On finite-dimensional pointed Hopf algebras over simple groups

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Plan of the talk.

I. The problem.

II. Main results.

III. The schemes of the proofs.

I. The problem. $\mathbb{C}$ alg. closed field char. 0

Definition. (H, m, Δ), **Hopf algebra:**

- (H, m) alg. with unit 1,
- $\Delta : H \to H \otimes H$ morphism of algebras **(coproduct)**,
- Δ coasociative with counit ε,
- $\exists \mathcal{S} : H \to H$ **"antipode"** s. t.

$$m(\mathcal{S} \otimes \mathrm{id})\Delta = \mathrm{id}_H = m(\mathrm{id} \otimes \mathcal{S})\Delta.$$

Examples.

• $\mathit{\Gamma}$ group, $\mathbb{C}\mathit{\Gamma}$ = group algebra = vector space with basis e_g $(g \in \mathit{\Gamma})$ and product $e_g e_h = e_{gh}$.

It becames a Hopf algebra with **coproduct** $\Delta(e_g) = e_g \otimes e_g$ and **antipode** $\mathcal{S}(e_g) = e_g^{-1}$

• $\mathfrak{g}$ Lie algebra, $U(\mathfrak{g})$ = universal algebra enveloping of $\mathfrak{g}$

It becames a Hopf algebra with **coproduct** $\Delta(x) = x \otimes 1 + 1 \otimes x$ and **antipode** $\mathcal{S}(x) = -x$, $x \in \mathfrak{g}$

Suppose now that the group Γ acts on a Lie algebra $\mathfrak{g}$ by Lie algebra automorphisms.

Let $U(\mathfrak{g})\#\mathbb{C}\Gamma = U(\mathfrak{g})\otimes\mathbb{C}\Gamma$ as vector space, with **tensor product comultiplication** and **semi-direct product multiplication**. This is a Hopf algebra.

Theorem. (Cartier, Kostant, Milnor-Moore).

If H is a *cocommutative* Hopf algebra, $H \simeq U(\mathfrak{g})\#\mathbb{C}\Gamma$.

Definition. A Hopf algebra H is *pointed* if any irreducible comodule (= representation of the dual Hopf algebra) has dimension 1.

For any Hopf algebra H,

$$G(H) = \{g \in H - 0 : \Delta(g) = g \otimes g\}$$

is a group. Then *'pointed'* means $\mathbb{C}G(H) \simeq$ the coradical of H.

- $H = U(\mathfrak{g}) \# \mathbb{C}\Gamma$ is pointed with $G(H) \simeq \Gamma$; in particular, the group algebra $\mathbb{C}\Gamma$ is pointed;

- the quantum groups of Drinfeld-Jimbo and the finite-dimensional variations of Lusztig are pointed.

An important part of the classification of all finite-dimensional Hopf algebras over $\mathbb{C}$ is the following.

Problem I. Classify all finite-dimensional *pointed* Hopf algebras.

Approach group-by-group. For a given finite group Γ, classify all fin.-dim. *pointed* Hopf algebras H such that $G(H) \simeq \Gamma$.

• If Γ is abelian and the prime divisors of Γ are > 5, then the classification is known. N. A. and H.-J. Schneider, *On the classification of finite-dimensional pointed Hopf algebras*, Ann. Math., to appear.

In this talk we shall consider Γ simple.

A landmark in mathematics is the classification of finite simple groups: any finite simple group is isomorphic to one of

- a cyclic group $\mathbb{Z}_p$, of prime order p;

- an alternating group $\mathbb{A}_n$, $n \geq 5$;

- a finite group of Lie type;

- a sporadic group – there are 27 of them (including the Tits group); the most prominent is the Monster.

II. Main results.

We shall say that a finite group Γ *collapses* if for any fin.-dim. pointed Hopf algebra H, with $G(H) \simeq \Gamma$, then $H \simeq \mathbb{C}\Gamma$.

Theorem I. If Γ is either

- the alternating group $\mathbb{A}_n$, $n \geq 5$;

- or else a sporadic simple group, different from the Fischer groups Fi_{22}, Fi_{23}, the Baby Monster B, or the Monster M;

then Γ collapses.

Comments.

The groups Fi_{22}, Fi_{23}, B and M do not admit so far any fin.-dim. pointed Hopf algebra (except the group algebra) but the computations are not finished yet.

We are working in the same problem for finite groups of Lie type.

The proof lies on reductions to questions on conjugacy classes; then thess questions are checked (for sporadic groups) using GAP.

In order to state our other main results, and also to explain these reductions, we need to present the notion of Nichols algebra.

Nichols algebras.

Suppose that the group Γ acts linearly on a vector space V, hence on the free Lie algebra $L(V)$ by Lie algebra automorphisms. Since $U(L(V)) \simeq T(V)$, we have the Hopf algebra $T(V)\#\mathbb{C}\Gamma$.

Variation.

What else is needed to have Hopf algebra $T(V)\#\mathbb{C}\Gamma = T(V)\otimes\mathbb{C}\Gamma$ as vector space, with with **semi-direct comultiplication** and **semi-direct product multiplication**?

Definition. A *Yetter-Drinfeld module* over Γ is a vector space V provided with

- a linear action of Γ,
- a Γ-grading $V = \oplus_{g\in\Gamma} V_g$,

such that $h \cdot V_g = V_{hgh^{-1}}$, for all $g, \in \Gamma$.

Then $T(V)\#\mathbb{C}\Gamma = T(V)\otimes\mathbb{C}\Gamma$ as vector space, with with **semi-direct product comultiplication** and **semi-direct product multiplication** is a Hopf algebra.

Here $\Delta(v) = v\otimes 1 + g\otimes v$, $g \in \Gamma$, $v \in V_g$.

If V is a Yetter-Drinfeld module over Γ, then $H = T(V)\#\mathbb{C}\Gamma$ is pointed with $G(H) \simeq \Gamma$.

Fact. There exists a homogeneous ideal $\mathcal{J} = \oplus_{n\geq 2}\mathcal{J}^n$ of $T(V)$ such that

- the quotient $T(V)\#\mathbb{C}\Gamma/\mathcal{J}\#\mathbb{C}\Gamma$ is a Hopf algebra,
- $\mathcal{J}$ is maximal with respect to this property.

The *Nichols algebra* is $\mathfrak{B}(V) = T(V)/\mathcal{J}$.
If Γ is finite and $\dim \mathfrak{B}(V) < \infty$, then

$$T(V)\#\mathbb{C}\Gamma/\mathcal{J}\#\mathbb{C}\Gamma \simeq \mathfrak{B}(V)\#\mathbb{C}\Gamma$$

is a fin.-dim. pointed Hopf algebra over Γ.

Conversely, if H is a fin.-dim. pointed Hopf algebra over a finite group Γ, then there exists a Yetter-Drinfeld module V canonically attached to H s. t. $\dim \mathfrak{B}(V) < \infty$.

Problem II. For a given finite group Γ, classify all Yetter-Drinfeld modules V s. t. $\dim \mathfrak{B}(V) < \infty$.

Since any Yetter-Drinfeld module is semisimple, the question splits into two cases:

(i) V irreducible,

(ii) V direct sum of (at least 2) irreducibles.

It turns out that irreducible Yetter-Drinfeld modules are parameterized by pairs $(\mathcal{O}, \rho)$, where

- $\mathcal{O}$ a conjugacy class of G,
- ρ an irreducible repr. of the centralizer $C_G(\sigma)$ of $\sigma \in \mathcal{O}$ fixed.

If $M(\mathcal{O}, \rho)$ denotes the irreducible Yetter-Drinfeld module corresponding to a pair $(\mathcal{O}, \rho)$ and V is the vector space affording the representation ρ, then $M(\mathcal{O}, \rho) \simeq \mathrm{Ind}_{C_G(\sigma)}^G \rho$ with the grading given by the identification $\mathrm{Ind}_{C_G(\sigma)}^G \rho = \mathbb{C}G \otimes_{C_G(\sigma)} V \simeq \mathbb{C}\mathcal{O} \otimes_{\mathbb{C}} V$.

The Nichols algebra of $M(\mathcal{O}, \rho)$ is denoted $\mathfrak{B}(\mathcal{O}, \rho)$.

Problem II bis. For a given finite group Γ, find all $(\mathcal{O}, \rho)$ s. t. $\dim \mathfrak{B}(\mathcal{O}, \rho) < \infty$.

Theorem II. Let $m \geq 5$. Let $\sigma \in \mathbb{S}_m$ be of type $(1^{n_1}, 2^{n_2}, \dots, m^{n_m})$, $\mathcal{O}$ the conjugacy class of σ; let $\rho \in \widehat{C_{\mathbb{S}_m}(\sigma)}$. If $\dim \mathfrak{B}(\mathcal{O}, \rho) < \infty$, then the type of σ and ρ are in the following list:

- $(1^{n_1}, 2)$, $\rho_1 = \text{sgn}$ or ϵ, $\rho_2 = \text{sgn}$.
- $(2, 3)$ in $\mathbb{S}_5$, $\rho_2 = \text{sgn}$, $\rho_3 = \overrightarrow{\chi_0}$.
- (2^3) in $\mathbb{S}_6$, $\rho_2 = \overrightarrow{\chi_1} \otimes \epsilon$ or $\overrightarrow{\chi_1} \otimes \text{sgn}$.

Actually, the orbit of type $(1^4, 2)$ in $\mathbb{S}_6$ is isomorphic to that of type (2^3), because of the outer automorphism of $\mathbb{S}_6$.

The Nichols algebras corresponding to types $(1^{n_1}, 2)$, and these representations, were considered by Fomin and Kirillov in relation with the quantum cohomology of the flag variety. They can not be treated by our methods.

Example: $\mathcal{O}=$ transpositions in $G=\mathbb{S}_n$,
$s=(12)$, $\rho=$ sgn

n	**rk**	**Relations**	$\dim \mathfrak{B}(V)$	**top**
3	3	5 relations in degree 2	$12 = 3.2^2$	$4 = 2^2$
4	6	16 relations in degree 2	576	12
5	10	45 relations in degree 2	8294400	40

$\mathbb{S}_3$, $\mathbb{S}_4$: A. Milinski & H. Schneider, Contemp. Math. **267** (2000), 215–236.

S. Fomin & K. Kirillov, Progr. Math. **172**, Birkhauser, (1999), pp. 146–182.

$\mathbb{S}_5$: [FK], plus web page of M. Graña. http://mate.dm.uba.ar/~matiasg/

$\mathbb{S}_n$, $n \geq 6$: open!

It is convenient to restate Problem II in terms of racks and cocycles.

For this, recall first that a *braided vector space* is a pair (V, c), where V is a vector space and $c \in \mathrm{GL}(V \otimes V)$ is a solution of the braid equation:

$$(c \otimes \mathrm{id})(\mathrm{id} \otimes c)(c \otimes \mathrm{id}) = (\mathrm{id} \otimes c)(c \otimes \mathrm{id})(\mathrm{id} \otimes c).$$

Any Yetter-Drinfeld module V is naturally a braided vector space:

$$c(v \otimes w) = g \cdot w \otimes v,$$

for $v \in V_g$ $(g \in G)$, $w \in V$.

If V is a Yetter-Drinfeld module, then the Nichols algebra $\mathfrak{B}(V)$ **depends only on the braiding c.**

This leads us to the consideration of a class of braided vector spaces where the study of the corresponding Nichols algebras is performed in a unified way.

Definition. A *rack* is a pair $(X, \triangleright)$ where X is a non-empty set and $\triangleright : X \times X \to X$ is an operation such that

- the map $\varphi_x = x \triangleright \_\_$ is invertible for any $x \in X$,
- $x \triangleright (y \triangleright z) = (x \triangleright y) \triangleright (x \triangleright z)$ for all $x, y, z \in X$.

Examples and basic notions.

• A group Γ is a rack with $x \triangleright y = xyx^{-1}$, $x, y \in \Gamma$. If $X \subset G$ is stable under conjugation by G, that is a union of conjugacy classes, then it is a subrack of G.

• A rack X is *abelian* iff $x \triangleright y = y$, $x, y \in \Gamma$.

• A rack X is *decomposable* iff there exist disjoint subracks $X_1, X_2 \subset X$ s. t. $X_i \triangleright X_j = X_j$, $1 \leq i, j \leq 2$ and $X = X_1 \coprod X_2$. Otherwise, X is *indecomposable*.

• A rack X is *simple* iff $\operatorname{card} X > 1$ and for any surjective morphism of racks $\pi : X \to Y$, either π is a bijection or $\operatorname{card} Y = 1$.

Cocycles. Let X be a rack, $n \in \mathbb{N}$. A map $q : X \times X \to \mathrm{GL}(n, \mathbb{C})$ is a *principal 2-cocycle of degree* n if

$$q_{x,y \triangleright z} q_{y,z} = q_{x \triangleright y, x \triangleright z} q_{x,z},$$

for all $x, y, z \in X$.

Here is an equivalent formulation: let $V = \mathbb{C}X \otimes \mathbb{C}^n$ and consider the linear isomorphism $c^q : V \otimes V \to V \otimes V$,

$$c^q(e_x v \otimes e_y w) = e_{x \triangleright y} q_{x,y}(w) \otimes e_x v,$$

$x \in X$, $y \in X$, $v \in \mathbb{C}^n$, $w \in \mathbb{C}^n$. Then q is a 2-cocycle iff c^q is a solution of the braid equation.

If this is the case, then the Nichols algebra of (V, c^q) is denoted $\mathfrak{B}(X, q)$.

Define g_x by $g_x(e_y w) = e_{x \triangleright y} q_{x,y}(w)$, $x \in X$, $y \in X$, $v \in \mathbb{C}^n$.

Fact. Let X be an indecomposable finite rack and q a 2-cocycle as above. If $\Gamma \subset$ GL(V) is the subgroup generated by $(g_x)_{x \in X}$, then V is a Yetter-Drinfeld module over Γ. If the image of q generates a finite subgroup, then Γ is finite.

Conversely, if Γ is finite and $V = M(\mathcal{O}, \rho)$ is a Yetter-Drinfeld module over Γ with the rack $\mathcal{O}$ indecomposable, then there exist a principal 2-cocycle q such that V is given as above and the braiding $c \in$ Aut$(V \otimes V)$ as Y.-D. module coincides with c^q.

There is a version of this result for *decomposable* in terms of *non-principal 2-cocycles*.

We can now reformulate Problem II in an **approach rack-by-rack.**

Problem III. For a given finite rack X, classify all cocycles q s. t. $\dim \mathfrak{B}(X, q) < \infty$.

If X is abelian, the classification is known:
I. Heckenberger, *Classification of arithmetic root systems*,
Adv. Math. **220** (2009) 59–124.

We are mainly interested in *indecomposable* racks.

It is natural to consider the class of finite simple racks; actually, the classification of these is known. In particular,

- Non-trivial conjugacy classes of a finite simple group are simple.

- The conjugacy class $\mathcal{O}$ of $\sigma \in \mathbb{S}_m - \mathbb{A}_m$, $m \geq 5$, is simple.

We shall say that a finite simple rack X *collapses* if for any cocycle q, $\dim \mathfrak{B}(X, q) = \infty$.

Let $\sigma \in \mathbb{S}_m$ be of type $(1^{n_1}, 2^{n_2}, \dots, m^{n_m})$ and let

$$\mathcal{O} = \begin{cases} (a) \text{ the conjugacy class of } \sigma \text{ in } \mathbb{S}_m, & \text{if } \sigma \notin \mathbb{A}_m, \\ (b) \text{ the conjugacy class of } \sigma \text{ in } \mathbb{A}_m, & \text{if } \sigma \in \mathbb{A}_m. \end{cases}$$

Theorem III. If the type of σ is not in the list below, then $\mathcal{O}$ collapses.

(a) $(2, 3)$; (2^3); $(1^n, 2)$.

(b) (3^2); $(2^2, 3)$; $(1^n, 3)$; (2^4); $(1^2, 2^2)$; $(1, 2^2)$.

(b) (open) $(1, m-1)$, if $m-1$ is prime; (m), if m is prime.

Theorem IV. If G is a sporadic simple group and $\mathcal{O}$ is a non-trivial conjugacy class of G NOT listed in the Tables below, then $\mathcal{O}$ collapses.

G	**# Classes**	**Classes**
M_{11}	10	8A, 8B, 11A, 11B
M_{12}	15	11A, 11B
M_{22}	12	11A, 11B
M_{23}	17	23A, 23B
M_{24}	26	23A, 23B
J_2	21	2A, 3A
Suz	43	3A
HS	24	11A, 11B
McL	24	11A, 11B
Co_3	42	23A, 23B
Co_2	60	2A, 23A, 23B

G	# **Classes**	**Classes**
Co_1	101	23A, 23B, 33A
J_1	15	15A, 15B, 19A, 19B, 19C
$O'N$	30	31A, 31B
J_3	21	5A, 5B, 19A, 19B
Ru	36	29A, 29B
He	33	all collapse
Fi_{22}	65	2A, 22A, 22B
Fi_{23}	98	2A, 23A, 23B
HN	54	all collapse
Th	48	3A, 31A, 31B
T	22	2B

G, #	**Classes**
Ly, 53	33A, 33B, 37A, 37B, 67A, 67B, 67C
J_4, 62	29A, 37A, 37B, 37C, 43A, 43B, 43C
Fi'_{24}, 108	9D, 23A, 23D, 27A, 27B, 27C,
	29A, 29B, 39A, 39B, 39C, 39D
B, 184	2A, 2C, 16C, 16D, 32A, 32B, 32C, 32D, 34A,
	40E, 46A, 46B, 47A, 47B, 60C
M, 194	32A, 32B, 33A, 46A, 46B, 46C, 47A, 47B,
	66B, 69A, 69B, 87A, 87B, 93A, 93B, 94A, 94B

III. The schemes of the proofs.

A basic property of Nichols algebras says that, if W is a braided subspace of a braided vector space V, then $\mathfrak{B}(W) \hookrightarrow \mathfrak{B}(V)$.

For instance, consider a simple $V = M(\mathcal{O}, \rho)$ – say $\dim \rho = 1$ for simplicity. If X is a proper subrack of $\mathcal{O}$, then $M(\mathcal{O}, \rho)$ has a braided subspace of the form $W = (\mathbb{C}X, c^q)$, which is clearly not a Yetter-Drinfeld submodule but can be realized as a Yetter-Drinfeld module over smaller groups, that could be reducible if X is decomposable. If we know that $\dim \mathfrak{B}(X, q) = \infty$, say because we have enough information on one of these smaller groups, then $\dim \mathfrak{B}(\mathcal{O}, \rho) = \infty$ too.

Proofs of Theorems I and II.

Assume Theorems III and IV.
Consider the remaining $M(\mathcal{O}, \rho)$; look at the abelian subracks of $\mathcal{O}$ and apply Heckenberger's result.

Examples. Say $\mathcal{O}$ real if $x \in \mathcal{O} \implies x^{-1} \in \mathcal{O}$. If x is not an involution, then $\{x, x^{-1}\}$ is an abelian subrack with 2 elements.

Fact. If $\mathcal{O}$ is a real conjugacy class of elements with odd order, then $\dim \mathfrak{B}(\mathcal{O}, \rho) = \infty$.

For instance, all remaining orbits in the case of the Janko group J_4 are real.

Proofs of Theorems III and IV.

It is based on the following result, a consequence of a Theorem by Heckenberger and Schneider.

We say that a rack X is *of type D* if there exists a decomposable subrack $Y = R \coprod S$ of X such that

$$r \triangleright (s \triangleright (r \triangleright s)) \neq s, \quad \text{for some } r \in R, s \in S.$$

Theorem. If X is a finite rack of type D, then X collapses.

Example: $G = \mathbb{Z}_n \rtimes \langle T \rangle$, $\rho =$ sgn.

n =**rk**	**Relations**	$\dim \mathfrak{B}(V)$	**top**
3	5 relations in degree 2	$12 = 3.2^2$	$4 = 2^2$
5	10 relations in degree 2 1 relation in degree 4	$1280 = 5.4^4$	$16 = 4^2$
7	21 relations in degree 2 1 relation in degree 6	$326592 = 7.6^6$	$36 = 6^2$

A few more examples: M. Graña, J. Algebra **231** (2000), pp. 235-257.
N. A. and M. Graña, Adv. in Math. **178**, 177–243 (2003).

Finite simple racks have been classified in [N. A. and M. Graña, Adv. in Math. **178**, 177–243 (2003)], see also [Joyce, JPAA]:

- $|X| = p$ a prime, $X \simeq \mathbb{F}_p$ a permutation rack: $x \triangleright y = y + 1$.

- $|X| = p^t$, p a prime, $t \in \mathbb{N}$, $X \simeq ({\mathbb{F}_p}^t, T)$ is an affine crossed set where T is the companion matrix of a monic irreducible polynomial of degree t, different from X and $X - 1$.

- Otherwise, there exist a non-abelian simple group L, $t \in \mathbb{N}$ and $x \in \mathrm{Aut}(L^t)$, where x acts by $x \cdot (l_1, \dots, l_t) = (\theta(l_t), l_1, \dots, l_{t-1})$ for some $\theta \in \mathrm{Aut}(L)$, such that $X = \mathcal{O}_x(n)$ is an orbit of the action $\rightharpoonup_x$ of L^t on itself; L and t are unique, and x only depends on its conjugacy class in $\mathrm{Out}(L^t)$. Here, the action $\rightharpoonup_x$ is given by $p \rightharpoonup_x n = p\, n\, (x \cdot p^{-1})$.