A quantum version of the algebra of distributions of $SL_2$

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Algebraic groups, positive characteristic

$G$: simple, simply connected algebraic group over an algebraically closed field $\mathbb{k}$, $\text{char} \mathbb{k} = p \geq 0$.

- $T$ a maximal torus, $X := \text{Hom}(T, \mathbb{k}^\times)$ its character lattice.

$\Delta \subset X$ the root system of $G$ with respect to $T$.

- $X^\vee := \text{Hom}(\mathbb{k}^\times, T)$ the cocharacter lattice, and for $\alpha \in \Delta \mapsto \alpha^\vee \in X^\vee$ the associated coroot.

$B \subset G$ a Borel subgroup that contains $T$, $\Delta^+ \subset \mathbb{R}$ its set of positive roots and $\Pi \subset \mathbb{R}^+ \subset \mathbb{R}$ be the set of simple roots.

$B^- \subset G$ the Borel subgroup that contains $T$ and is opposite to $B$, so that $\mathbb{R}^- = -\mathbb{R}^+$ is its set of roots.

$X^+ = \{ \lambda \in X | \langle \lambda, \alpha^\vee \rangle \geq 0 \text{ for all } \alpha \in \Pi \}$ the set of dominant weights.
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- \( X^+ = \{ \lambda \in X \mid \langle \lambda, \alpha^\vee \rangle \geq 0 \text{ for all } \alpha \in \Pi \} \) the set of dominant weights.
For $\lambda \in X^+$, the \textit{Weyl module} is

$$H^0(\lambda) := \text{ind}_{B^-}^G k_\lambda = \{ f : G \to \mathbb{k} \mid f \text{ is regular}, f(bg) = \lambda(b^{-1})f(g), \forall b \in B^-, g \in G \}.$$ 

$G$ acts on $H^0(\lambda)$ by $(gf)(h) = f(hg)$ for all $f \in H^0(\lambda), g, h \in G$. 

$L(\lambda)$ the socle of $H^0(\lambda)$, i.e. the maximal semisimple submodule. 

\textbf{Theorem (Chevalley)}

1. $L(\lambda)$ is simple $\forall \lambda \in X^+$. 
2. Each simple $G$-module is isomorphic to $L(\lambda)$ for a unique $\lambda \in X^+$. 

Iván Angiono

A quantum version of Dist $SL_2$
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\( G \) acts on \( H^0(\lambda) \) by \((gf)(h) = f(hg)\) for all \( f \in H^0(\lambda), g, h \in G \).

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Remark

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*T* is *diagonalisable*: for all representations *V* of *T*,

\[ V = \bigoplus_{\lambda \in \chi} V_{\lambda}, \quad V_{\lambda} = \{ v \in V \mid t.v = \lambda(t)v \quad \forall t \in T \}. \]

\[ [V] := \sum_{\lambda \in \chi} \dim_k V_{\lambda} \cdot e^{\lambda} \in \mathbb{Z}[\chi] = \bigoplus_{\lambda \in \chi} \mathbb{Z} \cdot e^{\lambda} \text{ the formal character}. \]
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- The Weyl group $\mathcal{W} = N_G(T)/T$ acts on $T$ and hence on $X$. 

Iván Angiono  
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Characters of Weyl and simple modules

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- \( w_0 \) longest element of \( \mathcal{W} \).
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- \( l: \mathcal{W} \rightarrow \mathbb{N} \) the length function for our choice of positive roots.
- \( w_0 \) longest element of *W*.
- \( \rho := 1/2 \sum_{\alpha \in R^+} \alpha: \langle \rho, \alpha^\vee \rangle = 1, \forall \alpha \in \Pi \), hence \( \rho \in X^+ \).
Theorem (Weyl’s character formula)

\[ [H^0(\lambda)] = \chi(\lambda) := \frac{\sum_{w \in W} (-1)^{l(w)} e^{w(\lambda + \rho)}}{\sum_{w \in W} (-1)^{l(w)} e^{w(\rho)}}, \quad \forall \lambda \in X^+. \]
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Problem

Obtain a formula for the character of the simple module \(L(\lambda), \lambda \in X^+\).
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Obtain a formula for the character of the simple module \( L(\lambda) \), \( \lambda \in X^+ \).

Remark

If \( p = 0 \), then \( L(\lambda) = H^0(\lambda) \) (Borel-Weil-Bott Theorem).
Let $p > 0$.

* $\alpha \in R^+$, $n \in \mathbb{Z}$: the affine transformation $s_{\alpha,n}$ on the lattice $X$ is

$$s_{\alpha,n}(\beta) := \beta - (\langle \beta, \alpha^\vee \rangle - pn)\alpha, \quad \beta \in X.$$
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- A set of simple affine reflections in $\hat{W}$ is $S = \{s_i\} \cup \{s_{\gamma,1}\}$, $\gamma \in R^+$ such that $\gamma^\vee$ is the highest coroot.
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- $\hat{W} \times X \to X$, $(w, \lambda) \mapsto w \cdot \lambda := w(\lambda + \rho) - \rho$, the $\rho$-shifted action of $\hat{W}$ on $X$. 

Iván Angiono  

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- $0 \in X$ is a regular point for the $\rho$-shifted action of $\hat{\mathcal{W}}$ on $X$ (i.e. $w \cdot 0 = 0$ only if $w = e$), if and only if $p \geq h := \max_{\alpha \in R^+} \{\langle \rho, \alpha^\vee \rangle + 1\}$, the Coxeter number. *This is our general assumption from now on.*
Lusztig’s conjecture

Conjecture (Lusztig, 1980)

For all $w \in \hat{W}$ and $\lambda \in X^+$ such that $w \cdot \lambda \in X_1$,

$$[L(w \cdot \lambda)] = \sum_{x \in \hat{W} : x \leq w, \ x \cdot \lambda \in X_1} (-1)^{l(w)-l(x)} h_{w_0 x, w_0 w}(1) \chi(x \cdot \lambda).$$
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$h_{a,b}(v) \in \mathbb{Z}[v]$ denotes the Kazhdan-Lusztig polynomial for the affine Weyl group at the parameters $a, b \in \widehat{W}$. 
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$h_{a,b}(\nu) \in \mathbb{Z}[\nu]$ denotes the Kazhdan-Lusztig polynomial for the affine Weyl group at the parameters $a, b \in \widehat{W}$.

$X_1 := \{\lambda \in X : 0 \leq \langle \lambda, \alpha^\vee \rangle < p, \forall \alpha \in \Pi\}$ the fundamental box.
Lusztig outlined a program for the proof of the conjecture. In *Modular representations and quantum groups*, 1989, Lusztig said:

*Here we shall describe a certain representation theory which seems to be governed by laws similar to those of the rational G-modules and which, at the same time, involves only characteristic 0; this is the theory of finite dim. representations of quantum groups whose parameter is a root of 1.*

In the same paper he started to study the representation theory of quantum groups at roots of unity.
\[ g \text{ s.s. fin. dim. Lie algebra with Cartan matrix } A, \lambda \text{ a root of unity of odd order } \ell. \]

- \( U_q(g) \) over \( \mathbb{C}(q) \): Hopf algebra generated by \( E_i, F_i, K_i^{\pm 1} \).
\( g \) s.s. fin. dim. Lie algebra with Cartan matrix \( A \), \( \lambda \) a root of unity of odd order \( \ell \).

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- \( A = \mathbb{C}[q, q^{-1}] \subset \mathbb{C}(q) \), \( U_A(g) \) the \( A \)-subalgebra of \( U_q(g) \) gen. by

\[
E_i^{(n)} = \frac{E_i^n}{(n)!_q}, \quad F_i^{(n)} = \frac{F_i^n}{(n)!_q}, \quad K_i^{\pm 1}.
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A s.s. fin. dim. Lie algebra with Cartan matrix $A$, $\lambda$ a root of unity of odd order $\ell$.

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- $U_\lambda(\mathfrak{g}) = U_A(\mathfrak{g}) \otimes_A \mathbb{C}_\lambda$, $\mathbb{C}_\lambda = \mathbb{C}$ with action $q \mapsto \lambda$. 

Iván Angioni
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- \( U_q(\mathfrak{g}) \) over \( \mathbb{C}(q) \): Hopf algebra generated by \( E_i, F_i, K_i^{\pm 1} \).
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- \( U_\lambda(\mathfrak{g}) \) is generated as \( \mathbb{C} \)-algebra by \( E_i, F_i, K_i, E_i^{(\ell)}, F_i^{(\ell)}, \{ K_i; 0 \}_{\ell} \)
$\mathfrak{g}$ s.s. fin. dim. Lie algebra with Cartan matrix $A$, $\lambda$ a root of unity of odd order $\ell$.

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- $E_i$, $F_i$, $K_i$ generate $u_\lambda(\mathfrak{g})$: same relations as $U_q(\mathfrak{g})$, but adding $E_\alpha^\ell = F_\alpha^\ell = 0$, $\alpha \in \Delta_+$, (in particular, $E_i^\ell = F_i^\ell = 0$), and $K_i^{2\ell}$.
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E_\alpha^{\ell} = F_\alpha^{\ell} = 0, \quad \alpha \in \Delta_+, \quad \text{(in particular, } E_i^{\ell} = F_i^{\ell} = 0) \text{, and } K_i^{2\ell}.
\]

- \( u_\lambda(\mathfrak{g}) \hookrightarrow U_\lambda(\mathfrak{g}) \twoheadrightarrow U(\mathfrak{g}) \) Hopf algebra extension:

\[
E_i, F_i \mapsto 0, \quad K_i \mapsto 1, \quad E_i^{(\ell)} \mapsto e_i, \quad F_i^{(\ell)} \mapsto f_i, \quad \left\{ K_i; 0 \atop \ell \right\} \mapsto h_i.
\]
• $U_\lambda(g)$ inherits \textit{triangular decomposition} of $U_q(g)$:

$$U_\lambda(g) \simeq U_\lambda^+(g) \otimes U_\lambda^0(g) \otimes U_\lambda^-(g).$$
• $U_\lambda(g)$ inherits **triangular decomposition** of $U_q(g)$:

\[ U_\lambda(g) \simeq U^+_\lambda(g) \otimes U^0_\lambda(g) \otimes U^-_\lambda(g). \]

• For $\beta = (b_1, \ldots, b_\theta) \in \mathbb{N}_0^\theta$, $\exists!$ simple module of $U_q(g)$ of h. wt. $\beta$:

\[ \nabla_q(\beta) = U_q(g)\nu, \quad E_i \nu = 0, \quad K_i \nu = q^{b_i} \nu. \]
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• $\nabla_q(\beta)$ same character as Weyl module of $\beta \in X^+$. 
- $U_\lambda(g)$ inherits *triangular decomposition* of $U_q(g)$:

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  $$\nabla_q(\beta) = U_q(g)v, \quad E_i v = 0, \quad K_i v = q^{b_i}v.$$  

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- $\nabla_\lambda(\beta) = \nabla_\lambda(\beta) \otimes \mathbb{C}_\lambda$ is a $U_\lambda(g)$-module.
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• $\nabla_\lambda(\beta) = \nabla_\lambda(\beta) \otimes_A \mathbb{C}_\lambda$ is a $U_\lambda(g)$-module.  
• $L_\lambda(\beta) =$ quotient of $\nabla_\lambda(\beta)$ by maximal proper submodule.
• $U_\lambda(g)$ inherits **triangular decomposition** of $U_q(g)$:

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• $\beta \mapsto L_\lambda(\beta)$ gives bijection between $\mathbb{N}_0^\theta$ and simple modules.
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**Conjecture (Lusztig, 1989)**

For all $w \in \hat{W}$ and $\lambda \in \mathbb{N}_0^\theta$ such that $w \cdot \lambda \in X_1$,

$$[L(w \cdot \lambda)] = \sum_{x \in \hat{W} : x \leq w, x \cdot \lambda \in X_1} (-1)^{l(w) - l(x)} h_{x, w}(1)[\nabla_\lambda(x \cdot \lambda)].$$
Relation between conjectures & answers

Remark

There is no restriction on the order of $\lambda$, a difference with conjecture on algebraic groups ($p \geq c_0^G$, i.e., think Coxeter number).

Lusztig's program was successfully carried out in a combined effort by Kashiwara & Tanisaki, Kazhdan & Lusztig, and Andersen, Jantzen & Soergel: positive answer of Conjecture on quantum groups.

Lusztig's program yielded a proof for almost all characteristics. More precisely, there exists a prime number $c_G \geq c_0^G$ depending only on the root datum of $G$ such that the conjectural formula is true if $p \geq c_G$ (not explicitly determined).

Fiebig showed that $c_G \leq c'_G$ where $c'_G$ is an explicitly known but very large constant.

Williamson showed that for infinitely many $G$, $c_G$ is much larger than $c_0^G$.
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5. Williamson showed that for infinitely many $G$, $c_G$ is much larger than $c_G^0$. 

### Relation between conjectures & answers

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Iván Angiono  
A quantum version of Dist SL₂
New approach, Lusztig 2015: series of character formulas \((E^i_\lambda)_{i \in \mathbb{N}_0}\) converging to the character formula \(E^\infty_\lambda\) of \(L(\beta)\) for algebraic groups. It takes into account

**Theorem (Steinberg’s tensor product decomposition)**

\[ \beta = \sum_{i \geq 0} \beta_i p^i, \beta_i \in X_1, \text{ then} \]

\[ L(\beta) \simeq L(\beta_0) \otimes L(\beta_1)^{[1]} \otimes \cdots \otimes L(\beta_k)^{[k]}, \]

\(M^{[i]}\) is the twist of \(M\) with the \(i^{th}\) power of the Frobenius endomorphism.
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\]

\(M[i]\) is the twist of \(M\) with the \(i^{th}\) power of the Frobenius endomorphism.

**Remark**

Only one step of (a kind of) Frobenius map and Steinberg decomposition for quantum groups:

\[
\beta = \beta_0 + \ell \beta_1: \quad L_\lambda(\beta) \simeq L_\lambda(\beta_0) \otimes L_\lambda(\ell \beta_1).
\]

Moreover \(L_\lambda(\ell \beta_1) \simeq L(\beta_1)\) as \(U(g)\)-module.
Definition

Let $I_e = \{ f \in k[G] | f(e) = 0 \}$.

$$\text{Dist} G := \{ \mu \in k[G]^* | \exists n \in \mathbb{N}_0 : \mu(I_{e}^{n+1}) = 0 \} = \bigcup_{n \geq 0} \text{Dist}_n G$$

$G$ is the algebra of distributions of $G$. It is a Hopf algebra.
Algebra of distributions of $G$

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Let $I_e = \{ f \in k[G] | f(e) = 0 \}$.

$\text{Dist } G := \{ \mu \in k[G]^* | \exists n \in \mathbb{N}_0 : \mu(I_e^{n+1}) = 0 \} = \bigcup_{n \geq 0} \text{Dist}_n G$ is the *algebra of distributions* of $G$. It is a Hopf algebra.

**Remark**

If $p = 0$, then $\text{Dist } G = U(g)$. If $p > 0$, then $\text{Dist } G$ contains $U[p](g)$, the restricted enveloping algebra.
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**Remark**

- $\text{Dist}_n^+ G = \{ \mu \in \text{Dist}_n G | \mu(1) = 0 \} \cong \text{Dist}_1^+ G \cong g$. 
Algebra of distributions of $G$

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- $\text{Dist}_n^+ G = \{ \mu \in \text{Dist}_n G | \mu(1) = 0 \} \sim \text{Dist}_1^+ G \simeq g$.
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Definition

Let $I_e = \{ f \in \mathbb{k}[G] | f(e) = 0 \}$. 

$\text{Dist}_G := \{ \mu \in \mathbb{k}[G]^* | \exists n \in \mathbb{N}_0 : \mu(I_e^{n+1}) = 0 \} = \bigcup_{n \geq 0} \text{Dist}_n G$ is the algebra of distributions of $G$. It is a Hopf algebra.

Remark

- $\text{Dist}_n^+ G = \{ \mu \in \text{Dist}_n G | \mu(1) = 0 \} \sim \text{Dist}_1^+ G \simeq g$. 
- If $p = 0$, then $\text{Dist} G = U(g)$. If $p > 0$, then $\text{Dist} G$ contains $U^{[p]}(g)$, the restricted enveloping algebra.
- $G$-modules $\leftrightarrow$ $\text{Dist} G$-modules
Theorem (Takeuchi, 1983)

Dist \( G \) is presented by generators \( H_i^{(n)}, X_i^{(n)}, Y_i^{(n)}, 1 \leq i \leq \theta, n \in \mathbb{N}_0 \), where \( H_i^{(0)} = X_i^{(0)} = Y_i^{(0)} = 1 \), and relations

\[
H_i^{(m)} H_i^{(n)} = \sum_{\ell=0}^{\min\{m, n\}} \binom{m+n-\ell}{m} \binom{m}{\ell} H_i^{(m+n-\ell)},
\]

\[
H_i^{(m)} H_j^{(n)} = H_j^{(n)} H_i^{(m)},
\]

\[
X_i^{(m)} X_i^{(n)} = \binom{m+n}{m} X_i^{(m+n)}, \quad Y_i^{(m)} Y_i^{(n)} = \binom{m+n}{m} Y_i^{(m+n)}
\]

\[
X_i^{(m)} Y_j^{(n)} = Y_j^{(n)} X_i^{(m)},
\]

\[
[X_i^{(p^m)}, Y_i^{(p^n)}] = \sum_{\ell=1}^{\min\{p^m, p^n\}} Y_i^{(p^m-\ell)} \left( \sum_{k=0}^{\ell} \binom{\ell+k}{\ell-k} H_i^{(k)} \right) X_i^{(p^n-\ell)},
\]

\[
[H_i^{(p^m)}, X_j^{(p^n)}] = \delta_{n,m} a_{ij} X_i^{(p^n)}, \quad [H_i^{(p^m)}, Y_j^{(p^n)}] = -\delta_{n,m} a_{ij} Y_i^{(p^n)},
\]
Theorem (Takeuchi, 1983)

\[
\text{ad} \left( X_i^{(n)} \right) \left( X_j^{(m)} \right) = \sum_{k=0}^{n} (-1)^k X_i^{(n-k)} X_j^{(m)} X_i^{(k)} = 0, \quad n > -ma_{ij},
\]

\[
\text{ad} \left( Y_i^{(n)} \right) \left( Y_j^{(m)} \right) = \sum_{k=0}^{n} (-1)^k Y_i^{(n-k)} Y_j^{(m)} Y_i^{(k)} = 0, \quad n > -ma_{ij}.
\]

for \( 1 \leq i \neq j \leq \theta \).

Let \( \mathcal{D}_n G := \text{Dist}_{p^n} G \) (a Hopf subalgebra), \( \pi_k : \mathcal{D}_{k+1} G \rightarrow \mathcal{D}_1 G = U^{[p]}(g) \)

\[
\pi_k(T_i^{(n)}) = \begin{cases} 
T_i^{(n')}, & \text{if } n = p^k n', \\
0, & \text{otherwise}, 
\end{cases} \quad T = X, Y, H.
\]
Theorem (Takeuchi, 1983)

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\]

for \(1 \leq i \neq j \leq \theta\).

Let \(D_n G \coloneqq \text{Dist}_{p^n} G\) (a Hopf subalgebra), \(\pi_k : D_{k+1} G \to D_1 G = \mathbb{U}^{[p]}(g)\)

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\pi_k(T_i^{(n)}) = \begin{cases} 
T_i^{(n')}, & \text{if } n = p^k n', \\
0, & \text{otherwise,}
\end{cases} \quad T = X, Y, H.
\]

Proposition

\(D_k G \hookrightarrow D_{k+1} G \xrightarrow{\pi_k} D_1 G\) is an exact sequence of Hopf algebras.
Let $\lambda$ a primitive root of unity of order $\ell$, $\ell > 1$ is odd. Let $N \in \mathbb{N}_0$.

$$\{m \choose n \}_{\lambda} := \prod_{i \geq 0} \left[ m_i \choose n_i \right]_{\lambda}, \quad m = \sum_{i \geq 0} m_i \ell^i, \quad n = \sum_{i \geq 0} n_i \ell^i, \quad 0 \leq m_i, n_i < \ell.$$

**Definition**

Let $\mathcal{D}_{\lambda,N}(\mathfrak{sl}_2)$ be the algebra defined by generators $E[i], F[i], K[i], 0 \leq i \leq N$ and relations

$$K[i]K[j] = K[j]K[i], \quad (K[i])^\ell = 1;$$

$$K[i]E[j] = \lambda^{2\delta_{ij}} E[j]K[i], \quad K[i]F[j] = \lambda^{-2\delta_{ij}} F[j]K[i];$$


$$\left( E[i] \right)^\ell = \left( F[i] \right)^\ell = 0; \quad E[i]F[j] = F[j]E[i], \quad j \neq i;$$
Definition

\[ E[i] F[j] = \sum_{t=0}^{\ell_j} F(\ell_j-t) \binom{K; 2t - 2\ell_j}{t} E(\ell_j-t). \]

Here, \( K[i] := (K[i])^{-1} \) and for \( m = \sum_{i=0}^{N} m_i \ell_i, \ s = \sum_{i=0}^{N} s_i \ell_i, \ 0 \leq m_i, s_i < \ell, \)

\[ E^{(m)} := \prod_{i=0}^{N} \frac{(E[i])^{m_i}}{[m_i]_{\lambda}^!}, \quad \{ K; s \ \ t \} = \prod_{i=0}^{N} \binom{K[i]; s_i}{t_i}_{\lambda}, \quad F^{(m)} := \prod_{i=0}^{N} \frac{(F[i])^{m_i}}{[m_i]_{\lambda}^!}. \]
Definition

\[ E[j] F[j] = \sum_{t=0}^{\ell^j} F(\ell^j - t) \left\{ \binom{K; 2t - 2\ell^j}{t} \right\} E(\ell^j - t). \]

Here, \( K^{-i} := (K[i])^{-1} \) and for \( m = \sum_{i=0}^{N} m_i \ell^i \), \( s = \sum_{i=0}^{N} s_i \ell^i \), \( 0 \leq m_i, s_i < \ell \),

\[ E^{(m)} := \prod_{i=0}^{N} \frac{(E[i])^{m_i}}{[m_i]_\lambda}, \quad \left\{ \binom{K; s}{t} \right\} = \prod_{i=0}^{N} \left[ \binom{K[i]; s_i}{t_i} \right]_\lambda, \quad F^{(m)} := \prod_{i=0}^{N} \frac{(F[i])^{m_i}}{[m_i]_\lambda}. \]

Remark

\( D_{\lambda,0}(sl_2) = u_{\lambda}(sl_2). \)
Some properties of $D_{\lambda, N}(\mathfrak{sl}_2)$

- A surjective algebra map $\pi_N : D_{\lambda, N}(\mathfrak{sl}_2) \to u_\lambda(\mathfrak{sl}_2)$,

\[
\pi_N(E[i]) = \delta_{iN}E, \quad \pi_N(F[i]) = \delta_{iN}F, \quad \pi_N(K[i]) = K^{\delta_{iN}}, \quad 0 \leq i \leq N.
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Some properties of $D_{\lambda,N}(\mathfrak{sl}_2)$

1. ∃ a surjective algebra map $\pi_N : D_{\lambda,N}(\mathfrak{sl}_2) \to u_\lambda(\mathfrak{sl}_2)$,

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2. ∃ an injective algebra map $\iota_N : D_{\lambda,N-1}(\mathfrak{sl}_2) \to D_{\lambda,N}(\mathfrak{sl}_2)$ which identifies the corresponding generators.
Some properties of $\mathcal{D}_{\lambda,N}(\mathfrak{sl}_2)$

1. ∃ a surjective algebra map $\pi_N : \mathcal{D}_{\lambda,N}(\mathfrak{sl}_2) \rightarrow u_\lambda(\mathfrak{sl}_2)$,

$$\pi_N(E^i) = \delta_i^N E, \quad \pi_N(F^i) = \delta_i^N F, \quad \pi_N(K^i) = K^\delta_i^N, \quad 0 \leq i \leq N.$$

2. ∃ an injective algebra map $\iota_N : \mathcal{D}_{\lambda,N-1}(\mathfrak{sl}_2) \rightarrow \mathcal{D}_{\lambda,N}(\mathfrak{sl}_2)$ which identifies the corresponding generators.

**Proposition**

$\mathcal{D}_{\lambda,N}(\mathfrak{sl}_2)$ is a $u_\lambda(\mathfrak{sl}_2)$-comodule algebra via

$\rho_N : \mathcal{D}_{\lambda,N}(\mathfrak{sl}_2) \rightarrow u_\lambda(\mathfrak{sl}_2) \otimes \mathcal{D}_{\lambda,N}(\mathfrak{sl}_2)$:

$$\rho_N(E^i) = 1 \otimes E^i, \quad i < N, \quad \rho_N(E^{[N]}) = E \otimes 1 + K \otimes E^{[N]},$$

$$\rho_N(F^i) = 1 \otimes F^i, \quad i < N, \quad \rho_N(F^{[N]}) = F \otimes K^{[-N]} + 1 \otimes F^{[N]},$$

$$\rho_N(K^i) = 1 \otimes K^i, \quad i < N, \quad \rho_N(K^{[N]}) = K \otimes K^{[N]}.$$

Moreover, $\mathcal{D}_{\lambda,N-1}(\mathfrak{sl}_2) \overset{\text{co} \rho_N}{=} \mathcal{D}_{\lambda,N}(\mathfrak{sl}_2)$, and $\mathcal{D}_{\lambda,N-1}(\mathfrak{sl}_2) \subset \mathcal{D}_{\lambda,N}(\mathfrak{sl}_2)$ is a $u_\lambda(\mathfrak{sl}_2)$-cleft extension.
Proposition

There exist algebra isomorphisms

\[ D_0^\lambda, N(sl_2) \cong k(Z_\ell)^{N+1}, \]

\[ D_\geq^\lambda, N(sl_2) \cong (u_\geq^\lambda(sl_2))^{N+1}, \]

\[ D_\pm^\lambda, N(sl_2) \cong (u_\pm^\lambda(sl_2))^{N+1}, \]

\[ D_\leq^\lambda, N(sl_2) \cong (u_\leq^\lambda(sl_2))^{N+1}. \]

\[ B_N := \{ F(m)K(n)E(p) | 0 \leq m, n, p < \ell^{N+1} \} \]

is a basis of \( D^\lambda, N(sl_2) \).

The multiplication induces a linear isomorphism

\[ D_-^\lambda, N(sl_2) \otimes D_0^\lambda, N(sl_2) \otimes D_+^\lambda, N(sl_2) \cong D^\lambda, N(sl_2). \]
Proposition

There exist algebra isomorphisms

\[ D^0_{\lambda, N}(\mathfrak{sl}_2) \cong \mathbb{k}(\mathbb{Z}_\ell)^{N+1}, \]
\[ D^{\geq 0}_{\lambda, N}(\mathfrak{sl}_2) \cong \left( u_{\lambda}^{\geq 0}(\mathfrak{sl}_2) \right)^{N+1}, \]
\[ D^{\pm}_{\lambda, N}(\mathfrak{sl}_2) \cong \left( u_{\lambda}^{\pm}(\mathfrak{sl}_2) \right)^{N+1}, \]
\[ D_{\lambda, N}^{\leq 0}(\mathfrak{sl}_2) \cong \left( u_{\lambda}^{\leq 0}(\mathfrak{sl}_2) \right)^{N+1}. \]
Proposition

There exist algebra isomorphisms

\[ \mathcal{D}_{\lambda, N}(\mathfrak{sl}_2) \cong \mathbb{k}(\mathbb{Z}_\ell)^{N+1}, \quad \mathcal{D}_{\lambda, N}^{\geq 0}(\mathfrak{sl}_2) \cong \left( \mathfrak{u}_\lambda^{\geq 0}(\mathfrak{sl}_2) \right)^{N+1}, \]
\[ \mathcal{D}_{\lambda, N}^{\pm}(\mathfrak{sl}_2) \cong \left( \mathfrak{u}_\lambda^{\pm}(\mathfrak{sl}_2) \right)^{N+1}, \quad \mathcal{D}_{\lambda, N}^{\leq 0}(\mathfrak{sl}_2) \cong \left( \mathfrak{u}_\lambda^{\leq 0}(\mathfrak{sl}_2) \right)^{N+1}. \]

2. \( B_N := \{ F^{(m)} K^{(n)} E^{(p)} | 0 \leq m, n, p < \ell^{N+1} \} \) is a basis of \( \mathcal{D}_{\lambda, N}(\mathfrak{sl}_2) \).
Proposition

1. There exist algebra isomorphisms

\[ D_{\lambda,N}(\mathfrak{sl}_2) \cong \mathbb{k}(\mathbb{Z}_\ell)^{N+1}, \quad D_{\lambda,N}(\mathfrak{sl}_2) \cong \left( u^\geq_{\lambda}(\mathfrak{sl}_2) \right)^{N+1}, \]
\[ D_{\lambda,N}(\mathfrak{sl}_2) \cong \left( u^\leq_{\lambda}(\mathfrak{sl}_2) \right)^{N+1}. \]

2. \( B_N := \{ F^{(m)} K^{(n)} E^{(p)} | 0 \leq m, n, p < \ell^{N+1} \} \) is a basis of \( D_{\lambda,N}(\mathfrak{sl}_2) \).

3. The multiplication induces a linear isomorphism

\[ D_{\lambda,N}(\mathfrak{sl}_2) \otimes D_{\lambda,N}(\mathfrak{sl}_2) \otimes D_{\lambda,N}(\mathfrak{sl}_2) \cong D_{\lambda,N}(\mathfrak{sl}_2). \]
Finite-dimensional irreducible $\mathcal{D}_{\lambda,N}(\mathfrak{sl}_2)$-modules

$V$ a finite dimensional $\mathcal{D}_{\lambda,N}(\mathfrak{sl}_2)$-module. As $\mathcal{D}_{\lambda,N}(\mathfrak{sl}_2) \simeq \mathbb{Z}^{N+1}$,

- each $K[i]$ acts by a scalar $\lambda^{p_i}$, $0 \leq p_i < \ell$. 

\textbf{Definition} \ Let $L_N(p) := M_N(p) / \hat{M}_N(p)$ the simple highest weight module obtained as a quotient of $M_N(p)$. 

Iván Angiono \ A quantum version of Dist SL$_2$
Finite-dimensional irreducible $D_{\lambda,N}(\mathfrak{sl}_2)$-modules

$V$ a finite dimensional $D_{\lambda,N}(\mathfrak{sl}_2)$-module. As $D_{\lambda,N}(\mathfrak{sl}_2) \simeq \mathbb{Z}_\ell^{N+1}$,

- each $K[i]$ acts by an scalar $\lambda^{p_i}$, $0 \leq p_i < \ell$.
- $V = \bigoplus_{0 \leq p < \ell^{N+1}} V_p$, where for $p = \sum_{i=0}^{N} p_i \ell^i$,

  $$V_p = \{ v \in V | K[i] \cdot v = \lambda^{p_i} v \text{ for all } 0 \leq i \leq N \}. $$
V a finite dimensional $\mathcal{D}_{\lambda,N}(\mathfrak{sl}_2)$-module. As $\mathcal{D}_{\lambda,N}(\mathfrak{sl}_2) \cong \mathbb{Z}_\ell^{N+1}$,

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Finite-dimensional irreducible $\mathcal{D}_{\lambda, N}(\mathfrak{sl}_2)$-modules

$V$ a finite dimensional $\mathcal{D}_{\lambda, N}(\mathfrak{sl}_2)$-module. As $\mathcal{D}_{\lambda, N}^0(\mathfrak{sl}_2) \simeq \mathbb{Z}_\ell^{N+1}$,

- each $K[i]$ acts by an scalar $\lambda^{p_i}$, $0 \leq p_i < \ell$.
- $V = \bigoplus_{0 \leq p < \ell^{N+1}} V_p$, where for $p = \sum_{i=0}^N p_i \ell^i$,

$$V_p = \{ \mathbf{v} \in V | K[i] \cdot \mathbf{v} = \lambda^{p_i} \mathbf{v} \text{ for all } 0 \leq i \leq N \}.$$ 

Given $0 \leq p < \ell^{N+1}$, let $k_p$ be the 1-dimensional representation of

$\mathcal{D}_{\lambda, N}^>^0(\mathfrak{sl}_2) \simeq \left( u_{\lambda}^>^0(\mathfrak{sl}_2) \right)^{N+1}$ such that $K[i] \cdot 1 = \lambda^{p_i}$ and $E[i] \cdot 1 = 0$. Let

$$\mathcal{M}_N(p) = \operatorname{Ind}_{\mathcal{D}_{\lambda, N}^>^0(\mathfrak{sl}_2)}^{\mathcal{D}_{\lambda, N}(\mathfrak{sl}_2)} k_p \simeq \mathcal{D}_{\lambda, N}(\mathfrak{sl}_2) \otimes \mathcal{D}_{\lambda, N}^>^0(\mathfrak{sl}_2) k_p.$$ 

**Definition**

Let $\mathcal{L}_N(p) := \mathcal{M}_N(p)/\overline{\mathcal{M}_N(p)}$ the simple highest weight module obtained as a quotient of $\mathcal{M}_N(p)$. 

Iván Angiono

A quantum version of Dist $\text{SL}_2$
Proposition

There exists a bijection between \( \{ p \mid 0 \leq p < \ell^{N+1} \} \) and the finite-dimensional simple modules of \( D_{\lambda, N}(sl_2) \) given by \( p \mapsto L_N(p) \).
Proposition
There exists a bijection between \( \{ p | 0 \leq p < \ell^{N+1} \} \) and the finite-dimensional simple modules of \( D_{\lambda,N}(\mathfrak{sl}_2) \) given by \( p \mapsto \mathcal{L}_N(p) \).

Proposition

**Let** \( 0 \leq p < \ell^{N+1} \). Then \( E[N] \cdot v = F[N] \cdot v = 0 \), \( K[N] \cdot v = v \), for all \( v \in \mathcal{L}_N(p) \).

Moreover, \( \mathcal{L}_N(p) \cong \mathcal{L}_N(p-1) \) as \( D_{\lambda,N-1}(\mathfrak{sl}_2) \)-modules.

\( \mathcal{L}_N(p) \) may be endowed of an \( D_{\lambda,N}(\mathfrak{sl}_2) \)-action by extending the \( D_{\lambda,N-1}(\mathfrak{sl}_2) \)-action, and \( \mathcal{L}_N(p) \cong \mathcal{L}_N(p) \) as \( D_{\lambda,N}(\mathfrak{sl}_2) \)-modules.
Proposition

There exists a bijection between \( \{ p \mid 0 \leq p < \ell^{N+1} \} \) and the finite-dimensional simple modules of \( \mathcal{D}_{\lambda,N}(\mathfrak{sl}_2) \) given by \( p \mapsto \mathcal{L}_N(p) \).

Proposition

Let \( 0 \leq p < \ell^N \). Then

\[
E^{[N]} \cdot v = F^{[N]} \cdot v = 0, \quad K^{[N]} \cdot v = v, \quad \text{for all } v \in \mathcal{L}_N(p).
\]

Moreover, \( \mathcal{L}_N(p) \simeq \mathcal{L}_{N-1}(p) \) as \( \mathcal{D}_{\lambda,N-1}(\mathfrak{sl}_2) \)-modules.
Proposition

There exists a bijection between \( \{ p | 0 \leq p < \ell^{N+1} \} \) and the finite-dimensional simple modules of \( \mathcal{D}_{\lambda, N}(\mathfrak{sl}_2) \) given by \( p \mapsto \mathcal{L}_N(p) \).

Proposition

1. Let \( 0 \leq p < \ell^N \). Then

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E^{[N]} \cdot v = F^{[N]} \cdot v = 0, \quad K^{[N]} \cdot v = v, \quad \text{for all } v \in \mathcal{L}_N(p).
\]

Moreover, \( \mathcal{L}_N(p) \simeq \mathcal{L}_{N-1}(p) \) as \( \mathcal{D}_{\lambda, N-1}(\mathfrak{sl}_2) \)-modules.

2. \( \mathcal{L}_{N-1}(p) \) may be endowed of an \( \mathcal{D}_{\lambda, N}(\mathfrak{sl}_2) \)-action by extending the \( \mathcal{D}_{\lambda, N-1}(\mathfrak{sl}_2) \)-action, and \( \mathcal{L}_{N-1}(p) \simeq \mathcal{L}_N(p) \) as \( \mathcal{D}_{\lambda, N}(\mathfrak{sl}_2) \)-modules.
A tensor product decomposition

\[ \pi_N : D_{\lambda,N}(\mathfrak{sl}_2) \to \mathfrak{u}_\lambda(\mathfrak{sl}_2) : \text{every } \mathfrak{u}_\lambda(\mathfrak{sl}_2)\text{-module is a } D_{\lambda,N}(\mathfrak{sl}_2)\text{-module.} \]

\[ \leadsto \text{each simple } \mathfrak{u}_\lambda(\mathfrak{sl}_2)\text{-module } L(p), \ 0 \leq p < \ell, \text{ is a } D_{\lambda,N}(\mathfrak{sl}_2)\text{-module.} \]

**Lemma**

Let \( p = p_N\ell^N, \ 0 \leq p_N < \ell. \) Then \( \mathcal{L}_N(p) \cong \mathcal{L}(p_N). \)
\( \pi_N : D_{\lambda, N}(\mathfrak{sl}_2) \rightarrow u_\lambda(\mathfrak{sl}_2) \): every \( u_\lambda(\mathfrak{sl}_2) \)-module is a \( D_{\lambda, N}(\mathfrak{sl}_2) \)-module.

\( \Rightarrow \) each simple \( u_\lambda(\mathfrak{sl}_2) \)-module \( L(p), 0 \leq p < \ell \), is a \( D_{\lambda, N}(\mathfrak{sl}_2) \)-module.

**Lemma**

Let \( p = p_N \ell^N, 0 \leq p_N < \ell \). Then \( L_N(p) \simeq L(p_N) \).

**Remark**

As \( D_{\lambda, N}(\mathfrak{sl}_2) \) is an \( u_\lambda(\mathfrak{sl}_2) \)-comodule algebra, given a \( u_\lambda(\mathfrak{sl}_2) \)-module \( M \) and a \( D_{\lambda, N}(\mathfrak{sl}_2) \)-module \( N \), \( M \otimes N \) is a \( D_{\lambda, N}(\mathfrak{sl}_2) \)-module via \( \rho \).
A tensor product decomposition

$$\pi_N : D_{\lambda,N}(\mathfrak{sl}_2) \to u_\lambda(\mathfrak{sl}_2):$$ every $$u_\lambda(\mathfrak{sl}_2)$$-module is a $$D_{\lambda,N}(\mathfrak{sl}_2)$$-module.

Each simple $$u_\lambda(\mathfrak{sl}_2)$$-module $$L(p), 0 \leq p < \ell$$, is a $$D_{\lambda,N}(\mathfrak{sl}_2)$$-module.

**Lemma**

Let $$p = p_N \ell^N, 0 \leq p_N < \ell$$. Then $$L_N(p) \simeq L(p_N)$$.

**Remark**

As $$D_{\lambda,N}(\mathfrak{sl}_2)$$ is an $$u_\lambda(\mathfrak{sl}_2)$$-comodule algebra, given a $$u_\lambda(\mathfrak{sl}_2)$$-module $$M$$ and a $$D_{\lambda,N}(\mathfrak{sl}_2)$$-module $$N$$, $$M \otimes N$$ is a $$D_{\lambda,N}(\mathfrak{sl}_2)$$-module via $$\rho$$.

**Theorem**

Let $$p = p_N \ell^N + \hat{p}$$, where $$0 \leq \hat{p} < \ell^N$$, $$0 \leq p_N < \ell$$. Then

$$L_N(p) \simeq L(p_N) \otimes L_N(\hat{p})$$

as $$D_{\lambda,N}(\mathfrak{sl}_2)$$-modules.
Future work

- $\mathcal{D}_{\lambda,N}(g)$ for any $g$ fin. dim. s.s.
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- $D_{\lambda, N}(g)$ for any $g$ fin. dim. s.s.
- Lusztig's isomorphisms at high level?
Future work

1. $D_{\lambda, N}(g)$ for any $g$ fin. dim. s.s.
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1. $\mathcal{D}_{\lambda,N}(g)$ for any $g$ fin. dim. s.s.
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4. Lusztig ’15 approach?