CONNECTEDNESS OF HECKE ALGEBRAS AND THE RAYUELA CONJECTURE: A PATH TO FUNCTORIALITY AND MODULARITY

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Abstract. Let ρ_1 and ρ_2 be a pair of residual, odd, absolutely irreducible two-dimensional Galois representations of a totally real number field F. In this article we propose a conjecture asserting existence of "safe" chains of compatible systems of Galois representations linking ρ_1 to ρ_2. Such conjecture implies the generalized Serre's conjecture and is equivalent to Serre's conjecture under a modular version of it. We prove a weak version of the modular variant using the connectedness of certain Hecke algebras, and we comment on possible applications of these results to establish some cases of Langlands functoriality.

1. Introduction

Let F be a totally real number field. In [KK03] Khare and Kiming conjectured that given ρ_1 and ρ_2 two odd, absolutely irreducible, two-dimensional residual representations of the absolute Galois group of F with values on finite fields of different prime characteristics ℓ and ℓ', satisfying certain local compatibilities (for all primes q different from ℓ and ℓ', there exists a Weil-Deligne parameter and a choice of integral model such that their reductions modulo ℓ and ℓ' are isomorphic to the restrictions of ρ_1 and ρ_2 to a decomposition group at q), there exists a modular form which residually coincides with ρ_1 at ℓ and ρ_2 at ℓ'. This is a very strong statement and no evidence for its truth is known. In particular it implies that given two Hilbert modular forms f and g over F, and two prime numbers ℓ and ℓ', if we assume suitable local compatibilities, then there exists a Hilbert modular form h which is congruent to f modulo ℓ and to g modulo ℓ'.

The purpose of this article is to state a weaker conjecture called **The Rayuela Conjecture** involving "chains of congruent systems" connecting two given Galois representations. The idea is not to link any given pair of residual Galois representations via a single compatible system of Galois representations but via a chain of such systems, meaning that each of them has to be congruent to the next one in the chain, modulo a suitable prime, and it will also be required that some reasonable properties are being satisfied at each of these congruences (this will be done in Section 2). Such conjecture has important consequences, like Serre's conjectures for totally real number fields or base change for totally real number fields (the second one follows from a weaker version that we will explain later). The conjecture for the field of rational numbers follows from the results proved by Dieulefait in [Die12b] (the reader can easily check that the method used to prove base change

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in loc. cit. implies, in fact is based on, the fact that this conjecture is true over $\mathbb{Q}$ at least for modular Galois representations, and this combined with the truth of Serre's conjecture over $\mathbb{Q}$ implies the conjecture).

Although we are not able to prove the conjecture itself, in Sections 4 and 5 we prove that some (weak) variant of it holds in the modular world. This is a generalization of a Theorem of Mazur, a result that he proved for the case of weight 2 modular forms of prime level N.

In this article, for general weights and levels, and for any totally real number field, we extend Mazur's result, and the control we get in the level and weights of the modular forms involved might be used to prove the modular version of the conjecture (although there are still some ingredients missing to get the full statement).

Also we will show that our Conjecture is equivalent to the generalized Serre's conjecture proposed in [BDJ10], at least if some strengthening of the available Modularity Lifting Theorems is assumed. What we show is how one can manipulate a pair of modular Galois representations to end up in a controlled situation where both representations have the same Hodge-Tate weights and ramification in a small controlled set of primes (and then this is combined with Mazur's result).

Combining the ideas of the present article with some nowadays standard tricks (like the use of Micro Good Dihedral primes), one can prove for some small real quadratic fields a Base Change theorem, all that is left is a finite computational check (to ensure that some Hecke Algebra of known level and weight is connected in a good way). This is part of a work in progress of the authors but more will be said at the end of the present article.

For most of the chains constructed in this paper, the construction carries on for the case of abstract Galois representations (the reader should keep in mind that the possibility of constructing congruences between abstract Galois representations where some local information changes and the existence of compatible systems containing most geometric p-adic Galois representations are the two main technical ingredients in the proofs of Serre's conjecture over $\mathbb{Q}$), and then again the use of Micro Good Dihedral primes combined with the results in this paper is enough to reduce the proof of Serre's conjecture over a given small real quadratic field to some special cases of Galois representations with small invariants (Serre's weights and conductor). Thus it is perfectly conceivable that one can give a complete proof of Serre's conjecture over a small real quadratic field F, by just completing this process with a few extra steps designed to remove the Micro Good Dihedral prime from the level (eventually relying on results of Skinner and Wiles if the residually reducible case has to be considered) and end up in some "base case for modularity" over F, such as the modularity/non-existence of semistable abelian varieties of conductor 3 over $\mathbb{Q}(\sqrt{5})$ proved by Schoof ([Sch12]). We plan to check over which real quadratic fields, such proof of Serre's conjecture can be completed in a future work.

Section 3 contains the Modularity Lifting Theorems (that will be denoted MLT) used and needed in the present article. We have included a mixed case that has only been proved in weight 2 situations, yet we assume its truth in more general situations. We believe that such a result can be deduced from the techniques in [BLGGT], but we haven't formally checked this.

Conventions and notations: If F is a number field, we denote by $\mathcal{O}_F$ its ring of integers. By G_F we denote the Galois group $\mathrm{Gal}(\bar{F}/F)$. All Galois representations (residual or ℓ-adic) are assumed to be continuous. Let $\mathfrak{p}$ be a prime ideal in $\mathcal{O}_F$. We denote by $\mathrm{Frob}_\mathfrak{p}$ a Frobenius element over $\mathfrak{p}$.

To ease notation, instead of specifying a prime π in the field of coefficients of a Galois representation and reducing mod π, sometimes we will simply say that we reduce "mod p", where p is the rational prime below π. We expect that this will be no cause of confusion.

Concerning local types of strictly compatible systems of Galois representations, we will use the words "Steinberg", "principal series" and "supercuspidal", to denote the local representations they correspond to under local Langlands.

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2. The Rayuela Conjecture

There are nowadays many MLT that can be applied to propagate modularity through a congruence between two ℓ-adic Galois representations. For us, the main interest in congruences between Galois representations is in the case where a MLT holds in both directions (i.e. modularity of either of the two representations implies modularity of the other one), so we will call a congruence an MLT congruence if the hypothesis of some MLT theorem are fulfilled in both directions. Since MLT hypothesis are becoming less restrictive with time, some of the proofs we give make use of tricks that are required to reduce to situations in which some of the available MLT applies, but some of these tricks are very likely to become obsolete in the near future (when new MLT are proved). There are also steps in the chains that we are going to build in our attempt to connect two given modular or abstract Galois representations that involve congruences that are not known to be MLT. This is the unique reason why we are not able to prove any strong result in this paper, such as relative base change (see the discussion at the end of the article).

Let us now state our conjecture for abstract Galois representations.

Conjecture 1. The Rayuela Conjecture: *Let F be a totally real number field, ℓ_0, ℓ_∞ prime numbers and*

$$\rho_i : \mathrm{Gal}(\bar{F}/F) \to \mathrm{GL}_2(\overline{\mathbb{F}}_{\ell_i}), \qquad i = 0, \infty,$$

two absolutely irreducible odd Galois representations. Then there exists a family of odd, absolutely irreducible, 2-dimensional strictly compatible systems of Galois representations $\{\rho_{i,\lambda}\}_{i=1}^n$ such that:

- $\rho_0 \equiv \rho_{1,\lambda_0} \pmod{\lambda_0}$, *with* $\lambda_0 \mid \ell_0$.
- $\rho_\infty \equiv \rho_{n,\lambda_\infty} \pmod{\lambda_\infty}$, *with* $\lambda_\infty \mid \ell_\infty$.
- *for* $i = 1, \dots, n-1$ *there exist* λ_i *prime such that* $\rho_{i,\lambda_i} \equiv \rho_{i+1,\lambda_i} \pmod{\lambda_i}$,
- *all the congruences involved are MLT.*

Remark 1. The primes involved always are taken to be in the field of coefficient of the corresponding system of representations (for the first and second condition) or in the compositum of the two relevant such fields (for the third conditions). From now on, this remark applies to all congruence between compatible systems appearing in this paper.

The last hypothesis of the conjecture implies that modularity of one of the given representations can be propagated through the chain of congruences produced by the conjecture allowing to prove modularity of the other one. It is thus clear that Conjecture 1 implies Serre's generalized conjecture over F by just taking ρ_∞ to be an irreducible residual representation attached to any cuspidal Hilbert modular form over F.

At this time we think Conjecture 1 is out of reach, but since it does not involve directly modular forms, it might lead to a different attack to Serre's conjecture. Note that in the hypothesis of the Conjecture, one can put ρ_0 inside a strictly compatible system of Galois representations as done by [Die04] for representations over $\mathbb{Q}$ and by [Sno09] for totally real fields (using Theorem 7.6.1 and Corollary 1.1.2). The main difficulty for abstract representations is to connect them. During this work we will show how to manipulate abstract representations such that if we start with any pair of them we can connect both, through suitable chains of MLT congruences, to representations having common values for their Serre's level and weights, but we cannot go any further so far.

For different purposes, such as applications to Langlands functoriality, it is interesting to study the previous conjecture in the modular setting. In this case we are able to prove part of it, namely, we are able to build the chain of congruences but we can not ensure that congruences are MLT at all steps. Still, we find this interesting because advances in MLT theorems may eventually lead to a proof of this modular variant of the conjecture, and this will be evidence for the truth of the Rayuela conjecture. In fact it will give a proof that this conjecture is equivalent to the generalized Serre's conjecture over F (the other implication being trivial as already remarked). The modular variant is the following:

Conjecture 2. *Let F be a totally real number field, ℓ_0, ℓ_∞ prime numbers and*

$$\rho_i : \mathrm{Gal}(\bar{F}/F) \to \mathrm{GL}_2(\overline{\mathbb{F}}_{\ell_i}), \qquad i = 0, \infty,$$

two absolutely irreducible odd Galois representations attached to Hilbert modular newforms f_0 and f_∞, respectively. Then there exists a family of odd, absolutely irreducible, 2-dimensional strictly compatible systems of Galois representations $\{\rho_{i,\lambda}\}_{i=1}^n$ such that:

- $\rho_0 \equiv \rho_{1,\lambda_0} \pmod{\lambda_0}$, *with* $\lambda_0 \mid \ell_0$.
- $\rho_\infty \equiv \rho_{n,\lambda_\infty} \pmod{\lambda_\infty}$, *with* $\lambda_\infty \mid \ell_\infty$.
- *for* $i = 1, \dots, n-1$ *there exist* λ_i *prime such that* $\rho_{i,\lambda_i} \equiv \rho_{i+1,\lambda_i} \pmod{\lambda_i}$,
- *all the congruences involved are MLT.*

The aim of this article is to prove part of Conjecture 2 and show some implications of it related to modularity and functoriality. It is clear that all the representations appearing in Conjecture 2 are modular. Clearly we have:

Theorem 2.1. *Serre's generalized conjecture + Conjecture 2 imply Conjecture 1.*

Proof. This is clear, since if ρ_0 and ρ_∞ are any two Galois representation and if Serre's conjecture holds, they are modular, and then they are connected in the right way by Conjecture 2. □

Conjecture 2 can be proved using standard arguments mainly due to Mazur, if we remove the condition that the congruences are MLT.

Theorem (Mazur)**.** *Let F be a totally real number field and f_0, f_∞ two Hilbert modular forms, whose weights are congruent modulo 2. Then there exists Hilbert modular forms $\{h_i\}_{i=1}^n$ such that $h_1 = f_0$, $h_n = f_\infty$ and such that for $i = 1, \dots, n-1$ there exist λ_i prime such that $\rho_{h_i,\lambda_i} \equiv \rho_{h_{i+1},\lambda_i} \pmod{\lambda_i}$.*

Remark 2. Although connectedness of the Hecke algebra was proved by Mazur for classical modular forms of weight 2 and prime level N using the curve $X_0(N)$, the proof we will give is strongly based in his argument.

Remark 3. If modularity would propagate via any congruence, the previous Theorem would be equivalent to Conjecture 2. Unfortunately this is not clear even for classical modular forms.

The main result of the present article is the following:

Theorem 2.2. *Let F be a totally real number field, ℓ_f, ℓ_g prime numbers and*

$$\rho_f : \mathrm{Gal}(\bar{F}/F) \to \mathrm{GL}_2(\overline{\mathbb{F}}_{\ell_f}), \qquad \rho_g : \mathrm{Gal}(\bar{F}/F) \to \mathrm{GL}_2(\overline{\mathbb{F}}_{\ell_g}),$$

two absolutely irreducible odd Galois representations attached to Hilbert modular newforms f and g, respectively. Then there exists prime ideals $\mathfrak{q} \subset \mathcal{O}_F$ and $p \in \mathbb{Q}$ which splits completely in F, Hilbert modular forms f_∞ and g_∞ of parallel weight 2 and level $\Gamma_0(p\mathfrak{q}^2)$ and two families of odd, absolutely irreducible, 2-dimensional strictly compatible systems of Galois representations $\{\rho^f_{i,\lambda}\}_{i=1}^{n_f}$, $\{\rho^g_{i,\lambda}\}_{i=1}^{n_g}$, such that:

- $\rho_f \equiv \rho^f_{1,\lambda_f} \pmod{\lambda_f}$, *with $\lambda_f \mid \ell_f$.*
- $\rho_{f_\infty,\lambda_\infty} \equiv \rho^f_{n,\lambda_\infty} \pmod{\lambda_\infty}$, *for some prime ideal λ_∞.*
- *for $i = 1, \dots, n-1$ there exist λ_i prime such that $\rho^f_{i,\lambda_i} \equiv \rho^f_{i+1,\lambda_i} \pmod{\lambda_i}$,*
- $\rho_g \equiv \rho^g_{1,\lambda_g} \pmod{\lambda_g}$, *with $\lambda_g \mid \ell_g$.*
- $\rho_{g_\infty,\lambda_\infty} \equiv \rho^g_{n,\lambda_\infty} \pmod{\lambda_\infty}$, *for some prime ideal λ_∞.*
- *for $i = 1, \dots, n-1$ there exist λ_i prime such that $\rho^g_{i,\lambda_i} \equiv \rho^g_{i+1,\lambda_i} \pmod{\lambda_i}$,*
- *all the congruences involved are MLT.*

Remark 4. The importance of our main result is that we can translate Conjecture 2 to a situation where the level and the weight are known. Some of the primes involved are not explicit, since they come from some application of Tchebotarev density theorem. See Section 6.

3. MLT theorems used

In this section we just enumerate the MLT theorems that will be used during this work. By E we denote a finite extension of $\mathbb{Q}_\ell$.

Theorem 3.1 (MLT1)**.** *Let F be a totally real number field, and $\ell \geq 5$ a prime number which splits completely in F. Let $\rho : G_F \to \mathrm{GL}_2(E)$ be a continuous irreducible representation such that:*

- *ρ ramifies only at finitely many primes,*
- *$\overline{\rho}$ is odd,*
- *$\rho|_{G_{F_v}}$ is potentially semi-stable for any $v \mid \ell$ with different Hodge-Tate weights.*
- *The restriction $\overline{\rho}|_{G_{F(\xi_\ell)}}$ is absolutely irreducible.*
- *$\bar{\rho} \sim \bar{\rho}_f$ for a Hilbert modular form f.*

Then ρ is automorphic.

Proof. This is Theorem 6.4 of [HT13]. □

Theorem 3.2 (MLT2 - ordinary case)**.** *Let F be a totally real field, p and odd prime and $\rho : G_F \to \mathrm{GL}_2(E)$ be a continuous irreducible representation such that:*

- *ρ is unramified at all but finitely many primes.*
- *ρ is de Rham at all primes above ℓ.*
- *The reduction $\bar{\rho}$ is irreducible and $\bar{\rho}(G_{F(\xi_\ell)}) \subset \mathrm{GL}_2(\overline{\mathbb{F}_\ell})$ is adequate.*
- *ρ is ordinary at all primes above ℓ.*
- *$\bar{\rho}$ is ordinarily automorphic.*

Then ρ is ordinarily automorphic. If ρ is also crystalline (resp. potentially crystalline), then ρ is ordinarily automorphic of level prime to ℓ (resp. potentially level prime to ℓ).

Proof. This is just Theorem 2.4.1 of [BLGGT]. □

Theorem 3.3 (MLT3 - pot. diagonalizable case)**.** *Let F be a totally real field, $\ell \geq 5$ be and odd prime and $\rho : G_F \to \mathrm{GL}_2(E)$ be a continuous irreducible representation such that:*

- *ρ is unramified at all but finitely many primes.*
- *ρ is de Rham at all primes above ℓ, with different Hodge-Tate numbers.*
- *$\rho|_{G_{F_\lambda}}$ is potentially diagonalizable for all $\lambda \mid \ell$.*
- *The restriction $\bar{\rho}|_{G_{F(\xi_\ell)}}$ is irreducible.*
- *$\bar{\rho}$ is either ordinarily automorphic or potentially diagonalizable automorphic.*

Then ρ is potentially diagonalizable automorphic (of level potentially prime to ℓ).

Proof. This is Theorem 4.2.1 of [BLGGT]. Note that although it is stated only for CM fields, one can chose a suitable CM extension and get the same result for totally real fields using solvable base change. □

Remark 5. The hardest condition to check in the previous Theorem is that of ρ being potentially diagonalizable, but this is satisfied if, in particular, ℓ is unramified in F, ρ is crystalline at all primes above ℓ and the Hodge-Tate weights are in the Fontaine-Laffaille interval.

We also need a mixed variant. Recall the following property.

Lemma 3.4. *Let F be a totally real field, and $\{\rho_\lambda\}$ be a strictly compatible system of continuous, odd, irreducible, parallel weight 2 representations, then ρ_λ is potentially Barsotti-Tate or ordinary at λ.*

Proof. If the restriction to λ is not potentially crystalline, then it is potentially semistable non-crystalline, in which case ordinariness is known. In fact, using potential modularity and the semistability at λ, the system is known to correspond to an abelian variety with potentially semistable reduction at λ, and potentially (over a suitable extension) to a parallel weight 2 Hilbert modular form which is Steinberg at λ. □

The following variant of the current MLT is not yet known:

Assumption 1 (MLT4 - mixed case)**.** *Let F be a totally real field, $\ell \geq 5$ be and odd prime and $\rho : G_F \to \mathrm{GL}_2(E)$ be a continuous irreducible representation such that:*

- *ρ is unramified at all but finitely many primes.*
- *ρ is de Rham at all primes above ℓ, with different Hodge-Tate numbers.*
- *$\rho|G_{F_\lambda}$ is potentially diagonalizable for some $\lambda \mid \ell$ and is ordinary at the others.*
- *The restriction $\bar{\rho}|_{G_{F(\xi_\ell)}}$ is irreducible.*
- *$\bar{\rho}$ is either ordinarily automorphic or potentially diagonalizably automorphic at the same places as $\bar{\rho}$.*

Then ρ is potentially automorphic (of level potentially prime to ℓ).

Remark 6. We believe that using the tools developed in [BLGGT] this result is accessible. Not only we consider the above result accessible, but also if we restrict to the case where both Galois representations involved are of parallel weight 2 (thus, because of the previous Lemma, locally potentially Barsotti-Tate or ordinary at all primes above p), it is a Theorem as proved in [BD], Theorem 3.2.2.

Remark 7. We will need also need Assumption 1 to hold for $p = 2$ and parallel weight 2.

4. Abstract representations:

As mentioned in the introduction, most of the level/weight manipulations that we are going to perform, work not only for modular representation but for abstract ones as well. In this section we will work in the greatest generality possible, and in the next section we will restrict to modular representations putting emphasis on the results that are nowadays only known for modular representations. The results in this section are enough to prove Mazur connectedness Theorem. We begin by recalling the following well-known definition (see [Ser98] for example). If λ_i is a prime in a number field K, we denote by ℓ_i the rational prime below λ_i and by L_i the set of primes in K dividing ℓ_i.

Definition 4.1. A *compatible system* of Galois representations over F is a family of continuous Galois representations

$$\rho_\lambda : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(K_\lambda),$$

where K is a finite extension of $\mathbb{Q}$ and λ runs through the prime ideals of $\mathcal{O}_K$, which satisfy:

1. There exists a finite set of primes S (independent of λ) such that ρ_λ is unramified outside $S \cup L$.
2. For each pair of prime ideals (λ_1, λ_2) in $\mathcal{O}_K$ and for each prime ideal $\mathfrak{p} \notin S \cup L_1 \cup L_2$, the characteristic polynomials $Q_\mathfrak{p}(x)$ of $\rho_{\lambda_i}(\mathrm{Frob}_\mathfrak{p})$ lie in $K[x]$ and are equal.

The most important examples of such families are the ones arising from the étale cohomology of a variety defined over F. In this case we also have some control on the roots of the characteristic polynomials, and some control on the λ-adic representation at primes in L. This motivate the following definition (see [BLGGT], Section 5).

Definition 4.2. A rank 2 *strictly compatible system* of Galois representations $\mathcal{R}$ of G_F defined over K is a 5-tuple

$$\mathcal{R} = (K, S, \{Q_\mathfrak{p}(x)\}, \{\rho_\lambda\}, \{H_\tau\}),$$

where

(1) K is a number field.
(2) S is a finite set of primes of F.
(3) for each prime $\mathfrak{p} \notin S$, $Q_{\mathfrak{p}}(x)$ is a degree 2 polynomial in $K[x]$.
(4) For each prime λ of K, the representation

$$\rho_\lambda : G_F \to \mathrm{GL}_2(K_\lambda),$$

is a continuous semi-simple representation such that:
  - If $\mathfrak{p} \notin S$ and $\mathfrak{p} \nmid \ell$, then ρ_λ is unramified at $\mathfrak{p}$ and $\rho_\lambda(\mathrm{Frob}_{\mathfrak{p}})$ has characteristic polynomial $Q_{\mathfrak{p}}(x)$.
  - If $\mathfrak{p} \mid \ell$, then $\rho|_{G_{F_{\mathfrak{p}}}}$ is de Rham and in the case $\mathfrak{p} \notin S$, crystalline.

(5) for $\tau : F \hookrightarrow \overline{K}$, H_τ contains 2 different integers such that for any $\overline{K} \hookrightarrow \overline{K}_\lambda$ over K, we have that $\mathrm{HT}_\tau(\rho_\lambda) = H_\tau$.
(6) For each finite place $\mathfrak{p}$ of F there exists a Weil-Deligne representation $\mathrm{WD}_{\mathfrak{p}}(\mathcal{R})$ of $W_{F_{\mathfrak{p}}}$ over $\overline{K}$ such that for each place λ of K not dividing the residue characteristic of $\mathfrak{p}$ and every K-linear embedding $\iota : \overline{K} \hookrightarrow \overline{K}_\lambda$, the push forward $\iota\,\mathrm{WD}_{\mathfrak{p}}(\mathcal{R}) \simeq \mathrm{WD}(\rho_\lambda|_{G_{F_{\mathfrak{p}}}})^{K\text{-ss}}$.

Remark 8. If one starts with a 2-dimensional continuous, odd, Galois representations over a totally real number field F, under some minor hypothesis (which are exactly the hypothesis for an MLT theorem to hold), one can prove that such representations is potentially modular. In particular, this implies that the representations is part of a strictly compatible system. Since all the congruences we will work with are where an MLT theorem works in both directions, without loss of generality, we will assume that all the representations come in strictly compatible systems.

Definition 4.3. Let $\{\rho_\lambda\}$ be a strictly compatible system of Galois representations. We say that the system is *dihedral* if the images are compatible dihedral groups, i.e. if there exists a quadratic extension L/F (independent of λ) such that ρ_λ is induced from a λ-adic character of L.

Lemma 4.4. *The family $\{\rho_\lambda\}$ is dihedral if and only if one representation is dihedral.*

Proof. It is clear that if the whole family is dihedral, in particular any of them is dihedral. For the converse, let λ be a prime ideal of $\mathcal{O}_F$, and suppose that ρ_{λ_0} is dihedral. Then there exists a quadratic extension L/F and a λ-adic character $\chi_{\lambda_0} : \mathrm{Gal}(\overline{L}/L) \to K_{\lambda_0}$ such that $\rho_{\lambda_0} = \mathrm{Ind}_{G_L}^{G_F} \chi_{\lambda_0}$. We know that χ_{λ_0} is part of a strictly compatible system of 1-dimensional representations $\{\chi_\lambda\}$, so we are led to prove that $\rho_\lambda \simeq \mathrm{Ind}_{G_L}^{G_F} \chi_\lambda$ for all primes $\lambda \subset \mathcal{O}_F$. This comes from a straightforward computation, since the values of the traces (an even the whole characteristic polynomial) of $\rho_{\lambda_0}(\mathrm{Frob}_{\mathfrak{p}})$ are given in terms of the values of χ_{λ_0}. For split primes in the extension L/F, the trace of $\rho_{\lambda_0}(\mathrm{Frob}_{\mathfrak{p}})$ equals $-\chi_{\lambda_0}(\mathfrak{p}_1) - \chi_{\lambda_0}(\mathfrak{p}_2)$, where $\mathfrak{p}\mathcal{O}_L = \mathfrak{p}_1\mathfrak{p}_2$ and for inert primes, the trace is zero. In particular, the same happens to $\rho_\lambda(\mathrm{Frob}_{\mathfrak{p}})$, for all λ so ρ_λ and $\mathrm{Ind}_{G_L}^{G_F} \chi_\lambda$ have the same trace and are thus isomorphic. $\square$

To apply most MLT's theorems we will need to have some control of the image of our residual Galois representations. In particular we will need the image of its restriction to a cyclotomic extension to be adequate. To avoid checking this particular condition at each step of our chain of congruences, we will move to families with "big image", in the sense that for all but finitely many primes p (and in most steps, for all primes of bounded size), the residual representation has an image containing $\mathrm{SL}_2(\mathbb{F}_p)$.

Proposition 4.5. *Let ρ_λ be a strictly compatible system of odd dihedral representations. Then there exists a strictly compatible system of odd non-dihedral representations $\{\rho_{2,\lambda}\}$ and a prime $\mathfrak{p}$ such that $\rho_{\mathfrak{p}} \equiv \rho_{2,\mathfrak{p}} \pmod p$ and MLT hods in both directions.*

Proof. It is well known that dihedral compatible families do not have Steinberg primes in the level, so we just need to add a Steinberg prime to our representation by some raising the level argument. Let λ be a prime over a prime $p > 5$ and such that:

- p splits completely in F.
- $\lambda \notin S$, i.e. $\rho_{\mathfrak{q}}$ is unramified at λ if $\mathfrak{q} \neq \lambda$.
- L and $F(\xi_p)$ are disjoint, i.e. $F(\xi_p) \cap L = F$.

We want to add a Steinberg prime $\mathfrak{q}$ modulo λ, and by the previous choice, we are in the hypothesis of Theorem 7.2.1 of [Sno09] which says that it is enough to raise the level locally. The local problem is standard, and can be achieve by Tchebotarev's Theorem as follows: take $\mathfrak{q}$ inert in the extension L/F, so that $a_q = 0$. Then the local raising the level condition becomes $\mathcal{N}\mathfrak{q} \equiv -1 \pmod p$, so we chose any such prime and get a global representation with the desired properties. The existence of a strictly compatible system attached to such global representation follows from Taylor's potential modularity result (see [BLGGT]) plus the argument from [Die04], as generalized in [BLGGT]. That MLT holds in both directions comes from Theorem 3.1. ∎

Proposition 4.6. *Let $\{\rho_\lambda\}$ be a strictly compatible system of odd, non-dihedral representations. Then there exists an integer B, such that if $\mathcal{N}(\lambda) > B$ then $\mathrm{SL}_2(\mathbb{F}_p) \subset \mathrm{Im}(\overline{\rho}_\lambda)$.*

Proof. According to Dickson's classifications of subgroups of $\mathrm{PGL}_2(\mathbb{F}_\lambda)$, when we consider the residual representations, they might:

(1) contain $\mathrm{PSL}_2(\mathbb{F}_\lambda)$,
(2) be reducible,
(3) be dihedral,
(4) be isomorphic to A_4, S_4 or A_5.

We want to prove that in our hypothesis, there are only finitely many primes where the first case does not hold. But the second case is exactly Lemma 5.4 of [CG11], the third case Corollary 5.2 of [CG11] (here we use the assumption that the system is not dihedral), and the last case is Lemma 5.3 of [CG11]. ∎

Remark 9. Another way to prove the last Proposition (following the classical approach of Ribet) it to first apply some potential automorphy result (like in [BLGGT]) to deduce that the system is potentially modular. Then its restriction is isomorphic to that of a Hilbert modular form. Furthermore, since our abstract representations are not dihedral, the respective Hilbert modular form has no CM. But for Hilbert modular forms, such result is proven in [Dim05] Proposition 0.1.

By the above considerations, from now on we will only consider non-dihedral families, so we will skip writing this hypothesis in the next results.

Proposition 4.7. *Let $\{\rho_{1,\lambda}\}$ be a strictly compatible system of odd irreducible Galois representations. Then there exists a compatible system $\{\rho_{2,\lambda}\}$ of parallel weight 2 representations and a prime $\mathfrak{p}$ such that $\rho_{1,p} \equiv \rho_{2,p} \pmod{p}$ and MLT holds in both directions.*

Proof. Let $p > 5$ be a prime which splits completely in F, does not divide the level of ρ, is larger than all weights of the system and such that the image of $\overline{\rho}_{1,\mathfrak{p}}$ is large. Let $\det(\overline{\rho}) = \overline{\psi}\overline{\chi_p}$, where $\overline{\chi_p}$ is the reduction of the cyclotomic character, and let ψ be any lift of $\overline{\psi}$. Then Theorem 7.6.1 of [Sno09] implies that $\overline{\rho}$ admits a weight two lift to $\overline{\mathbb{Q}}_p$ which ramifies at the same primes as ρ and ψ, with determinant $\psi\chi_p$. MLT hold in both directions by the same proof as the previous Proposition. $\square$

4.1. **Adding a good dihedral prime.** As already mentioned, while working with MLT one needs to ensure that residual images are big enough to be in the hypothesis of such a theorem. Sometimes, one needs the restriction of the residual representation to the cyclotomic extension of p-th roots of unity to have adequate image, but for most MLT requiring irreducibility of this restriction is enough (and for $p > 5$ it is known that both properties are equivalent). A way to get this property guaranteed at most steps is by introducing to the level an extra prime which forces the image modulo all primes up to a certain bound to be "non-exceptional", i.e. it is irreducible, and its projectivization is not dihedral, nor the exceptional groups A_4, S_4, A_5. A way to get this, is by adding a "good dihedral prime" (with respect to the given bound) as was introduced by Khare and Wintenberger in their work on Serre's conjecture. The difference with the classical setting is that since we work with two strictly compatible systems at the same time, we need to add the same good dihedral prime to both of them.

Proposition 4.8. *Let $\{\rho_{1,\lambda}\}$ and $\{\rho_{2,\lambda}\}$ be two strictly compatible systems of continuous, odd, irreducible representations of parallel weight 2. Fix B a positive integer, larger than 5 and than all primes in the conductors of both systems. Let $p \equiv 1 \pmod{4}$ be a rational prime such that:*

- *p is bigger than B.*
- *p splits completely in the compositum of the coefficient fields of ρ_1 and ρ_2.*
- *p is relatively prime to the conductors of both systems.*
- $\mathrm{Im}(\overline{\rho}_{i,\mathfrak{p}}) = \mathrm{GL}_2(\mathbb{F}_p)$*, $i = 1, 2$, for some prime $\mathfrak{p}$ over p.*

Then there exists a prime q not dividing the conductor of the systems such that:

- *$q \equiv -1 \pmod{p}$.*
- *q splits completely in the extension F' given by the compositum of all quadratic extensions of F ramified only at primes above rational primes $\ell < B$.*
- *$q \equiv 1 \pmod{8}$.*
- *There exists a prime ideal $\mathfrak{q}$ in F over q such that the image of $\overline{\rho}_{\mathfrak{p}}(\mathrm{Frob}_{\mathfrak{q}})$ has eigenvalues 1 and -1.*

With this choice of primes, there exists two strictly compatible systems of continuous representations $\{\varrho_{1,\lambda}\}$ and $\{\varrho_{2,\lambda}\}$ of parallel weight 2 such that:

1. $\overline{\rho}_{1,\mathfrak{p}} \simeq \overline{\varrho}_{1,\mathfrak{p}}$.
2. $\overline{\rho}_{2,\mathfrak{p}} \simeq \overline{\varrho}_{2,\mathfrak{p}}$.
3. *$\{\varrho_{i,\lambda}\}$, for $i = 1, 2$, is locally good dihedral at $\mathfrak{q}$ (w.r.t. the bound B).*
4. *$\varrho_{i,\mathfrak{p}}$, is Barsotti-Tate at all primes dividing p and has the same type as $\rho_{i,\mathfrak{p}}$ locally at any prime other that $\mathfrak{q}$ for $i = 1, 2$.*

(5) *The congruences are MLT.*

Remark 10. We do not want to make precise what condition (4) of the last statement means (the local type), since it will not be important for our purposes, but what we prove is the following: for abstract representations, we will use Theorem 7.2.1 of [Sno09], where a "type" is understood as a Weil type (no information on the monodromy), while for modular representations (actually residually modular ones), we will use Theorem 3.2.2 of [BD], which uses the complete notion of type.

Proof. The existence of the primes $\mathfrak{p}$ and $\mathfrak{q}$ follows with almost the same arguments as [Die12a] (Lemma 3.3). The only difference is that we need to consider the compositum of $\mathbb{Q}(i), F$ and the coefficient field of ρ_1 and ρ_2. Take p big enough (for the images to be large) and split in such extension. Then q is chosen using Tchebotarev density Theorem, with the condition that it hits complex conjugation in the same suitable field.

We are lead to prove the existence of a lift of $\bar{\rho}_{1,\mathfrak{p}}$ with the desired properties. By Theorem 7.2.1 of [Sno09], or by Theorem 3.2.2 of [BD] (see the last remark), we know that a global representation with the desired properties exists if and only if locally the corresponding lifts do exist, so we only need to show which are the local deformation conditions:

- At the primes $\mathfrak{l} \neq \mathfrak{q}$, that of $\rho_i|_{G_\mathfrak{l}}$.
- $\varrho_i|_{D_\mathfrak{q}} = \mathrm{Ind}_{\mathbb{Q}_q}^{\mathbb{Q}_q[\sqrt{\epsilon}]}(\chi)$, where $\mathbb{Q}_q[\sqrt{\epsilon}]$ is the unique quadratic unramified extension of $\mathbb{Q}_q$, and χ is a character with order p (this is called type C in [Sno09]).

This proves the existence of $\varrho_{i,\mathfrak{p}}$. The congruence is MLT in both directions because of Theorem 3.1 (in this case both forms are of parallel Hodge-Tate weight 2 which is smaller than the prime p). Since we are in the hypothesis of an MLT theorem, we can put such representation into a strictly compatible system and get the result. □

Remark 11. Although we stated the result for representations of parallel weight 2, in general we can first move to parallel weight 2 (using Proposition 4.7) and then add the good dihedral prime.

With this result, we can prove Mazur's Theorem.

Theorem (Mazur). *Let F be a totally real number field, f_0 and f_∞ two Hilbert modular forms, whose weights are congruent modulo 2, then there exists Hilbert modular forms $\{h_i\}_{i=1}^n$ such that $h_1 = f_0$, $h_n = f_\infty$ and such that for $i = 1, \dots, n-1$ there exist λ_i prime such that $\rho_{h_i,\lambda_i} \equiv \rho_{h_{i+1},\lambda_i} \pmod{\lambda_i}$.*

Proof. By the previous results, we can assume that both forms are of parallel weight 2, and do not have complex multiplication (or we move to such a situation using Proposition 4.5 and Proposition 4.7), for some common level $\Gamma_1(\mathfrak{n})$ (of course they might not be new at the same level). The idea now is to find both modular forms in the cohomology of a Shimura curve, and for this purpose we need to add an auxiliary prime to the level where both forms are not principal series (in case $[F : \mathbb{Q}]$ is odd). So what we do is to raise the level of both forms at an auxiliary prime $\mathfrak{p}$ as in Proposition 4.8 (we could also add a Steinberg prime). Now we consider the Shimura curve $X^{\mathfrak{p}}(\mathfrak{n})$ ramified at all infinite places of F but one, and at the auxiliary prime $\mathfrak{p}$ if needed (depending whether $[F : \mathbb{Q}]$ is odd or even) and with level $\mathfrak{n}$.

We apply Mazur's argument just as in [Maz77] (proposition 10.6, page 98) with some minor adjustments. The main idea of his proof is the following: if the Jacobian J is a product of two abelian varieties $A \times B$, since J decomposes (up to isogeny) as a product of simple factors with multiplicity one, there are no nontrivial homomorphisms from A to $\hat{B}$ nor from B to $\hat{A}$. Then the principal polarization of J induces principal polarizations in A and B, but a Jacobian cannot decompose as a nontrivial direct product of principally polarized abelian varieties. From this it follows that $\operatorname{Spec} \mathbb{T}$ is connected, where $\mathbb{T}$ denotes the Hecke algebra acting on J.

In his original article Mazur was dealing with the curve $X_0(N)$, with N a prime number, so there are no old forms appearing in $J_0(N)$. To use the same argument in our context, we have to deal with old forms as well, and the problem is that the abelian varieties A_f corresponding to old forms do not appear with multiplicity one in the decomposition (up to isogeny) of the Jacobian of a modular or a Shimura curve. But this is not a problem if we observe that for what we want it is not necessary to prove the connectedness of $\operatorname{Spec} \mathbb{T}$, it is enough to show that the anemic Hecke algebra $\mathbb{T}_0$ generated only by the Hecke operators with index prime to the level is connected. Therefore, what we need is to discard the cases where the Jacobian of $X^{\mathfrak{p}}(\mathfrak{n})$ decomposes as a product of abelian varieties $A \times B$ with every simple factor in A and every simple factor in B being orthogonal (recall that now these simple factors need not appear with multiplicity one). In such case, the same proof as in Mazur's article applies, and gives the connectedness we are looking for. □

5. Killing the level

Now that we have a good dihedral prime, which controls the image of the residual representations in the families of the two representations for small primes, we want to connect them at a chosen level. From know on, we will only consider modular representations, pointing out in each case the needed result for abstract ones.

5.1. **Modifying the non-Steinberg primes.**

From now on we work under the assumption that MLFMT (Assertion 1) is true. We call an abstract strictly compatible system of continuous, odd, irreducible Galois representations *modular* if there exists a Hilbert modular form, whose attached Galois representations matches the abstract one.

Theorem 5.1. *Let $\{\rho_{1,\lambda}\}$ be a strictly compatible system of continuous, odd, irreducible, Galois representations attached to a Hilbert newform f over a totally real number field F of parallel weight 2, containing in its conductor a locally good dihedral prime $\mathfrak{q}$ (w.r.t. some sufficiently large bound B). Then:*

- *There exists a strictly compatible system of continuous, odd, irreducible, parallel weight 2 representations $\{\rho_{2,\lambda}\}$, which is semistable at all primes except the same good dihedral prime $\mathfrak{q}$ and such that the Steinberg ramified primes are bounded in norm by B.*
- *There exists a chain of congruences of compatible systems linking $\{\rho_{1,\lambda}\}$ and $\{\rho_{2,\lambda}\}$ such that all congruences involved occur in residual characteristics bounded by B and are MLT.*

In particular the system $\{\rho_{2,\lambda}\}$ is also modular.

Proof. Let λ be a prime which is supercuspidal or principal series. Let p be a prime number dividing the order of the character corresponding to ramification at λ. We consider two different cases: if p is relative prime to λ, we call it the *tamely ramified case*, while the other case we call it the *wildly ramified case*.

The tamely ramified case. Let $\mathfrak{p}$ be a prime ideal in K (the coefficient field) dividing p, and consider the residual mod $\mathfrak{p}$ representation. Then the p-part of the ramification is lost, so we take a minimal lift with the same parallel weight 2 (it exists by Theorem 3.2.2 in [BD]). Observe that at this step we are not only modifying the ramification type at λ, but also at any other prime supercuspidal or principal series in the prime-to-p part of the conductor with ramification given by a character of order divisible by p. Then by Lemma 3.4, we might be in a mixed situation of potentially Barsotti-Tate and ordinary representations. Using Theorem 3.2.2 of [BD] (recall that this is a special case of our Assumption 1), we get rid of this p-part of the inertia with an MLT congruence. Iterating this process, we end up killing all tamely ramified ramification given by characters, i.e. the prime-to-p part of the ramification at primes dividing p is killed. So we are reduced to the case where all primes in the conductor are either Steinberg or with ramification given by a prime order character whose order is divisible by the ramified prime.

The wildly ramified case. In this case, we will move the wildly ramified primes (up to twist) to tamely ramified primes, so the previous argument ends our proof. For a prime $\mathfrak{t}$ in the conductor of the wildly ramified case, let us call t the rational prime below $\mathfrak{t}$ and consider a mod t congruence, up to twist by some finite order character ψ, with the Galois representation corresponding to a Hilbert newform H of parallel weight 2 with at most $\mathfrak{t}$ to the first power in the level (i.e. Γ_1 at $\mathfrak{t}$), and the same for the other primes dividing t. The existence of such a form is proved in [BDJ10], Corollary 2.12.

By level-lowering, we can assume that the only extra primes in the level of H are those primes that have been introduced to the residual conductor while twisting by a character ψ. It is easy to see that such character can be chosen such that at primes other than those dividing t it has square-free conductor. To this congruence, Theorem 3.2.2 of [BD] applies (the conditions for this theorem are preserved by twisting, and modularity too) so we are reduced to a case where we have a system that is either Steinberg or tamely ramified principal series at primes dividing t, and tamely ramified principal series at all extra primes introduced by ψ. If we iterate this process at all wildly ramified primes, we end up with a system with no wildly ramified primes. We repeat the previous case procedure of killing all ramification given by tamely ramified characters, but now in the absence of wild ramification we finish with a compatible system such that all its ramified primes other than the good-dihedral prime are Steinberg. It is not hard to see that in all this process, for a suitably chosen bound B, all auxiliary primes can be taken to be smaller than B. □

Remark 12. Except for the application of Corollary 2.12 of [BDJ10] at a key point, and a better control of the local types (which seems reasonable for abstract representations), all congruences in the above proof are known to exist for abstract Galois representation, so an analogue of the above result for abstract compatible systems can be proved assuming that this result from [BDJ10] generalizes to the abstract setting (thus relating the modularity of any geometric compatible system to that of a system with only Steinberg primes).

5.2. **Killing the Steinberg primes.** This part of the process is a little more delicate, and the MLT theorems are more restrictive, so what we do first is move the Steinberg primes to primes which split completely in F.

Theorem 5.2. *Suppose that Assumption 1 is true. Let $\{\rho_{1,\lambda}\}$ be a system of modular continuous, odd, irreducible, parallel weight 2 representations with a big locally good dihedral prime $\mathfrak{q}$. Let $\mathcal{L} \mid \ell$ ($\ell \in \mathbb{Z}$) be a Steinberg prime of the system which does not split completely in F. We also assume that the system is either unramified or Steinberg at all primes dividing ℓ. Then:*

- *there exists a strictly compatible system $\{\rho_{2,\lambda}\}$ of continuous, odd, irreducible, parallel weight 2 representations, which has the same ramification behavior at all primes except those dividing ℓ, where it is unramified, and has at most a set of extra Steinberg primes, all of them dividing the same rational prime which splits completely in F.*
- *there exists a chain of MLT congruences linking the two systems.*

In particular, the system $\{\rho_{2,\lambda}\}$ is also modular.

Proof. We look at $\rho_{1,\lambda'}$ for a prime λ' dividing ℓ and we reduce it modulo ℓ. We can construct then a modular lift which is unramified at $\mathcal{L}$ and at all primes dividing ℓ, and with weights among those predicted by the Serre's weights of the residual representation. This follows from the results of [BDJ10], since in the Steinberg case the determinant locally at primes above ℓ is a fixed power of the cyclotomic character and in this case it follows from Proposition 2.5 in [BDJ10] that a lift with level prime to ℓ exists. Note that this congruence is MLT, at least under Assumption 1, which applies because at Steinberg primes, weight 2 representations are ordinary, and the crystalline lift of higher weight can also be taken to be ordinary (observe that to deduce modularity of the weight 2 family from the other one we can apply Theorem 3.2.2 in [BD]).
Let p be a big prime (bigger than the weights of this second family) which splits completely in F, then by [Sno09] we can construct a third family with parallel weight 2 by looking modulo p, and this congruence is MLT because of the results in [BLGGT] (we are comparing a Fontaine-Laffaille with a potentially Barsotti-Tate representation, so they are both potentially diagonalizable). We apply the previous section method to this new family to end up with a representation which is at most Steinberg at all primes dividing p as desired. Observe that in this last step no extra ramified primes are introduced because we do not have wild ramification at p (the parallel weight 2 lift implies tamely ramified principal series at all primes above p). $\square$

Now that we have only totally split Steinberg primes in our family, we can get rid of the Steinberg primes.

Theorem 5.3. *Let $\{\rho_{1,\lambda}\}$ be a system of modular, irreducible, parallel weight 2 representations, whose ramification consists of a big locally good dihedral prime $\mathfrak{q}$ and Steinberg primes which split completely in F. Then:*

- *There exists a strictly compatible system $\{\rho_{2,\lambda}\}$ of continuous, odd, irreducible representations which are only ramified at the locally good dihedral prime $\mathfrak{q}$.*

- *There exists a chain of MLT congruences linking the two systems.*

In particular, the system $\{\rho_{2,\lambda}\}$ is also modular.

Proof. The procedure is quite similar to the one in the previous Theorem, but now we can use Theorem 3.1. For each Steinberg prime of residual characteristic ℓ we look at $\overline{\rho}_{1,\lambda}$ with $\lambda \mid \ell$. There exists a minimal lift which is crystalline at all primes of residual characteristic ℓ, as follows again by the results in [BDJ10]. Now, since the prime is split, the congruence is MLT by the cited Theorem. The only delicate point here is that to apply Theorem 3.1 we need the residual characteristic to be different from 2 and 3, so if we have such a small Steinberg prime, we first apply Theorem 5.2 to transfer the ramification to Steinberg ramification at some larger split prime. □

Remark: In the previous two Theorems, modularity of the given system was only used to ensure existence of the lift corresponding to a system with no ℓ in its conductor, and this was deduced from results of [BDJ10]. In the case of abstract Galois representations, after computing the Serre's weights of the residual representation one should be able to propose locally at all primes above ℓ a crystalline lift of the residual local representation, but then the problem is that in order to apply results such as those in [BLGGT] that guarantee existence of a global lift with such local conditions, one should be in a potentially diagonalizable case. In particular, under the conjecture that all potentially crystalline representations are potentially diagonalizable, the previous two theorems shall generalize to the abstract setting (of course, the conclusion that the last system is modular shall also be removed).

Proof of Theorem 2.2. With the machinery developed in the previous chapters, starting with two modular representations, we can take each of them to representations which are only ramified at the good dihedral prime $\mathfrak{q}$. The only problem is that while killing the Steinberg primes, we lost control over the weights, so to take the families to parallel weight 2, we chose an auxiliary prime p which splits completely in F and consider a parallel weight 2 lift of each family modulo p, with the same local type at $\mathfrak{q}$. Note that we cannot assure that the forms we get at the end of the process will be newforms for $\Gamma_0(p\mathfrak{q}^2)$, because at some of the prime ideals of F dividing p our representations could be unramified. □

6. Further developments

As mentioned before, although we have some good control on the level of the forms we started with, the primes in the level are not explicit. One can go further and change the primes in the level for smaller and concrete ones, so as to check whether the connectedness of the Hecke algebras in Mazur's Theorem corresponds to a chain where all congruences are MLT or not. For this purpose, one can use the notion of *micro good dihedral primes*. Adding a micro good dihedral prime (which is chosen asking some splitting behavior in the base field) one can get rid of the good dihedral prime, and also bound the Steinberg primes in the level. These ideas, although standard (see for example [Die12a]) are more delicate, and involve some technicalities that we prefer to avoid in the present article. With this control, one can give an algorithm that given a totally real number field, checks whether our approach implies Base Change over that field via a finite computation involving Hilbert modular forms. See [DP] for more details.

Concerning Serre's conjecture over a specified small real quadratic field, one should carefully check that the chain of congruences constructed carries on to the case of abstract representations, and then once having reduced the problem to the case of representations of concrete small invariants (those where the above process concludes, after the introduction of the micro good dihedral prime) one should connect such representations to some "base case" where modularity or residual reducibility is known (applying for example the result of Schoof recalled in the introduction), checking on the way that all congruences are MLT (with the advantage that over quadratic fields there are also MLT of Skinner and Wiles that deal with the residually reducible case, under suitable assumptions). We plan to check in a future work if this strategy succeeds in giving a proof of Serre's conjecture over some small real quadratic field.

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