h\right) \triangleleft \gamma\right)x\alpha(h)^{-1}\right) \triangleleft g^{-1}\right)\alpha(u)^{-1}
\end{aligned}
$$

On the other hand, (6.3) acted by $\triangleleft f$ equals

$$
\begin{aligned}
&\left(\alpha(((u \triangleleft \beta(g))h) \triangleleft \gamma)x\alpha((u \triangleleft \beta(g))h)^{-1}\right) \triangleleft (\alpha(u) \triangleright g)^{-1} \\
&\qquad = \left(\alpha(((u \triangleleft \beta(g))h) \triangleleft \gamma)x\alpha(h)^{-1}(\alpha(u) \triangleleft g)^{-1}\right) \triangleleft (\alpha(u) \triangleright g)^{-1} \\
&\qquad = \left(\left(\alpha(((u \triangleleft \beta(g))h) \triangleleft \gamma)x\alpha(h)^{-1}\right) \triangleleft \left((\alpha(u) \triangleleft g)^{-1} \triangleright (\alpha(u) \triangleright g)^{-1}\right)\right) \\
&\qquad\qquad \left((\alpha(u) \triangleleft g)^{-1} \triangleleft (\alpha(u) \triangleright g)^{-1}\right) \\
&\qquad = \left(\left(\alpha(((u \triangleleft \beta(g))h) \triangleleft \gamma)x\alpha(h)^{-1}\right) \triangleleft g^{-1}\right)\alpha(u)^{-1}.
\end{aligned}
$$

Therefore (1.3) holds. The verification of (1.4) is straightforward, and the claim is proved. $\square$

Claim 2. The map $\leftharpoondown$ given by (1.37) is a right action of $\mathbb{V} \boxtimes \mathcal{H}^{op}$ on $\mathcal{V} \bowtie \mathbb{H}$.

Proof. Let $(g,h) \in \mathcal{V} \bowtie \mathbb{H}$, $(\gamma, x) \in \mathbb{V} \boxtimes \mathcal{H}^{op}$. Action (1.37) is well-defined since

$$
\begin{aligned}
b\left(((\alpha(h \triangleleft \gamma)x\alpha(h)^{-1}) \triangleleft g^{-1}) \triangleright g\right) &= l\left(((\alpha(h \triangleleft \gamma)x\alpha(h)^{-1}) \triangleleft g^{-1}) \triangleleft g\right) \\
&= l\left(\alpha(h \triangleleft \gamma)x\alpha(h)^{-1}\right) = l\left(h \triangleleft \gamma\right).
\end{aligned}
$$

We check (1.8):

$$
\begin{aligned}
r\left((g,h) \leftharpoondown (\gamma, x)\right) &= r\left(((\alpha(h \triangleleft \gamma)x\alpha(h)^{-1}) \triangleleft g^{-1}) \triangleright g, h \triangleleft \gamma\right) \\
&= r(h \triangleleft \gamma) = b(\gamma) = l(x) = b(\gamma, x).
\end{aligned}
$$

We now check (1.9). We have

$$
\begin{aligned}
&((g,h) \leftharpoondown (\gamma, x)) \leftharpoondown (\tau, y) \\
&\qquad = \left(((\alpha(h \triangleleft \gamma)\, x\, \alpha(h)^{-1}) \triangleleft g^{-1}) \triangleright g, h \triangleleft \gamma\right) \leftharpoondown (\tau, y) \\
&= (((\alpha(h \triangleleft \gamma) \triangleleft \tau)\, y\, \alpha(h \triangleleft \gamma)^{-1}) \triangleleft ((\alpha(h \triangleleft \gamma)\, x\, \alpha(h)^{-1}) \triangleright g^{-1}) \\
&\qquad \triangleright (((\alpha(h \triangleleft \gamma)\, x\, \alpha(h)^{-1}) \triangleleft g^{-1}) \triangleright g), (h \triangleleft \gamma) \triangleleft \tau) \\
&= ((\alpha(h \triangleleft \gamma\tau)\, y\, \alpha(h \triangleleft \gamma)^{-1}) \triangleleft ((\alpha(h \triangleleft \gamma)\, x\, \alpha(h)^{-1}) \triangleright g^{-1}) \\
&\qquad ((\alpha(h \triangleleft \gamma)\, x\, \alpha(h)^{-1}) \triangleleft g^{-1}) \triangleright g, h \triangleleft \gamma\tau)
\end{aligned}
$$

$$= \left(\left((\alpha(h \triangleleft \gamma\tau)\, y\, \alpha(h \triangleleft \gamma)^{-1}\alpha(h \triangleleft \gamma)\, x\, \alpha(h)^{-1}) \triangleleft g^{-1}\right) \triangleright g, h \triangleleft \gamma\tau\right)$$
$$= \left(\left((\alpha(h \triangleleft \gamma\tau)\, yx\, \alpha(h)^{-1}) \triangleleft g^{-1}\right) \triangleright g, h \triangleleft \gamma\tau\right)$$
$$= (g,h) \leftharpoonup (\gamma\tau, yx).$$

Here we have used $((\alpha(h \triangleleft \gamma)x\alpha(h)^{-1}) \triangleleft g^{-1}) \triangleright g)^{-1} = (\alpha(h \triangleleft \gamma)x\alpha(h)^{-1}) \triangleright g^{-1}$ in the second equality, which follows from (1.12). Hence, (1.9) is valid for (1.37). A straightforward computation, using (1.16), shows that also (1.10) holds. $\square$

Claim 3. The groupoids $\mathcal{V} \bowtie \mathbb{H}$ and $\mathbb{V} \boxtimes \mathcal{H}^{op}$, together with the actions (1.36) and (1.37), form a matched pair.

Proof. We check (1.11):

$$\begin{aligned} b\left((g,h) \rightharpoonup (\gamma,x)\right) &= b\left(\beta(g)(h \triangleright \gamma)\beta\left(\left(\alpha(h \triangleleft \gamma)x\alpha(h)^{-1}\right) \triangleright g^{-1}\right)\right) \\ &= b\left(\left(\alpha(h \triangleleft \gamma)x\alpha(h)^{-1}\right) \triangleright g^{-1}\right) \\ &= l\left(\left(\alpha(h \triangleleft \gamma)x\alpha(h)^{-1}\right) \triangleleft g^{-1}\right); \\ &= t\left(((\alpha(h \triangleleft \gamma)x\alpha(h)^{-1}) \triangleleft g^{-1}) \triangleright g\right) \\ &= l\left((g,h) \leftharpoonup (\gamma,x)\right). \end{aligned}$$

We check (1.12):

$$\begin{aligned} &(g,h) \rightharpoonup ((\gamma,x)(\tau,y)) = (g,h) \rightharpoonup (\gamma\tau, yx) \\ &\quad = \left(\beta(g)(h \triangleright \gamma\tau)\beta\left(\left(\alpha(h \triangleleft \gamma\tau)yx\alpha(h)^{-1}\right) \triangleright g^{-1}\right), \left(\alpha(h \triangleleft \gamma\tau)yx\alpha(h)^{-1}\right) \triangleleft g^{-1}\right), \end{aligned}$$

while $((g,h) \rightharpoonup (\gamma,x))\,(((g,h) \leftharpoonup (\gamma,x)) \rightharpoonup (\tau,y))$ equals

$$\begin{aligned} &(\beta(g)(h \triangleright \gamma)\beta((\alpha(h \triangleleft \gamma)x\alpha(h)^{-1}) \triangleright g^{-1}), \\ &\qquad (\alpha(h \triangleleft \gamma)x\alpha(h)^{-1}) \triangleleft g^{-1})((((\alpha(h \triangleleft \gamma)x\alpha(h)^{-1}) \triangleleft g^{-1}) \triangleright g, h \triangleleft \gamma) \rightharpoonup (\tau,y)) \end{aligned}$$

The first component of this expression is

$$\begin{aligned} &\beta(g)(h \triangleright \gamma)\beta\left(\left(\alpha(h \triangleleft \gamma)x\alpha(h)^{-1}\right) \triangleright g^{-1}\right)\beta\left(((\alpha(h \triangleleft \gamma)x\alpha(h)^{-1}) \triangleleft g^{-1}) \triangleright g\right)((h \triangleleft \gamma) \triangleright \tau) \\ &\qquad \beta\left(\left(\alpha(h \triangleleft \gamma\tau)y\alpha(h \triangleleft \gamma)^{-1}\right) \triangleright \left((\alpha(h \triangleleft \gamma)x\alpha(h)^{-1}) \triangleright g^{-1}\right)\right) \\ &\qquad\qquad = \beta(g)(h \triangleright \gamma\tau)\beta\left((\alpha(h \triangleleft \gamma\tau)yx\alpha(h)^{-1}) \triangleright g^{-1}\right); \end{aligned}$$

and the second component is

$$\begin{aligned} &\left(\left(\alpha(h \triangleleft \gamma\tau)y\alpha(h \triangleleft \gamma)^{-1}\right) \triangleleft \left((\alpha(h \triangleleft \gamma)x\alpha(h)^{-1}) \triangleright g^{-1}\right)\right)\left(\left(\alpha(h \triangleleft \gamma)x\alpha(h)^{-1}\right) \triangleleft g^{-1}\right) \\ &\qquad = \left(\alpha(h \triangleleft \gamma\tau)y\alpha(h \triangleleft \gamma)^{-1}\alpha(h \triangleleft \gamma)x\alpha(h)^{-1}\right) \triangleleft g^{-1}. \end{aligned}$$

Hence (1.12) holds. We finally check (1.13). It is convenient to set

$$Q := \alpha\left(((u \triangleleft \beta(g))h) \triangleleft \gamma\right) x\alpha(h^{-1}).$$

Then the first component of

$$((f,u)(g,h)) \leftharpoonup (\gamma,x) = (f(\alpha(u) \triangleright g), (u \triangleleft \beta(g))h) \leftharpoonup (\gamma,x)$$

is

$$
\begin{aligned}
&\left(\left(\alpha\left(((u\triangleleft\beta(g))h)\triangleleft\gamma\right)x\alpha\left((u\triangleleft\beta(g))\,h\right)^{-1}\right)\triangleleft\left(f(\alpha(u)\triangleright g)\right)^{-1}\right)\triangleright\left(f(\alpha(u)\triangleright g)\right)\\
&=\left(\left(\left(Q\left(\alpha(u)\triangleleft g\right)^{-1}\right)\triangleleft\left(\alpha(u)\triangleright g\right)^{-1}\right)\triangleleft f^{-1}\right)\triangleright\left(f(\alpha(u)\triangleright g)\right)\\
&=\left(\left(\left(Q\triangleleft g^{-1}\right)\alpha(u)^{-1}\right)\triangleleft f^{-1}\right)\triangleright\left(f(\alpha(u)\triangleright g)\right)\\
&=\left(\left(\left(\left(Q\triangleleft g^{-1}\right)\alpha(u)^{-1}\right)\triangleleft f^{-1}\right)\triangleright f\right)\left(\left(\left(\left(\left(Q\triangleleft g^{-1}\right)\alpha(u)^{-1}\right)\triangleleft f^{-1}\right)\triangleleft f\right)\triangleright(\alpha(u)\triangleright g)\right)\\
&=\left(\left(\left(\left(Q\triangleleft g^{-1}\right)\alpha(u)^{-1}\right)\triangleleft f^{-1}\right)\triangleright f\right)\left(\left(\left(Q\triangleleft g^{-1}\right)\alpha(u)^{-1}\right)\triangleright(\alpha(u)\triangleright g)\right)\\
&=\left(\left(\left(\left(Q\triangleleft g^{-1}\right)\alpha(u)^{-1}\right)\triangleleft f^{-1}\right)\triangleright f\right)\left(\left(Q\triangleleft g^{-1}\right)\triangleright g\right),
\end{aligned}
$$

where we have used (1.22); while the second is $((u\triangleleft\beta(g))h)\triangleleft\gamma$. Set

$$P=\alpha(h\triangleleft\gamma)x\alpha(h)^{-1}\,.$$

Then the first component of $((f,u)\triangleleft((g,h)\triangleright(\gamma,x)))((g,h)\triangleleft(\gamma,x))$ is

$$
\begin{aligned}
&\left(\left(\alpha\left(u\triangleleft\left(\beta(g)(h\triangleright\gamma)\beta(P\triangleright g^{-1}\right)\right)\left((P\triangleleft g^{-1})\alpha(u)^{-1}\right)\triangleleft f^{-1}\right)\triangleright f\right)\\
&\qquad\left(\alpha\left(u\triangleleft\left(\beta(g)(h\triangleright\gamma)\beta(P\triangleright g^{-1}\right)\right)\triangleright\left((P\triangleleft g^{-1})\triangleright g\right)\right)\right)
\end{aligned}
$$

The first factor in the last expression is

$$
\begin{aligned}
&\left(\left(\alpha\left(u\triangleleft(\beta(g)(h\triangleright\gamma))\triangleleft(P\triangleright g^{-1}\right)\right)\left((P\triangleleft g^{-1})\alpha(u)^{-1}\right)\triangleleft f^{-1}\right)\triangleright f\\
&=\left(\left(\left(\left(\alpha\left(u\triangleleft(\beta(g)(h\triangleright\gamma))\right)P\right)\triangleleft g^{-1}\right)\alpha(u)^{-1}\right)\triangleleft f^{-1}\right)\triangleright f\\
&=\left(\left(\left(\left(\alpha\left(u\triangleleft(\beta(g)(h\triangleright\gamma)\right)(h\triangleleft\gamma)\right)x\alpha(h)^{-1}\right)\triangleleft g^{-1}\right)\alpha(u)^{-1}\right)\triangleleft f^{-1}\right)\triangleright f\\
&=\left(\left(\left(\left(Q\triangleleft g^{-1}\right)\alpha(u)^{-1}\right)\triangleleft f^{-1}\right)\triangleright f\right);
\end{aligned}
$$

while the second factor is

$$
\begin{aligned}
&\left(\alpha\left(u\triangleleft\left(\beta(g)(h\triangleright\gamma)\beta(P\triangleright g^{-1}\right)\right)\right)\left(P\triangleleft g^{-1}\right)\right)\triangleright g\\
&=\left(\alpha\left(u\triangleleft(\beta(g)(h\triangleright\gamma)\right)P\right)\triangleleft g^{-1}\right)\triangleright g\\
&=\left(\left(Q\triangleleft g^{-1}\right)\triangleright g\right).
\end{aligned}
$$

This shows that the first components of both sides of (1.13) in the present setting are equal. Finally, the second component of $((f,u)\triangleleft((g,h)\triangleright(\gamma,x)))((g,h)\triangleleft(\gamma,x))$ is

$$
\begin{aligned}
&\left(\left(u\triangleleft\left(\beta(g)(h\triangleright\gamma)\beta(P\triangleright g^{-1}\right)\right)\right)\triangleleft\beta\left((P\triangleleft g^{-1})\triangleright g\right)\right)(h\triangleleft\gamma)\\
&=\left((u\triangleleft(\beta(g)(h\triangleright\gamma)))\triangleleft\beta\left((P\triangleright g^{-1})((P\triangleleft g^{-1})\triangleright g)\right)\right)(h\triangleleft\gamma)\\
&=(u\triangleleft(\beta(g)(h\triangleright\gamma)))\,(h\triangleleft\gamma)\\
&=((u\triangleleft\beta(g))h)\triangleleft\gamma
\end{aligned}
$$

Hence, also the second components of both sides of (1.13) in the present setting are equal. □

The proof of Theorem 1.15 is complete.

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