1. Introduction

Nichols algebras are graded connected algebras with a comultiplication in a braided sense. In particular, the Jordan plane and the super Jordan plane are two Nichols algebras that play an important role in the classification of pointed Hopf algebras with finite Gelfand-Kirillov dimension [AAH1, AAH2].

The Jordan plane was first defined in [G] and considered in many papers, e.g. [AS], see also the references in [AAH2, I]. Its representation theory was studied in [I].

The purpose of this note is to begin the study of the representation theory of the super Jordan plane $B$: we classify the simple finite-dimensional $B$-modules (all of dimension 1, Theorem 2.6) and the indecomposable $B$-modules of dimension 2 (Theorem 3.2) and 3 (Theorem 3.11). We also observe that one of the generators of $B$ has at most two eigenvalues in every indecomposable $B$-module (Theorem 2.11) and describe two families of indecomposable modules in every dimension.

2. Basic facts

2.1. Notations and conventions. Fix an algebraically closed field $k$ of characteristic 0; all vector spaces, tensor products, Hom spaces, algebras are over $k$. All algebras are associative and all modules are left, unless explicitly stated. Let $A$ be a $k$-algebra; then $[ , ]$ denotes the Lie bracket given by the commutator. As customary we use indistinctly the languages of modules
and representations. Denote by $\mathcal{A}$ the category of finite dimensional $A$-modules. Given a $k$-vector space $V$, $\mathfrak{gl}(V)$ denotes the Lie algebra of all linear operators on $V$. The Jacobson radical of an algebra $A$ it will be denoted by $\text{Jac} A$.

2.2. The Jordan plane. The Jordan plane is the free associative algebra $A$ in generators $y_1$ and $y_2$ subject to the quadratic relation

$$y_1y_2 - y_2y_1 - y_2^2.$$ 

The algebra $A$ is a Nichols algebra, $\text{GKdim} A = 2$ and $\{y_1^ay_2^b : a, b \in \mathbb{N}_0\}$ is a basis of $A$. By Proposition 3.4 of [I], $A$ is a Koszul algebra.

2.3. The super Jordan plane. Let $x_{21} = x_1x_2 + x_2x_1$ in the free associative algebra in generators $x_1$ and $x_2$. Let $B$ be the algebra generated by $x_1$ and $x_2$ with defining relations

\begin{align*}
(2.1) & \quad x_1^2, \\
(2.2) & \quad x_2x_{21} - x_{21}x_2 - x_1x_{21}.
\end{align*}

The algebra $B$ (which is graded by $\deg x_1 = \deg x_2 = 1$) was introduced in [AAH1, AAH2] and is called the super Jordan plane. Since $B$ is not a quadratic algebra, it follows that $B$ is not Koszul; see e. g. § 2.1 of [PP].

Proposition 2.1. [AAH2] The algebra $B$ is a Nichols algebra, $\text{GKdim} B = 2$ and $\{x_1^ax_2^bx_3^c : a \in \{0, 1\}, b, c \in \mathbb{N}_0\}$ is a basis of $B$. \hfill \Box

The following identities are valid in $B$:

\begin{align*}
(2.3) & \quad x_{21}x_1 = x_1x_{21}, \\
(2.4) & \quad x_2x_1 = x_1x_2 + x_1x_2x_1, \\
(2.5) & \quad x_{21}x_3 = (x_2 - x_2)x_{21}.
\end{align*}

Indeed, in presence of (2.1), (2.2) is equivalent to (2.4).

By (2.5) and Proposition 2.1, the subalgebra of the super Jordan plane $B$ generated by $x_2^2$ and $x_{21}$, is isomorphic to the Jordan plane via $y_1 \mapsto x_2^2$ and $y_2 \mapsto x_{21}$.

It is convenient to introduce $s = x_{21}$ and $t = x_2^2$. By (2.5), $st = ts - s^2$ and whence

$$[t, s^n] = ns^{n+1}, \quad n \geq 1; \quad x_1s = sx_1; \quad x_2t = tx_2; \quad tx_1 = x_1(t + s).$$

Lemma 2.2. Given $b, c \in \mathbb{N}$, we have that

$$x_{21}^b x_2^c = (x_2 - bx_1)x_{21}^b x_2^c - 1, \quad x_1 x_{21}^b x_2^c = x_1 x_{21} x_2^b x_2^c - 1.$$ 

Proof. We prove $*$ by induction. For $b = c = 1$, the relation is valid by (2.2). Suppose that $*$ is valid for $b - 1 > 0$ and $c = 1$. Then

\begin{align*}
 x_{21}^b x_2 & = x_{21}^{b-1} x_2 x_{21} x_2 - x_{21}^{b-1} x_2 x_{21} x_2 x_{21} x_2 \\
 & = (x_2 - (b - 1)x_1)x_{21}^{b-1} x_2 x_{21} x_2 x_{21} x_2 - (x_2 - bx_1)x_{21}^{b-1} x_2 x_{21} x_2 x_{21} x_2.
\end{align*}
Fix \( b \in \mathbb{N} \) and assume that the relation is true for \( c - 1 \), with \( c > 1 \). Thus
\[
(x_{21}^b x_{2}^c) = (x_{21}^b x_{2}^{c-1})x_{2} = (x_{2} - bx_{1})x_{21}^b x_{2}^{c-2}x_{2} = (x_{2} - bx_{1})x_{21}^b x_{2}^{c-1}.
\]
The proof of \( \heartsuit \) is similar.

The next result follows immediately from Proposition 2.1 and Lemma 2.2.

**Proposition 2.3.** The set \( \{1, x_1, x_2, x_1 x_2\} \) generates \( B \) as a right \( A \)-module.

2.4. **Simple modules.** Let \((V, \rho)\) be a finite-dimensional representation of \( B \); set \( X_1 = \rho(x_1) \), \( X_2 = \rho(x_2) \), \( S = \rho(s) \) and \( T = \rho(t) \) and
\[
V_0 = \ker X_1.
\]
Then \( V_0 \) is always \( \neq 0 \) and it is stable under \( S \) and \( T \) by (2.6). In fact, let \( E_{1/2}(n) \in \mathfrak{gl}(\mathbb{k}^n) \) (or \( E_{1/2} \) if \( n \) is clearly from de context) the matrix whose the entry 1 \( \times 2 \) is equal to 1 and all other entries are equal to 0. Then the Jordan form of \( X_1 \) consists of \( r \) blocks like \( E_{1/2}(2) \) and \( s \) blocks of size 1 filled by 0. Hence \( \dim V = 2r + s; r = 0 \iff V = V_0. \)

**Lemma 2.4.** Assume the previous notations. Then:

(i) \( S \) and \( T \) have a simultaneous eigenvector in \( V_0 \).

(ii) \( W = X_2 V_0 \cap V_0 \) is a submodule of \( V \).

(iii) \( U = X_2 V_0 + V_0 \) is a submodule of \( V \).

**Proof.** (i): The subspace of \( \mathfrak{gl}(V) \) generated by \( T \) and \( S^n, n \in \mathbb{N}_0 \), is a solvable Lie subalgebra by (2.6); then Lie Theorem applies.

(ii): Clearly \( X_1 W \subseteq X_1 V_0 = \{0\} \subseteq W. \) It remains to show that \( X_2 W \subseteq W \). In fact, let \( w \in W \), this is, \( w \in V_0 \) and \( w = X_2 v \) for some \( v \in V_0 \). Clearly \( X_2 w \in X_2 V_0 \). Moreover,
\[
X_1(X_2 w) = X_1 X_2^2 v \overset{(2.4)}{=} (X_2^2 X_1 - X_1 X_2 X_1)v = 0 \implies X_2 w \in W.
\]

(iii): Since \( B \cdot V_0 \subseteq X_2 V_0 \) and \( B \cdot (X_2 V_0) \subseteq V_0 \), the claim follows.

**Lemma 2.5.** If \( V \in \mathcal{B} \mathcal{M} \) is simple, then \( V = V_0 \).

**Proof.** Assume that \( V \neq V_0 \). By Lemma 2.4 we have that \( W = X_2 V_0 \cap V_0 = 0 \) and \( V = X_2 V_0 + V_0 \), so that \( V = X_2 V_0 \oplus V_0 \). By Lemma 2.4 (i), there exists a simultaneous eigenvector \( v \in V_0 \) of \( S \) and \( T \), i.e. there exist \( \alpha, \tau \in \mathbb{k} \) such that \( Sv = \alpha v, Tv = \tau v \).

Now \( M = \text{span}\{v, X_2 v\} \neq 0 \) is a \( B \)-submodule of \( V \) and by simplicity of \( V \), \( M = V \). By our assumption, \( X_2 v \notin V_0 \); hence \( \Lambda = \{v, X_2 v\} \) is a basis of \( V \). Note that \( [X_1]_\Lambda = \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix} \) and \( [X_2]_\Lambda = \begin{pmatrix} 0 & \tau \\ 1 & 0 \end{pmatrix} \). The relation \( X_2^2 X_1 = X_1 X_2^2 + X_1 X_2 X_1 \) is satisfied if and only if \( \tau \alpha = \alpha \tau + \alpha^2 \). Therefore \( \alpha = 0 \) and \( V = V_0 \), a contradiction.

Let \( A \in \text{End}(\mathbb{k}^n) \). Denote by \( \mathbb{k}_A^n \) the \( B \)-module defined by \( X_1 = 0 \) and \( X_2 = A \). Every \( B \)-module \( V \) with \( V = V_0 \) is isomorphic to \( \mathbb{k}_A^n \) for some \( A \). If \( B \in \text{End}(\mathbb{k}^m) \), then \( \mathbb{k}_A^n \simeq \mathbb{k}_B^m \) iff \( n = m \) and \( A \) and \( B \) are similar matrices.
**Theorem 2.6.** Every simple $\mathcal{B}$-module is isomorphic to $k^1_a$ for a unique $a \in k$.

*Proof.* This follows from Lemma 2.5 and the preceding considerations. \qed

**Corollary 2.7.** Let $\rho : \mathcal{B} \to \text{End} V$ a finite dimensional representation of $\mathcal{B}$ and $B = \rho(\mathcal{B})$. Then there exists an integer $s$ such that $B/Jac B \simeq k^s$ and $\text{Jac } B = \{x \in B : x \text{ is nilpotent}\}$.

*Proof.* Since $B/Jac B$ is semisimple and $k$ is algebraically closed, there are positive integers $n_1, \ldots, n_s$ such that $B/Jac B = M_{n_1}(k) \times \cdots \times M_{n_s}(k)$. The composition

$$\mathcal{B} \xrightarrow{\rho} B \xrightarrow{\pi} B/Jac B \xrightarrow{\pi_j} M_{n_j}(k)$$

is a finite dimensional simple representation of $\mathcal{B}$. Hence, by Theorem (2.6), $n_1 = \cdots = n_s = 1$. Thus, $B/Jac B \simeq k^s$. Let $x \in B$ a nilpotent element. Then $\pi(x)$ is a nilpotent element of $B/Jac B$. Since $B/Jac B$ is commutative, we obtain that $\pi(x) \in \text{Jac } (B/Jac B) = \{0\}$. Hence, $x \in \text{Jac } B$. On the other hand, $B$ finite dimensional implies that $\text{Jac } B$ is a nilpotent ideal. Consequently, $\text{Jac } B = \{x \in B : x \text{ is nilpotent}\}$. \qed

We also remark:

**Proposition 2.8.** If $V$ is an indecomposable $\mathcal{B}$-module with $V = V_0$, then there exist $n \in \mathbb{N}$ and $\lambda \in k$ such that $V$ is isomorphic to $k^n_A$ where $A$ is the Jordan block of size $n$ with eigenvalue $\lambda$. \qed

If $A$ is the Jordan block of size $n$ with eigenvalue $\lambda$, then denote $A_\lambda = k^n_A$.

### 2.5. Indecomposable modules.

Throughout this subsection, $V, X_1, X_2, T$ and $S$ are as in §2.4. When $V$ is indecomposable, we will prove that $T$ has a unique eigenvalue. In order to do this, the following relations are useful.

**Lemma 2.9.** Let $\lambda \in k$, $z := t - \lambda \text{id} \in \mathcal{B}$ and $n \in \mathbb{N}$. Then

$$z^n x_1 = x_1 \sum_{j=0}^{n} \frac{n!}{(n-j)!} s^j z^{n-j}, \quad z^n x_1 x_2 = x_1 x_2 \sum_{j=0}^{n} \frac{n!}{(n-j)!} s^j z^{n-j}.$$

*Proof.* We prove $\clubsuit$ by induction on $n$; the proof of $\lozenge$ is similar. We will use that $x_1 z s^n = x_1 x z^{n+1} + n x_1 s^{n+1}$, which can be verified easily. Note that

$$zx_1 = x_1 z + x_1 x_2 x_1 = x_1 z + x_1 s = x_1 (z + s),$$
and whence the formula is true for \( n = 1 \). Denote \( \zeta_{n,j} := \frac{m!}{(n-1)!}, 0 \leq j \leq n \).
Consider \( n > 1 \) and assume that the formula is true for \( n - 1 \). Then

\[
\sum_{j=0}^{n-1} \zeta_{n-1,j} x^{n-1-j} = \sum_{j=0}^{n-1} \zeta_{n-1,j} x^{n-1-j} + \sum_{j=0}^{n-1} \zeta_{n-1,j} x^{n-2-j} = x_1 \sum_{j=0}^{n} \zeta_{n,j} x^{n-j}.
\]

\[\square\]

Let \( \lambda \) be an eigenvalue of \( T \). Denote \( V_\lambda^T \) the generalized eigenspace of \( V \) associated to \( \lambda \), i.e. \( V_\lambda^T := \bigcup_{j \geq 0} \ker (T - \lambda \text{id})^j \).

**Lemma 2.10.** \( V_\lambda^T \) is a \( B \)-submodule of \( V \), for all eigenvalue \( \lambda \) of \( T \).

**Proof.** Clearly \( V_\lambda^T = \ker (T - \lambda \text{id})^r = \ker (X_2^2 - \lambda \text{id})^r \), where \( r \) is the maximal size of \( \lambda \)-blocks in the Jordan normal form of \( T \). Thus \( V_\lambda^T \) is stable by \( X_2 \). It remains to show that it stable by \( X_1 \). By Lemma 2.9, if \( u \in V_\lambda^T \) then

\[
(T - \lambda \text{id})^n X_1 u = X_1 \sum_{j=0}^{n} \zeta_{j,n} S^j (T - \lambda \text{id})^{n-j} u.
\]

By Lemma 2.1 of [I], \( S \) is nilpotent. Taking \( n \) big enough, it follows that \((T - \lambda \text{id})^n X_1 u = 0\) and whence \( X_1 u \in V_\lambda^T \). \( \square \)

Now Lemma 2.10 implies the next result.

**Theorem 2.11.** Let \( \lambda_1, \ldots, \lambda_t \) be the different eigenvalues of \( T \). Then \( V \) decomposes into the direct sum of the \( B \)-submodules \( V_\lambda^T \).

In particular, if \( V \) is indecomposable then \( T \) has a unique eigenvalue. Hence either \( X_2 \) has a unique eigenvalue or else the eigenvalues of \( X_2 \) are \( \lambda \) and \( -\lambda \), with \( \lambda \in k^\times \). \( \square \)

Given \( \lambda \in k \), denote by \( B \mathcal{M}_\lambda \) the full subcategory of \( B \mathcal{M} \) whose objects are the \( B \)-modules \( V \) such that \( V = \ker (T - \lambda \text{id})^m \), for some \( m \in \mathbb{N}_0 \). With this notation, the next result follows immediately from Theorem 2.11.

**Corollary 2.12.** \( B \mathcal{M} \simeq \prod_{\lambda \in k} B \mathcal{M}_\lambda \). \( \square \)

The next result will be useful in §3.

**Lemma 2.13.** Let \( \Lambda = \{v_1, \ldots, v_n\} \) be a basis of \( V \) such that \( [X_1]_\Lambda = E_{12} \) and \( W \) a one-dimensional \( B \)-submodule of \( V \). Then:

(i) If \( L \) is a complement (as a \( B \)-module) of \( W \) in \( V \) then \( L \cap V_0 = \langle v_1 \rangle \).

(ii) If \( W = \langle v_1 \rangle \) does not have a complement (as a \( B \)-submodule) in \( V \).

**Proof.** (i): Assume that \( W = \langle w \rangle \) and \( \{u_1, u_2, \cdots, u_{n-1}\} \) is a basis of \( L \). Since \( W_0 = W \cap V_0 \neq 0 \), it follows that \( W \subset V_0 \). Using that \( v_2 \) is a linear combination of \( w, u_1, u_2, \cdots, u_{n-1} \) we see that \( v_1 = X_1 v_2 \in L \); hence \( v_1 \in V_0 \cap L \).

(ii): It follows at once from (i). \( \square \)
3. Indecomposable Representations of Dimension 2 and 3

3.1. Dimension 2. In this subsection we describe all 2-dimensional indecomposable representations of $B$. Fix $(V, \rho)$ a 2-dimensional representation of $B$.

Lemma 3.1. If $V \neq V_0$ then $V$ is indecomposable.

Proof. Suppose that $V$ is decomposable, i.e. there are non-trivial submodules $U$ and $W$ such that $V = U \oplus W$. Then $V_0 = U_0 \oplus W_0 = U \oplus W = V$. □

Define representations of $B$ on the vector space $k^2$ given by $X_1 = E_{12}$ and the following action of $x_2$:

- $X_2 = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$, $a, b \in k$. This is denoted by $U_{a,b}$.
- $X_2 = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}$, $a \in k^\times$. This is denoted by $V_a$.

It is easy to check that these are indecomposable modules pairwise non-isomorphic.

Theorem 3.2. Every 2-dimensional indecomposable representation of $B$ is isomorphic either to $U_{a,b}$, or to $V_a$, or to $k^2_\lambda$ for unique $a, b, \lambda \in k$.

This confirms Theorem 2.11.

Proof. If $V = V_0$, then Proposition 2.8 applies. Assume that $V_0 \neq 0$; then there exists a basis $\Lambda = \{v_1, v_2\}$ of $V$ such that $[X_1]_\Lambda = E_{12}$. Let $[X_2]_\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then (2.4) is satisfied if and only if

$$c(a + d) = 0 \quad \text{and} \quad d^2 + c = a^2.$$  

Suppose that $c \neq 0$. Then by the first equation it follows that $d = -a$. Replacing in the second equation we have that $c = 0$, which is a contradiction. Therefore $c = 0$ and consequently $d = a$ or $d = -a$.

If $d = a$ then $V \simeq U_{a,b}$. Assume that $d = -a \neq 0$ and take $w_1 = v_1$ and $w_2 = \frac{1}{2a}v_1 + v_2$. Then $\Omega = \{w_1, w_2\}$ is a basis of $V$ such that $[X_1]_\Omega = E_{12}$ and $[X_2]_\Omega = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}$. Thus $V \simeq V_a$. □

Corollary 3.3. If $\text{Ext}^1(k_\lambda^1, k_\lambda^1) \neq 0$ then $a = \pm b$. □

3.2. Dimension 3. Let $V$ be a $B$-module of dimension 3 such that $V \neq V_0$. Throughout this subsection, $\Lambda = \{v_1, v_2, v_3\}$ denotes a basis of $V$ such that $[X_1]_\Lambda = E_{12}$. We define four families of representations of $B$ on the vector
space \( V \) determined by the following action of \([X_2]_\Lambda\), for all \( a, b, c, d, e \in k \):

\[
\Theta_1 : \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & a^2 - d^2 & -d \end{pmatrix}, \quad \Theta_2 : \begin{pmatrix} a & b & c \\ 0 & a & 0 \\ 0 & d & e \end{pmatrix};
\]

\[
\Theta_3 : \begin{pmatrix} a & b & \frac{c^2-a^2}{d} \\ 0 & c & 0 \\ d & e & -a \end{pmatrix}, \quad \Theta_4 : \begin{pmatrix} a & b & c \\ 0 & -a & 0 \\ 0 & d & e \end{pmatrix}, \quad a, b, c, d, e \in k^\times.
\]

**Lemma 3.4.** The families \( \Theta_1, \Theta_2, \Theta_3 \) and \( \Theta_4 \) contain all 3-dimensional representations of \( B \), up to isomorphism.

**Proof.** Let \([X_2]_\Lambda = \begin{pmatrix} \alpha & \beta & \gamma \\ \delta & \epsilon & \zeta \\ \eta & \theta & \iota \end{pmatrix}\). Then (2.4) is valid if and only if

\[
\begin{align*}
\delta(\alpha + \epsilon) &= -\zeta \eta \\
\zeta(\epsilon + \iota) &= -\gamma \delta \\
\eta(\alpha + \iota) &= -\delta \theta \\
e^2 - \alpha^2 &= -\delta + \gamma \eta - \zeta \theta
\end{align*}
\]

(3.1)

**Claim:** If the system (3.1) has solution then \( \delta = 0 \).

Assume that \( \delta \neq 0 \). If \( \zeta = 0 \) then \( \gamma = 0 \) and \( \epsilon = -\alpha \). Thus, \( \delta = 0 \) which is a contradiction. If \( \zeta \neq 0 \) then

\[
\gamma = -\frac{\zeta(\epsilon + \iota)}{\delta}, \quad \eta = -\frac{\delta(\alpha + \epsilon)}{\zeta} \quad \text{and} \quad \theta = \frac{(\alpha + \epsilon)(\alpha + \iota)}{\zeta}.
\]

From the last equation of (3.1), \( \delta = 0 \) which is again a contradiction.

Assume \( \delta = 0 \). Thus \( \zeta \eta = 0 \). If \( \zeta \neq 0 \) then \( \eta = 0 \), \( \iota = -\epsilon \) and \( \theta = \frac{\alpha^2 - \epsilon^2}{\zeta} \). Hence \( V \) belongs to the family \( \Theta_1 \). When \( \zeta = 0 \) and \( \eta \neq 0 \), it follows that \( \iota = -\alpha \) and \( \gamma = \frac{\epsilon^2 - \alpha^2}{\eta} \). Thus, \( V \) belongs to the family \( \Theta_3 \). If \( \zeta = 0 \) and \( \eta = 0 \) then \( \epsilon = |\alpha| \). In this case, \( V \) belongs to the families \( \Theta_2 \) or \( \Theta_4 \). \( \square \)

**Remark 3.5.** Let \( L \) a \( B \)-submodule of \( V \) of dimension 2 such that \( L \cap V_0 \) is one-dimensional. Fix \( \overline{L} := L/(L \cap V_0) = \langle \overline{v}_1 \rangle \). Since \( u \notin V_0 \), we can suppose that \( u = \alpha v_1 + v_2 + \gamma v_3 \in L \), with \( \alpha, \gamma \in k \).

**Proposition 3.6.** Let \( V \) be a \( B \)-module. Then:

(i) the representations in the family \( \Theta_1 \) are always indecomposable;

(ii) a representation in the family \( \Theta_2 \) is indecomposable if and only if \( c \neq 0 \) and \( e = a \) or \( d \neq 0 \) and \( e = a \);

(iii) the representations in the family \( \Theta_3 \) are always indecomposable;

(iv) a representation in the family \( \Theta_4 \) is indecomposable if and only if \( c \neq 0 \) and \( e = a \) or \( d \neq 0 \) and \( e = -a \).

**Proof.** (i): The unique one-dimensional \( B \)-submodule of \( V \) is \( \langle v_1 \rangle \) which does not have complement by Lemma 2.13 (ii).
(ii): Let $V$ be a representation of $\mathcal{B}$ of the type $\Theta_2$. Suppose that $W = \langle w \rangle$ is a one-dimensional $\mathcal{B}$-submodule of $V$. Since $W \subset V_0$, see §2.4, $w = \alpha v_1 + \beta v_3$, with $\alpha, \beta \in \mathbb{k}$. Note that $X_2w = \gamma w$, $\gamma \in \mathbb{k}$, if and only if $\beta(\gamma - e) = 0$ and $\alpha(\gamma - a) = \beta c$. Consequently, the one-dimensional $\mathcal{B}$-submodules of $V$ are:

- $\langle v_1 \rangle, \langle v_3 \rangle, c = 0, e \neq a$,
- $\langle \alpha v_1 + \beta v_3 \rangle, c = 0, e = a$,
- $\langle v_1, v_1 + \frac{e-a}{c} v_3 \rangle, c \neq 0, e \neq a$,
- $\langle v_1 \rangle, c \neq 0, e = a$.

Assume $e \neq a$. If $c \neq 0$, $V = \langle v_1 + \frac{e-a}{c} v_3 \rangle \oplus \langle v_1, v_1 + v_2 + \frac{d}{a-e} v_3 \rangle$. If $c = 0$, $V = \langle v_3 \rangle \oplus \langle v_1, v_1 + v_2 + \frac{d}{a-e} v_3 \rangle$. If $c = d = 0$, $V = \langle v_1, v_2 \rangle \oplus \langle v_3 \rangle$. Hence, $V$ is decomposable.

Conversely, suppose $e = a$ and $c \neq 0$. Then the unique one-dimensional $\mathcal{B}$-submodule of $V$ is $\langle v_1 \rangle$ which does not have complement. Suppose that $e = a$ and $d \neq 0$. Assume that $W$ is a one-dimensional $\mathcal{B}$-submodule of $V$ which admits a complement $L = \langle u_1, u_2 \rangle$. Then by Lemma 2.13 (ii), $v_1 \in L \cap V_0$. By Remark 3.5, $\mathcal{L} = \langle \mathfrak{m} \rangle$ where $u = \alpha v_1 + v_2 + \beta v_3$. Thus $X_2 \mathfrak{m} = \gamma \mathfrak{m}$, $\gamma \in \mathbb{k}$, if and only if $\gamma = a$ and $\beta(a - e) = d$. Since $d \neq 0$ and $e = a$ then $W$ does not have complement in $V$.

(iii): Suppose that $W = \langle w \rangle$ is a one-dimensional $\mathcal{B}$-submodule of $V$ which admits a complement $L$. Then by Lemma 2.13 (ii), $\langle v_1 \rangle = L \cap V_0$, which is a contradiction because $d \neq 0$.

(iv): Analogous to item (ii). \hfill \square

### 3.2.1. Isomorphism classes in $\Theta_1$.

Assume $V$ in the family $\Theta_1$. We distinguish: for all $a, b, c, d, e \in \mathbb{k}$

- $X_2 = \begin{pmatrix} a & b & c \\ 0 & d & e \\ \frac{a^2 - c^2}{e} & -d \end{pmatrix}$, $e \in \mathbb{k}^\times$. This is denoted by $\mathcal{Y}_{a,b,c,d,e}$.
- $X_2 = \begin{pmatrix} a & b & 0 \\ 0 & a & 1 \\ 0 & 0 & -a \end{pmatrix}$. This is denoted by $\mathcal{U}^{a,b}$.

By Proposition 3.6 (i), these representations are indecomposable. Note that $\mathcal{U}^{a,b} = \mathcal{Y}_{a,b,0,a,1}$.

### Proposition 3.7.

Every 3-dimensional indecomposable representation $V$ of $\mathcal{B}$ in $\Theta_1$ is isomorphic either to $\mathcal{U}^{a,b}$, or to $\mathcal{Y}_{a,b,c,d,e}$. Moreover,

$$\mathcal{Y}_{a,b,c,d,e} \simeq \mathcal{Y}_{a',b',c',d',e'} \text{ if and only if } (a - d') \frac{e' - c' e}{e'} = e(b' - b) + c(d' - d).$$

In particular, $\mathcal{U}^{a,b} \simeq \mathcal{U}^{a,b'}$ if and only if $b = b'$.

**Proof.** Since $\langle X_2 v_1 \rangle = \text{Im} X_1$, we obtain that $a$ is invariant. Consider the indecomposable representation $\mathcal{Y}_{a,b',c',d',e'}$ of $\mathcal{B}$. If $d' = a$, taking the basis $\{v_1 + \frac{e'}{c'}v_1 + v_2, \frac{1}{c'}v_3\}$ we conclude that $\mathcal{Y}_{a,b',c',d',e'} \simeq \mathcal{U}^{a,b'}$. 

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Note that $Y_{a,b,c,d,e}$ and $Y_{a',b',c',d',e'}$ are isomorphic if and only if there exists a basis $\{w_1, w_2, w_3\}$ of $V$ such that $X_1 w_1 = X_1 w_3 = 0$, $X_1 w_2 = w_1$, $X_2 w_1 = aw_1$, $X_2 w_2 = b' w_1 + d' w_2 + \frac{a^2 - d'^2}{e'} w_3$ and $X_2 w_3 = c' w_1 + e' w_2 - d' w_3$. Since $\langle v_1 \rangle = \text{Im} X_1$ and $V_0$ has dimension 2, then we can consider $w_1 = v_1$, $w_2 = \lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3$ and $w_3 = \beta_1 v_1 + \beta_3 v_3$, $\lambda_1, \lambda_2, \lambda_3, \beta_1, \beta_3 \in k$. Then, $Y_{a,b,c,d,e} \simeq Y_{a',b',c',d',e'}$ if and only if $(a - d') \frac{ce' - d'c}{e'} = e(b' - b) + c(d' - d)$. 

\[ \Theta \]

3.2.2. Isomorphism classes in $\Theta_2$. Consider $V$ in the family $\Theta_2$ and the following distinguish representations: for all $a \in k$

- $\bigotimes_2 = \left( \begin{array}{ccc} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{array} \right)$. This is denoted by $\mathcal{R}_a$.
- $\bigotimes_2 = \left( \begin{array}{ccc} a & 0 & 1 \\ 0 & a & 0 \\ 0 & 0 & a \end{array} \right)$. This is denoted by $\mathcal{S}_a$.
- $\bigotimes_2 = \left( \begin{array}{ccc} a & 0 & b \\ 0 & a & 0 \\ 0 & c & a \end{array} \right)$, $b \in k^\times$ or $c \in k^\times$. This is denoted by $\mathcal{T}_{a,b,c}$.

By Proposition 3.6 (ii), these are indecomposable representations. Notice that $\mathcal{R}_a = \mathcal{T}_{a,0,1}$ and $\mathcal{S}_a = \mathcal{T}_{a,1,0}$.

**Proposition 3.8.** Every 3-dimensional indecomposable representation $V$ of $B$ in $\Theta_2$ is isomorphic either to $\mathcal{R}_a$, or to $\mathcal{S}_a$ or to $\mathcal{T}_{a,b,c}$. Moreover, $\mathcal{T}_{a,b,c}$ and $\mathcal{T}_{a',b',c'}$ are isomorphic if and only if $bc = b'c'$.

**Proof.** Let $V'$ the representation of $B$ given by

$$[X_2]_A = \left( \begin{array}{ccc} a & d' & b' \\ 0 & a & 0 \\ 0 & c' & a \end{array} \right).$$

If $b' = 0$, then by Proposition 3.6 (ii) we have that $c' \neq 0$. In this case, taking the basis $\{v_1, v_2, d' v_1 + c' v_3\}$ of $V'$, we conclude that $V' \simeq \mathcal{R}_a$. Similarly, if $c' = 0$ then $b' \neq 0$. Taking the basis $\{v_1, v_2 - \frac{d'}{c'} v_3, \frac{1}{b'} v_3\}$ of $V'$, we obtain that $V' \simeq \mathcal{S}_a$. If $b, b', c, c' \in k^\times$, taking the basis $\{v_1, v_2, \frac{d'}{c'} v_1 + v_3\}$, it follows that $V' \simeq \mathcal{T}_{a,b,c'}$.

Finally, notice that $\mathcal{T}_{a,b,c} \simeq \mathcal{T}_{a,b',c'}$ if and only if there exists a basis $\{w_1, w_2, w_3\}$ of $k^3$ such that $X_1 w_1 = X_1 w_3 = 0$, $X_1 w_2 = w_1$, $X_2 w_1 = aw_1$, $X_2 w_2 = aw_2 + cw_3$ and $X_2 w_3 = bw_1 + aw_3$. We can assume $w_1 = v_1$, $w_2 = \lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3$ and $w_3 = \beta_1 v_1 + \beta_3 v_3$, $\lambda_1, \lambda_2, \lambda_3, \beta_1, \beta_3 \in k$. Note that $X_2 w_2 = w_1$ if and only if $\lambda_2 = 1$. Moreover, $X_2 w_2 = aw_2 + cw_3$ and $X_2 w_3 = bw_1 + aw_3$ if and only if $bc = b'c'$. 

3.2.3. Isomorphism classes in $\Theta_3$. Consider $V$ in the family $\Theta_3$ and the following distinguished representations: for all $a, b, c, d, e \in k$
Proof. Let \( B \). Every Proposition 3.10. By Proposition 3.6 (iv), these are indecomposable representations pairwise distinguish representations: for all \( a \). Isomorphism classes in \( X \). Taking the basis \( \{ 3 \} \) such that \( = \). Note that \( k \). In particular, \( X \) \( \) if and only if \( \). With this choose of \( w \) \( v \) \( X \) \( \). However \( X \) \( \) if and only if \( \). Finally, \( X \) \( \) if and only if \( d \). □

3.2.4. Isomorphism classes in \( \Theta_4 \). Consider \( V \) in \( \Theta_4 \) and the following distinguish representations: for all \( a \in k^\times \)

\( X_2 = \begin{pmatrix} a & 0 & 1 \\ 0 & -a & 0 \\ 0 & 0 & a \end{pmatrix} \). This is denoted by \( Y^a \).

\( X_2 = \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 1 & -a \end{pmatrix} \). This is denoted by \( Y_a \).

By Proposition 3.6 (iv), these are indecomposable representations pairwise non-isomorphic.

Proposition 3.10. Every 3-dimensional indecomposable representation \( V \) of \( B \) in \( \Theta_4 \) is isomorphic either to \( Y^a \) or to \( Y_a \) for unique \( a \in k^\times \).

Proof. Let \( V' \) be a 3-dimensional indecomposable representation of \( B \) such that

\[ [X_2]_\Lambda = \begin{pmatrix} a & b & c \\ 0 & -a & 0 \\ 0 & d & e \end{pmatrix}, \ a \in k^\times.\]
Since $V'$ is indecomposable, by Proposition 3.6 (iv) we have that $c \neq 0$ and $e = a$ or $d \neq 0$ and $e = -a$. If $c \neq 0$ and $e = a$, taking the basis 
\{v_1, \frac{cd-2ab}{da^2}v_1 + v_2 - \frac{a}{2a}v_3, v_1 + \frac{1}{c}v_3\} of $V'$, we obtain $V' \simeq \Lambda a$. If $d \neq 0$ and $e = -a$, taking the basis \{v_1, -\frac{2ab+ad}{da}v_1 + v_2, -\frac{dc}{2a}v_1 + dv_3\} of $V'$, it follows that $V' \simeq \Lambda a$. \hfill \Box

3.2.5. Classification of indecomposable 3-dimensional $B$-modules.

**Theorem 3.11.** Every 3-dimensional indecomposable $B$-module is isomorphic either to $\mathbb{k}^3\Lambda$ for a unique $\lambda$, or else to a representation in one of the families $\Theta_j$, $j = 1, 2, 3, 4$, with the constraints described in Proposition 3.6. The isomorphism classes are described in Propositions 3.7, 3.8, 3.9 and 3.10.

Again, this agrees with Theorem 2.11.

**Remark 3.12.** It is straightforward to verify that two 3-dimensional indecomposable representations of $B$ that belong to different families $\Theta_i$, $i = 1, 2, 3, 4$, are not isomorphic.

4. Families of indecomposable $B$-modules

Throughout this section $(V, \rho)$ is an $n$-dimensional representation of $B$, $\Lambda = \{v_1, \ldots, v_n\}$ is a basis of $V$, $X_1 = \rho(x_1)$, $X_2 = \rho(x_2)$ and $[X_1]_{\Lambda} = E_{12}$.

4.1. The family $U_a$. Let $a \in \mathbb{k}$. Consider the following action of $X_2$ on $V$:

\[
[X_2]_{\Lambda} = \begin{pmatrix}
a & 0 & 0 & 0 & \ldots & 0 & 0 \\
0 & a & 0 & 0 & \ldots & 0 & 0 \\
0 & 1 & a & 0 & \ldots & 0 & 0 \\
0 & 0 & 1 & a & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 1 & a & 0 \\
0 & 0 & 0 & \ldots & 0 & 1 & a
\end{pmatrix}.
\]

Clearly $V$ with this action is a $B$-module which will be denoted by $U_a$.

**Lemma 4.1.** Let $W$ be a proper $B$-submodule of $U_a$. Then:

(i) $v_2 \notin W$;

(ii) If $v = \sum_{i=1}^{n} \lambda_i v_i \in W$ then $\lambda_2 = 0$.

**Proof.** (i): Suppose $v_2 \in W$. Then $v_1 = X_1 v_2 \in W$ and $X_2 v_2 = av_2 + v_3 \in W$. Hence $v_3 \in W$. Again, $X_2 v_3 = av_3 + v_4 \in W$ and consequently $v_4 \in W$. With this procedure, we obtain that $\Lambda \subset W$. Thus, $W = U_a$ and we have a contradiction.

(ii): Assume $\lambda_2 \neq 0$ and fix $w_1 = \lambda_2^{-1} v$. Thus $w_1 = \alpha_1 v_1 + v_2 + \ldots + \alpha_n v_n$, where $\alpha_i = \lambda_2^{-1} \lambda_i$, for all $1 \leq i \leq n$. Consider the following elements of $V$:

\[w_j := v_{j+1} + \alpha_3 v_{j+2} + \ldots + \alpha_{n-j+1} v_n, \text{ for all } 2 \leq j \leq n - 2.\]
By a straightforward calculation, we obtain that $X_2w_j = aw_j + w_{j+1}$, for all $1 \leq j \leq n - 2$. Thus, $w_1, \ldots, w_{n-2} \in \mathcal{W}$ and $X_2w_{n-2} = aw_{n-2} + v_n$. Therefore, $v_n \in \mathcal{W}$. But $w_{n-2} = v_{n-1} + \alpha_3v_n$ and whence $v_{n-1} \in \mathcal{W}$. By this procedure, it follows that $v_3, \ldots, v_n \in \mathcal{W}$. From $v_1 = X_1w_1 \in \mathcal{W}$, it follows that $v_2 \in \mathcal{W}$ which contradicts (i).

**Theorem 4.2.** $\mathcal{U}_a$ is an indecomposable $\mathcal{B}$-module, for all $n \geq 2$.

**Proof.** Suppose $\mathcal{U}_a$ decomposable. Let $\mathcal{W}$, $\widetilde{\mathcal{W}}$ be nontrivial $\mathcal{B}$-submodules of $\mathcal{U}_a$ such that $\mathcal{U}_a = \mathcal{W} \oplus \widetilde{\mathcal{W}}$. Consider $\{w_1, \ldots, w_r\}$ and $\{w_{r+1}, \ldots, w_n\}$ basis of $\mathcal{W}$ and $\widetilde{\mathcal{W}}$ respectively. By Lemma 4.1, $w_i = \lambda_iw_1 + \lambda_i2w_2 + \ldots + \lambda_inw_n$, for all $1 \leq i \leq n$. Since $v_2 \in \mathcal{U}_a$, there exist $\alpha_1, \ldots, \alpha_n \in k$ such that $v_2 = \alpha_1w_1 + \ldots + \alpha_nw_n$, a contradiction. \(\square\)

4.2. The family $\mathcal{V}_a$. Let $a \in k^X$. Consider the following action of $X_2$ on $V$:

$$[X_2]_\lambda = \begin{pmatrix} a & 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & -a & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & -a & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & -a & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & -a & 0 \\ 0 & 0 & 0 & \ldots & 0 & 1 & -a \end{pmatrix}.$$  

Notice that $V$ is a $\mathcal{B}$-module which will be denoted by $\mathcal{V}_a$. Since $a \neq 0$, $\mathcal{U}_a$ and $\mathcal{V}_a$ are not isomorphic.

**Theorem 4.3.** $\mathcal{V}_a$ is an indecomposable $\mathcal{B}$-module, for all $n \geq 2$.

**Proof.** Let $\mathcal{W}$ a proper $\mathcal{B}$-submodule of $\mathcal{V}_a$. As in Lemma 4.1 (i), we can show that $v_2 \notin \mathcal{W}$. Let $v \in \mathcal{W}$ such that $v = \sum_{i=1}^{n} \lambda_iv_i$. Assume that $\lambda_2 \neq 0$ and consider $u := \lambda_2^{-1}v \in \mathcal{W}$. Then $v_1 = X_1u \in \mathcal{W}$. Take $w_1 := u - \lambda_2^{-1}\lambda_1v_1$ and note that $w_1 = \alpha_2v_2 + \ldots + \alpha_nv_n$, where $\alpha_i = \lambda_2^{-1}\lambda_i$, for all $2 \leq i \leq n$. Considering the following elements of $V$

$$w_j := v_{j+1} + \alpha_3v_{j+2} + \ldots + \alpha_{n-j+1}v_n,$$

it follows $X_2w_j = -aw_j + w_{j+1}$, for all $1 \leq j \leq n - 2$. As in Lemma 4.1, this implies that $v_2 \in \mathcal{W}$ which is a contradiction. Thus, the result follows as in Theorem 4.2. \(\square\)

**References**


FAAMF-Universidad Nacional de Córdoba, CIEM (CONICET), Medina Allende s/n, Ciudad Universitaria, 5000 Córdoba, República Argentina.

E-mail address: andrus@famaf.unc.edu.ar

Departamento de Matemática, Universidade Federal de Santa Maria, 97105-900, Santa Maria, RS, Brazil.

E-mail address: bagio@smail.ufsm.br, saradia.flora@ufsm.br, flores@ufsm.br