ON THE NUMBER OF GALOIS ORBITS OF NEWFORMS

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Abstract. Counting the number of Galois orbits of newforms in $S_k(\Gamma_0(N))$ and giving some arithmetic sense to this number is an interesting open problem. The case $N = 1$ corresponds to Maeda's conjecture (still an open problem) and the expected number of orbits in this case is 1, for any $k \geq 16$. In this article we give local invariants of Galois orbits of newforms for general N and count their number. Using an existence result of newforms with prescribed local invariants we prove a lower bound for the number of non-CM Galois orbits of newforms for $\Gamma_0(N)$ for large enough weight k (under some technical assumptions on N). Numerical evidence suggests that in most cases this lower bound is indeed an equality, thus we leave as a Question the possibility that a generalization of Maeda's conjecture could follow from our work. We finish the paper with some natural generalizations of the problem and show some of the implications that a generalization of Maeda's conjecture has.

Contents

Introduction

A conjecture of Maeda predicts that there is a unique Galois orbit of level 1 newforms for all weights $k \geq 16$. A natural problem is to study what happens while working with modular forms of arbitrary level N. For small weights, the number of Galois orbits in $S_k(\Gamma_0(N)$ is hard to understand, for example in weight 2 (which is not in the original

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Maeda's conjecture) there are many elliptic curves of the same conductor N. However, while computing spaces of modular forms of a fixed level and varying the weight k, the situation changes completely. Surprisingly, the number of orbits tends to stabilize very fast, and the numbers obtained follow some pattern (see for example the data in [Tsa14]).

While proving Maeda's conjecture of newforms for $\mathrm{SL}_2(\mathbb{Z})$ is a very hard problem, it is fairly easy to prove the lower bound 1 for the number of Galois orbits when $k \geq 16$, which corresponds to the "easy" inequality. The purpose of the present article is to present invariants of Galois orbits of eigenforms, and use them to give a lower bound for the number of Galois orbits of newforms in $S_k(\Gamma_0(N))$ for k large enough (i.e. for all $k \geq k_0$, for some $k_0 \geq 2$). In many instances, the numerical data seems to indicate that such inequality is in fact an equality.

The invariants introduced are of two different natures: a local one, namely the Galois orbit of the *local type* of the automorphic representation at each prime dividing N; and a local-global one, coming from the Atkin-Lehner eigenvalue at p of the modular form f. Recall that the local type can be thought of (via the Local-Langlands correspondence) as the isomorphism class of the restriction of the Weil-Deligne representation to the inertia subgroup (see Section 1). The Atkin-Lehner sign is more subtle, and it is not clear how to obtain it from the Weil-Deligne representation.

The lower bound we prove is of the following form. Let $\mathbf{NCM}(N,k)$ denote the number of Galois orbits of non-CM newforms of level N and weight k. If N is a prime power or if N is square-free, then

$$\prod_{q \mid N} \mathbf{LO}(q^{\mathrm{val}_q(N)}) \leq \mathbf{NCM}(N,k), \tag{1}$$

for all k large enough, where the values of $\mathbf{LO}(q^r)$ are given in Theorem 24. Let us explain a little bit all the ingredients of the formula and its proof.

In Section 1, we recall the theory of local types for GL_2, and consider Galois conjugacy classes of them. Since we want to count the number of Galois orbits of modular forms, a naive idea is that while conjugating a modular form f, one also conjugates the local types, hence while identifying global conjugates one should do the same locally. The section contains a detailed description of local types and their number, the main result being a formula for the number of Galois orbits of local types of level p^n for any prime p (the case $p = 2$ being the hardest one).

Section 2 considers local types coming from modular forms. There are two advantages on doing so: first we prove (see Lemma 15 and Lemma 16) that if a modular form f has a local type $\tilde{\tau}$, then its coefficient field is an extension of $\mathbb{Q}$ with enough endomorphisms. In particular, this shows that the naive approach (looking at local Galois orbits) is correct in most instances. This is not true in general, but it is true under the hypothesis on N stated before, i.e. N is a prime power or a square free integer (see Remark 26 to understand the general case). The second advantage of working with modular forms of trivial Nebentypus is that we have the theory of Atkin-Lehner involutions. Clearly their eigenvalues are constant on Galois orbits (see Lemma 18), thus they give an extra invariant. There is an interesting phenomenon while computing Atkin-Lehner eigenvalues: a modular form of level p (prime) might have any Atkin-Lehner eigenvalue (for different values of p and k both are attained) but its twist by the quadratic character unramified outside p does not! Then we might have two different Galois orbits of level p (distinguished by the Atkin-Lehner eigenvalue) whose twists (of level p^2) still give two different orbits, but both of them having the same Atkin-Lehner eigenvalue. This phenomenon suggests that we do not have to consider the Atkin-Lehner sign as an invariant, but what we call the *minimal Atkin-Lehner sign* (see Definition 19).

An important result in this direction is the determination of what are the possible Atkin-Lehner signs for each local type. Such description is given in Theorem 20, which describes when the local type determines the minimal Atkin-Lehner sign uniquely, and when it does not. For the latter, we prove that the local sign varies while twisting by the unramified quadratic character at p. Then we can count the number of pairs $(\tilde{\tau}, \epsilon)$ consisting of an isomorphism class of local types of level p^n and its compatible minimal Atkin-Lehner sign. This number is denoted by $\mathbf{LO}(p^n)$ and is the one appearing in (1). An important result in this section is a precise formula for such value (see Theorem 24).

Section 3 considers the problem of the existence of pairs $(\tilde{\tau}, \epsilon)$ as before, for large values of k. The main result is Theorem 25, in the case N a prime power or square-free. The proof is based on results from Weinstein ([Wei09]) and Kim-Shin-Templier [KST16]. The latter article proves an existence result of modular forms with a fixed local representation at p (not being principal series), not just its type!. Such a result is very strong, but it implies Theorem 25 under our hypothesis. For general N, a different approach must be taken, as principal series would need to be included (see Remark 26). We want to stress that if Theorem 25 holds for general N, then (1) holds in general (since the restriction on N is only used in such result).

It is natural to ask why we discard the CM modular forms in our result. The reason is twofold: first of all, modular forms with complex multiplication do form an orbit on their own. The second one is that (for k large enough) when the space of newforms of a given level N contains a CM Galois orbit, there is another Galois orbit with the same local type without complex multiplication.

Example 1. Let $N = 9$ and $k = 16$. This space contains a unique modular form with complex multiplication, whose q-expansion starts $q - 32768q^4 + 1244900q^7 + O(q^{12})$. The local characters giving the local representation can be computed with [S$^+$13]; they correspond to the character over the unramified quadratic extension of $\mathbb{Q}_3$, sending a generator s of $\mathbb{F}_9^\times$ to $\sqrt{-1}$. There is another form, with q-expansion $q + aq^2 + 87112q^4 + 464aq^5 - 2591260q^7 + 54344aq^8 + O(q^{10})$, where $a^2 = 119880$ whose characters (at inertia) are exactly the same hence both representations have the same local type. Note that the latter form does not have complex multiplication (as the 5-th coefficient is non-zero).

This same situation holds in general and is part of Theorem 25, whose proof uses the fact that the number of non-CM forms with prescribed local types grows linearly on the weight k, while the number of CM forms is constant. With all these ingredients, the proof of the stated bound (Theorem 27) is straightforward.

In [Tsa14] the author proposed a generalization of Maeda's conjecture (Conjecture 2.2) to arbitrary levels N as follows:

- the function $\mathbf{NCM}(N,k)$ is constant in the variable k for k large enough
- the limit function $\mathbf{NCM}(N) := \lim_{k\to\infty} \mathbf{NCM}(N,k)$ is multiplicative.
- some values of $\mathbf{NCM}(p^n)$ were tabulated based on numerical experiments.

The present article started from the effort to prove that the tabulated numbers have some meaning, and to express them as Galois orbits invariants. While doing so, we realized that we do not expect the function $\mathbf{NCM}(N)$ to be multiplicative (see Remark 26). The reason is that the automorphisms of the coefficient field are not enough in general to conjugate two different local types independently. Examples for this involve huge levels which are nowadays unfeasible to compute with nowadays resources (this was probably the reason why this phenomena went unobserved).

We end the article with some possible generalizations of the present ideas, and some applications. We propose a question (Question 29) which is in the spirit of Maeda's original conjecture. Numerical evidence (gathered by the third named author) suggests that in most of the considered cases this lower bound is indeed an equality (for large enough weight k) to the number of such Galois orbits, thus we leave as a Question the possibility that a generalization of Maeda's conjecture could follow from our work; in which case, for historical reasons, it should be called the "Maeda-Tsaknias" conjecture. In Example 2 we present a discrepancy between the experimental values of $\mathbf{NCM}(256,12)$ and our lower bound which seems to persist for all weights greater than 12. We could not find any extra invariant that justifies this discrepancy (it is an interesting problem to investigate). In particular, if the value of $\mathbf{NCM}(256)$ is indeed 12, Question 29 needs to be reformulated taking into account the missing invariants.

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1. Inertial types for GL_2

Let $\mathcal{A}_p$ denote the set of isomorphism classes of complex-valued irreducible admissible representations of $\mathrm{GL}_2(\mathbb{Q}_p)$. The local Langlands correspondence gives a bijection between $\mathcal{A}_p$ and the isomorphism classes of two-dimensional Frobenius-semisimple Weil-Deligne representations of $\mathbb{Q}_p$, say $\pi \leftrightarrow \tau(\pi)$. Furthermore, the equivalence preserves L-functions and ϵ-factors (see [Kut80] and [BH06]). Via the local-Langlands correspondence, we will move to-and-from $\mathcal{A}_p$ indistinctly.

Definition 1. A local inertial type of a Weil-Deligne representation τ is the isomorphism class of its restriction to the inertia subgroup. We denote it by $\tilde{\tau}$. We say that a type is *trivial* or *unramified* if $\tilde{\tau}$ is the trivial representation.

Remark 2. The inertial type can also be described in terms of the restriction $\pi|_{\mathrm{GL}_2(\mathbb{Z}_p)}$, as explained in [Hen02]. See also[Wei09, Section 2.1].

While working with local types, the maximal ideal is always clear from the context. For this reason, and to ease notation, for the rest of the article we will use the term *conductor* (of a representation, of a character, etc) to denote the exponent of the conductor. We hope this will not create any confusion.

Definition 3. A global inertial type is a collection $(\tilde{\tau}_p)_p$ with p running over all prime numbers, where each $\tilde{\tau}_p$ is a local type at p and $\tilde{\tau}_p$ is trivial for all primes but finitely many.

Theorem 4. *Any element π of $\mathcal{A}_p$ is one of the following:*

- **Principal series:** *given characters $\chi_1, \chi_2 : \mathbb{Q}_p^* \to \mathbb{C}^*$ such that $\chi_1\chi_2^{-1} \neq |\ |^{\pm 1}$, the representation $\pi(\chi_1, \chi_2)$ is the induction of a 1-dimensional representation of the Borel subgroup of $\mathrm{GL}_2(\mathbb{Q}_p)$, with action given by $\chi_1 \otimes \chi_2$. The central character of $\pi(\chi_1, \chi_2)$ equals $\chi_1\chi_2$ and its conductor equals $\mathrm{cond}(\chi_1) + \mathrm{cond}(\chi_2)$.*
- **Special representations or Steinberg:** *if $\chi_1\chi_2^{-1} = |\ |^{\pm 1}$, the representation $\pi(\chi_1, \chi_2)$ contains an irreducible codimension 1 subspace/quotient. Such representations are called* Steinberg *and they are twists of a "primitive" (or standard) one denoted* St. *The central character of $\mathrm{St} \otimes \chi$ equals χ^2 and its conductor equals*

$$\mathrm{cond}(\mathrm{St} \otimes \chi) = \begin{cases} 2\,\mathrm{cond}(\chi) & \text{if } \chi \text{ is ramified,} \\ 1 & \text{otherwise.} \end{cases}$$

- **Supercuspidal representations***: the remaining ones, see* [Kut78a, Kut78b].

Using the previous classification the local Langlands correspondence is given explicitly by:

(1) The Weil-Deligne representation attached to $\pi(\chi_1, \chi_2)$ via the local Langlands correspondence consists of the pair $(\chi_1 \oplus \chi_2, 0)$, i.e. the Weil representation is given by the direct sum $\chi_1 \oplus \chi_2$ (recall that we are identifying characters of the Weil group and of $\mathbb{Q}_p^\times$ via local class field theory) and the monodromy is trivial.
(2) The Weil-Deligne representation attached to the representation $\mathrm{St} \otimes \chi$ consists of the pair $(\chi\omega_1 \oplus \chi, \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right))$, where ω_1 is the unramified character giving the action of $W(\mathbb{Q}_p)$ on the roots of unity. This is the only case of non-trivial monodromy.
(3) If $p \neq 2$, the Weil representation attached to the supercuspidal representations via the local Langlands correspondence equals $\mathrm{Ind}_{W(\mathbb{Q}_p)}^{W(E)} \theta$, where $E/\mathbb{Q}_p$ is a quadratic extension, and $\theta : W(E) \to \mathbb{C}^\times$ is a character. Furthermore, regarding θ as a character of $E^\times$, such representation is irreducible precisely when θ does not factor through the norm map $\mathrm{Norm} : E^\times \to \mathbb{Q}_p^\times$. Let ϵ_E denote the quadratic character of $\mathbb{Q}_p^\times$ associated by local class field theory to the extension $E/\mathbb{Q}_p$. The central

character of $\mathrm{Ind}_{\mathrm{W(E)}}^{\mathrm{W(\mathbb{Q}_p)}}(\theta)$ equals $\theta|_{\mathbb{Q}_p} \cdot \epsilon_E$ and its conductor equals

$$\mathrm{cond}(\mathrm{Ind}_{\mathrm{W(E)}}^{\mathrm{W(\mathbb{Q}_p)}}(\theta)) = \begin{cases} 2\,\mathrm{cond}(\theta) & \text{if } E/\mathbb{Q}_p \text{ is unramified,} \\ \mathrm{cond}(\theta) + \mathrm{cond}(\epsilon_E) & \text{otherwise.} \end{cases}$$

If $p = 2$, besides the cases described above, the projective image of the Weil representation can be one of the sporadic groups A_4 or S_4 corresponding to the *sporadic supercuspidal representations* (as studied by Weil in [Wei74]), see 1.2.1 for more details.

Remark 5. The image of the inertia subgroup of a Weil representation lies in a finite extension of $\mathbb{Q}$, hence it makes sense to look at its Galois conjugates.

Definition 6. Given $\pi_1, \pi_2 \in \mathcal{A}_p$ they have *Galois conjugate local inertial type* if there exists $\sigma \in \mathrm{Aut}_{\mathbb{Q}}(\mathbb{C})$ such that the local inertial type of $\tau(\pi_1)$ and $\sigma(\tau(\pi_2))$ agree. By a *local type Galois orbit* we mean an equivalence class of Galois conjugate local inertial types.

Remark 7. Elements in the same local type Galois orbit need not have the same central character.

1.1. **Counting local type Galois orbits.** Let p be a prime number, and denote by $\mathbf{LT}(p^n)$ the number of local type Galois orbits of conductor n with trivial Nebentypus. For a a positive integer, let $\sigma_0(a)$ denote the number of positive divisors of a.

Theorem 8. *Let $p \neq 2$ be a prime number. Then the values of $\mathbf{LT}(p^n)$ are given in table 1.*

n	*P.S.*	St	*S.C.U.*	*S.C.R*
1	—	1	—	—
2	$\sigma_0(p-1)-1$	1	$\sigma_0(p+1)-2$	—
$n \overset{p\neq 3}{\geq} 3$ *odd*	—	—	—	2
$n \overset{p=3}{\geq} 3$ *odd*	—	—	—	4
$n \geq 3$*even*	$\sigma_0(p-1)$	—	$\sigma_0(p+1)$	—

TABLE 1. Values for $\mathbf{LT}(p^n)$ for $p \neq 2$.

Remark 9. There exists a ramified supercuspidal representation of conductor 2 for $p \equiv 3 \pmod 4$, but its local type matches that of an unramified supercuspidal representation (see for example [Gér75, Theorem 2.7]), which is why we do not count it in the previous table.

By Theorem 4, to compute $\mathbf{LT}(p^n)$ it is enough to count the number of Galois orbits for the Principal Series, the Steinberg and the Supercuspidal types. The Steinberg type is the easy one (they are all twists of St), while the Principal Series count comes from the well known group structure of $(\mathbb{Z}_p/p^n)^\times$.

Supercuspidal representations are induced from a character θ of a quadratic extension E of $\mathbb{Q}_p$. By Theorem 4 such induction has trivial Nebentypus precisely when the restriction of θ to $\mathbb{Q}_p^\times$ is fixed (and matches that of ϵ_E). Clearly two induced representations have Galois conjugate inertial types precisely when the quadratic field E is the same for both of them, and the two characters are Galois conjugate. This occurs precisely when one is a power (prime to the order) of the other.

Let $E = \mathbb{Q}_p(\sqrt{d})/\mathbb{Q}_p$ be a quadratic extension, let e denote the ramification degree of $E/\mathbb{Q}_p$, let $\mathscr{O}_E$ denote the ring of integers of E and $\mathfrak{p}$ its maximal ideal. Let k denote the residual field $\mathscr{O}_E/\mathfrak{p}$, and $q = p^r = \#k$. For n a positive integer let ξ_n denote a primitive n-th root of unity.

Theorem 10. *Let n be a positive integer and let $d \in \{\pm 1, \pm 3\}$. Then the group structure of $(\mathscr{O}/\mathfrak{p}^n)^\times$ is given in Table 2 where: – means no condition, and the pair (a, b) satisfies the following two conditions (which determines them uniquely):*

- $a + b = n - 1$,
- $a = b$ *if n is odd,*
- $a = b + 1$ *if n is even.*

E	e	p	n	*Structure*	*Generators*
—	1	$\neq 2$	—	$\mathbb{F}_q^\times \times \mathbb{Z}/p^{n-1} \times \mathbb{Z}/p^{n-1}$	$\{\xi_{p^2-1}, 1+p, 1+p\sqrt{d}\}$
—	1	2	≥ 2	$\mathbb{F}_4^\times \times \mathbb{Z}/2 \times \mathbb{Z}/2^{n-2} \times \mathbb{Z}/2^{n-1}$	$\{\xi_3, -1, 5+4\sqrt{5}, \sqrt{5}\}$
$\neq \mathbb{Q}_3(\sqrt{-3})$	2	$\neq 2$	—	$\mathbb{F}_p^\times \times \mathbb{Z}/p^a \times \mathbb{Z}/p^b$	$\{\xi_{p-1}, 1+p, 1+p\sqrt{d}\}$
$\mathbb{Q}_3(\sqrt{-3})$	2	3	≥ 2	$\mathbb{F}_3^\times \times \mathbb{Z}/3 \times \mathbb{Z}/3^{a-1} \times \mathbb{Z}/3^b$	$\{-1, \xi_3, 4, 1+3\sqrt{-3}\}$
$\mathbb{Q}_2(\sqrt{-1})$	2	2	≥ 3	$\mathbb{Z}/4 \times \mathbb{Z}/2^{b-1} \times \mathbb{Z}/2^{a-1}$	$\{\sqrt{-1}, 5, 1+2\sqrt{-1}\}$
$\mathbb{Q}_2(\sqrt{3})$	2	2	≥ 5	$\mathbb{Z}/2 \times \mathbb{Z}/2^{a-1} \times \mathbb{Z}/2^b$	$\{-1, \sqrt{3}, 1+2\sqrt{3}\}$
$\mathbb{Q}_2(\sqrt{2d})$	2	2	≥ 5	$\mathbb{Z}/2 \times \mathbb{Z}/2^{b-1} \times \mathbb{Z}/2^a$	$\{-1, 5, 1+\sqrt{2d}\}$

TABLE 2. Group structure of $(\mathscr{O}/\mathfrak{p}^n)^\times$.

Proof. See for example [Ran10] or [Neu99, Chapter II]. □

Remark 11. For completeness, the missing small values are: $(\mathscr{O}/2)^\times \simeq \mathbb{Z}/2$ if $E/\mathbb{Q}_2$ is ramified; if $E = \mathbb{Q}_2(\sqrt{3})$ or $\mathbb{Q}_2(\sqrt{2d})$, $(\mathscr{O}/\mathfrak{p}_2^3)^\times \simeq \mathbb{Z}/4$ and $(\mathscr{O}/\mathfrak{p}_2^4)^\times \simeq \mathbb{Z}/4 \times \mathbb{Z}/2$.

Lemma 12. *Let $E/\mathbb{Q}_p$ be a quadratic extension and let δ denote the valuation of the discriminant of E. The number of inertial type Galois orbits of primitive characters $\theta : E^\times \to \mathbb{C}^\times$ of conductor n whose restriction to $\mathbb{Q}_p^\times$ matches the character of the extension $E/\mathbb{Q}_p$ is given in Table 3.*

Proof. Given the group structure and generators of Table 2, it is enough to define a character in each of them.

Suppose that $E/\mathbb{Q}_p$ is unramified and $p \neq 3$. The condition $\theta|_{(\mathbb{Z}_p)^\times} = 1$ implies that θ is trivial in the second generator. The primitive condition implies that its value at the third generator must be a primitive p^{n-1} root of unity, and its value on ξ_{p^2-1}, is an element of order dividing $p+1$. Up to conjugation, the last value is the only free one, hence the total

E	e	p	n	# Prim. Char.
—	1	$\neq 2$	—	$\sigma_0(p+1)$
—	1	2	≥ 3	4
—	2	$\neq 2$	1	1
$\neq \mathbb{Q}_3(\sqrt{-3})$	2	$\neq 2$	$\overset{n\geq 2}{odd \mid even}$	$0 \mid 1$
$\delta = 3$	2	2	$\overset{n\geq 6}{odd \mid even}$	$0 \mid 1$
$\delta = 3$	2	2	$n = 5$	3
$\mathbb{Q}_3(\sqrt{-3})$	2	3	2	1
$\mathbb{Q}_3(\sqrt{-3})$	2	3	$\overset{n\geq 3}{odd \mid even}$	$0 \mid 3$
$\delta = 2$	2	2	$3, 4$	1
$\delta = 2$	2	2	$\overset{n\geq 6}{odd \mid even}$	$0 \mid 2$

TABLE 3. Number of primitive characters

number equals $\sigma_0(p+1)$. The case $p = 3$ works the same. For $p = 2$ there is 1 for $n = 1$, 2 for $n = 2$ and 4 for $n \geq 3$.

If $E/\mathbb{Q}_p$ is ramified, either $n = 1$ (hence $\mathscr{O}_E/\mathfrak{p} \simeq \mathbb{Z}/p$) in which case there is a unique character (namely that of ϵ_E) or primitive characters only appear for even exponents. The reason is that for odd conductor exponents $(1 + p)$ increases its order but θ is trivial on such element giving non-primitive characters. There are some exceptions, namely when $E/\mathbb{Q}_2$ is ramified. For example: if $\mathrm{Disc}(E/\mathbb{Q}_2) = 2^2$, the condition $\theta|_{\mathbb{Q}_2^\times} = \epsilon_E$ implies that $n \geq 3$ hence characters of conductor 3 are primitive. If $E = \mathbb{Q}_2(\sqrt{2d})$ then $\epsilon_E(5) = -1$ so $n \geq 5$ and characters of conductor 5 are also primitive. The number of characters in each case follows easily from the generators and the group structure given in Table 2. □

Supercuspidal automorphic representations correspond via local Langlands to irreducible induced representations of a character θ from a quadratic extension E, and the irreducibility condition is equivalent to θ not factoring through the norm map.

Lemma 13. *Let $E/\mathbb{Q}_p$ be a quadratic extension. The inertia type Galois orbits of characters θ that factor through the norm map are:*

i) The trivial one (of conductor 0).

ii) A conductor 1 one if $E/\mathbb{Q}_p$ is unramified.

iii) A conductor 2 and two of conductor 3 if $E/\mathbb{Q}_2$ is unramified.

iv) A conductor 1 one if $E/\mathbb{Q}_p$ is ramified and $p \equiv 1 \pmod 4$.

v) Two quadratic of conductor 5 for $E = \mathbb{Q}_2(\sqrt{2})$ or $\mathbb{Q}_2(\sqrt{-6})$.

Proof. Clearly the trivial character factors through the norm map. Let ϵ_E be the quadratic character giving the extension $E/\mathbb{Q}_p$. Suppose that $\theta(\alpha) = \phi(\mathrm{Norm}(\alpha))$ for some character ϕ of $\mathbb{Q}_p^\times$. Since $\theta|_{\mathbb{Q}_p^\times} = \epsilon_E$, $\theta(a) = \phi(a^2) = \epsilon_E(a)$ for any $a \in \mathbb{Q}_p^\times$. In particular, ϵ_E is a square and if $p \neq 2$, $\mathrm{cond}(\phi) = \mathrm{cond}(\epsilon_E)$.

- If $E/\mathbb{Q}_p$ is unramified, the norm map is surjective, hence ϕ is uniquely determined by θ (and vice-versa). Since ϵ_E is trivial on $\mathbb{Z}_p^\times$, ϕ is trivial on $(\mathbb{Z}_p^\times)^2$. If $p \neq 2$, $\mathbb{Z}_p^\times/(\mathbb{Z}_p^\times)^2$ is of order two, which give two possible characters ϕ namely the trivial one (with conductor 0) and a ramified one of conductor 1.
- If $E/\mathbb{Q}_2$ is unramified, $\mathbb{Z}_2^\times/(\mathbb{Z}_2^\times)^2$ has index 4, we get one case of conductor 0 (the trivial one), one case of conductor 2 and two cases of conductor 3.
- If $E/\mathbb{Q}_p$ is ramified and $p \neq 2$, the norm map is not surjective, being the image of $\mathscr{O}_E^\times$ equal to $(\mathbb{Z}_p^\times)^2$. This determines ϕ uniquely, since if $\alpha \in \mathscr{O}_E^\times$, there exists $a \in \mathbb{Z}_p^\times$ such that $\mathrm{Norm}(\alpha) = a^2$ hence $\theta(\alpha) = \phi(a^2) = \epsilon_E(a)$. In particular, ϕ gives a square root of $\varepsilon_E|_{\mathbb{Z}_p^\times}$ so $p \equiv 1 \pmod 4$. Clearly there are two conjugate characters ϕ (of conductor p) whose square equals ϵ_E.
- If $E/\mathbb{Q}_2$ is ramified, the condition $\epsilon_E(-1) = 1$ implies that $\mathrm{cond}(\epsilon_E) = 3$ and $\epsilon_E(3) = \epsilon_E(5) = -1$ (so $E = \mathbb{Q}_2(\sqrt{2})$ or $\mathbb{Q}_2(\sqrt{-6})$). The image of the norm map contains the squares with index 2; since ϵ has order 2, ϕ has order at most 4, hence it factors through $(\mathbb{Z}_2/16)^\times$. Each field gives two possible quadratic characters ϕ as stated.

□

Proof of Theorem 8. The number of Galois conjugate local types of level p^n is the following:

• **Principal Series:** the local representation is of the form $\pi(\chi_1, \chi_2)$. The Nebentypus being trivial implies that $\chi_2|_{\mathbb{Z}_p^\times} = \chi_1^{-1}|_{\mathbb{Z}_p}^\times$ hence $n = 2\,\mathrm{cond}(\chi_1)$, i.e. they only contribute at even exponents. Let $d = n/2$. The restriction to inertia of χ_1 is a primitive character of $(\mathbb{Z}/p^d\mathbb{Z})^\times$, a cyclic group of order $(p-1)p^{d-1}$. The number of such characters (up to conjugation) is precisely $\sigma_0(p-1)$ for $d > 1$ and $\sigma_0(p-1)-1$ for $d = 1$ (to avoid the trivial character).

• **Special representations or Steinberg:** since the Nebentypus is trivial, there are exactly two different types, of level p and p^2 respectively, with one type being the twist of the other by the quadratic character ramified at p.

• **Supercuspidal Representations:** By Theorem 4 they are obtained by inducing a character θ, that does not factor through the norm map, from a quadratic extension E of $\mathbb{Q}_p$. As before, let ϵ_E denote the character corresponding to the quadratic extension $E/\mathbb{Q}_p$.

∗ If $E/\mathbb{Q}_p$ is unramified (denoted S.C.U. in Table 1), $n = 2\,\mathrm{cond}(\theta)$ and $\theta|_{(\mathbb{Z}_p)^\times} = 1$. By Lemma 12 the total number of such characters equals $\sigma_0(p+1)$, and by Lemma 13 only two of them factor through the norm map (the trivial one and a conductor p one) for $n = 2$.

∗ If $E/\mathbb{Q}_p$ is ramified (denoted S.C.R. in Table 1), $n = \mathrm{cond}(\theta) + \mathrm{cond}(\epsilon_E)$ and $\theta|_{(\mathbb{Z}_p)^\times} = \epsilon_E|_{(\mathbb{Z}_p)^\times}$. If $\mathrm{cond}(\theta) = 1$, there is a unique type by Lemma 12 and by Lemma 13 the one for $p \equiv 1 \pmod 4$ factors though the norm map. If $p \equiv 3 \pmod 4$, the local type matches that of an unramified supercuspidal representation (see for example [Gér75, Theorem 2.7]). Otherwise, Lemma 12 implies that primitive characters have even conductor (hence n is odd) and there is a unique Galois inertial type orbit for each conductor except when $p=3$ and $E=\mathbb{Q}_3(\sqrt{-3})$, in which case there are three. □

1.2. **The case $p=2$.** This case is more delicate, and includes the types corresponding to the *sporadic supercuspidal series.*

1.2.1. *Sporadic supercuspidal representations:* The projective image of the Weil group of $\mathbb{Q}_2$ might be one of the sporadic cases A_4 or S_4. This phenomenon was studied by Weil in [Wei74], where he proved that the case A_4 does not occur over $\mathbb{Q}$, while the case S_4 does. He also proved that there are precisely 3 extensions of $\mathbb{Q}_2$ with Galois group isomorphic to S_4 and that there are precisely eight different cases with projective image S_4 that correspond to the field extension of $\mathbb{Q}_2$ obtained by adding the 3-torsion points of the elliptic curves

$$E_1^{(r)}: ry^2 = x^3+3x+2, \qquad r \in \{\pm 1, \pm 2\}, \tag{2}$$

$$E_2^{(r)}: ry^2 = x^3-3x+1, \qquad r \in \{\pm 1, \pm 2\}. \tag{3}$$

A way to understand the problem is as follows: given an S_4 extension (equivalently, a representation $\rho: G \to S_4$, where $G = \mathrm{Gal}(\overline{K}/K)$, for $K=\mathbb{Q}$ or $\mathbb{Q}_2$), compute all (if any) representations $\tilde{\rho}$ of K into $\mathrm{GL}_2(\mathbb{C})$ whose projectivization is isomorphic to ρ. Such general problem was studied by Serre in [Ser84]. Let $\tilde{S_4} \simeq \mathrm{GL}_2(\mathbb{F}_3)$ denote the quadratic extension of S_4, where transpositions lift to involutions (see [Ser84] page 654). Then two of the S_4 extensions lift to a representation of $\tilde{S_4}$ while the other one does not (see Section 8 of [BR99]). The 2-dimensional representations come from (composing with) the faithful 2-dimensional representation of $\tilde{S_4}$. Note that the representations obtained from (2) (respectively from (3)) are quadratic twists of each other, hence have the same projective image (and correspond to the two extensions mentioned before).

Recall that two representations $\rho_i: G \to \mathrm{GL}_n(K)$, $i=1,2$ whose projectivizations $\tilde{\rho_i}: G \to \mathrm{PGL}_n(K)$ are isomorphic are twist of each other, i.e. there exists a character $\chi: G \to K^\times$ such that $\rho_1 \simeq \rho_2 \otimes \chi$. Since we only consider forms with trivial Nebentypus, all sporadic supercuspidal representations are unramified twists of (2) and (3) so they cover all local types. The level of such types is computed in [Rio06] (section 6). It equals 2^7 for the curves $E_1^{(r)}$, 2^4 for $E_2^{(1)}$, 2^3 for $E_2^{(-1)}$ and 2^6 for $E_2^{(\pm 1)}$.

Theorem 14. *The values of* $\mathbf{LT}(2^n)$ *are given in table 4.*

Proof. The strategy is the same as before, but more delicate.

• **Principal Series:** this case mimics the odd prime one with the difference that $(\mathbb{Z}/2^n)^\times$ is cyclic for $n=2$ but isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2^{n-2}$ if $n \geq 3$. Hence there is a unique local type of conductor 4, and two types for all other even exponents.

• **Special representations or Steinberg:** there is a unique automorphic form St of conductor 2. There is one quadratic character of conductor 2 and two of conductor 3 whose twists give types of conductor 4 and 6 respectively.

• **Supercuspidal Representations:** As in the odd case, we distinguish each possible extension $E/\mathbb{Q}_2$.

d	$P.S.$	Stb	$S.C.U.$	$S.C.R(2)$	$S.C.R(3)$	$Sporadic$
1	—	1	—	—	—	—
2	—	—	1	—	—	—
3	—	—	—	—	—	1
4	1	1	1	—	—	1
5	—	—	—	2	—	—
6	2	2	2	2	—	2
7	—	—	—	—	—	4
8	2	—	4	4	—	—
≥ 9 *odd*	—	—	—	—	4	—
≥ 10 *even*	2	—	4	4	—	—

TABLE 4. Types for $p = 2$.

$*$ If $E/\mathbb{Q}_2$ is unramified (denoted S.C.U. in Table 4), the level of the form equals $2\,\mathrm{cond}(\theta)$. There is one local type Galois orbit for $\mathrm{cond}(\theta) = 1$ (θ being a cubic character), one for $\mathrm{cond}(\theta) = 2$ (as the other one factors through the norm map), two for $\mathrm{cond}(\theta) = 3$ (two factor through the norm map) and four for $\mathrm{cond}(\theta) > 3$ (see Lemmas 12 and 13).

$*$ If $E/\mathbb{Q}_2$ is ramified with conductor 2 (denoted S.C.R.(2) in Table 4), the level of the form equals $2 + \mathrm{cond}(\theta)$. There are two such fields E, namely $\mathbb{Q}_2(\sqrt{-1})$ and $\mathbb{Q}_2(\sqrt{3})$. By Lemmas 12 and 13, the number of such types equals:

$$\begin{cases} 0 & \text{if } \mathrm{cond}(\theta) = 1, 2 \text{ or } \mathrm{cond}(\theta) \geq 4 \text{ and odd,} \\ 1 & \text{if } \mathrm{cond}(\theta) = 3, \\ 1 & \text{if } \mathrm{cond}(\theta) = 4, \\ 2 & \text{if } \mathrm{cond}(\theta) \geq 5 \text{ and even.} \end{cases}$$

$*$ If $E/\mathbb{Q}_2$ is ramified with conductor 3 (denoted S.C.R.(3) in Table 4), the level of the form equals $3+\mathrm{cond}(\theta)$. There are four such fields, namely $\mathbb{Q}_2(\sqrt{2}), \mathbb{Q}_2(\sqrt{-2}), \mathbb{Q}_2(\sqrt{6})$ and $\mathbb{Q}_2(\sqrt{-6})$. By Lemma 12 the number of Galois orbits equals:

$$\begin{cases} 0 & \text{if } \mathrm{cond}(\theta) = 2 \text{ or } \mathrm{cond}(\theta) \text{ is odd,} \\ 3 & \text{if } \mathrm{cond}(\theta) = 5, \\ 1 & \text{if } \mathrm{cond}(\theta) \geq 5 \text{ and is even.} \end{cases}$$

Recall that for odd primes $p \equiv 3 \pmod 4$, a ramified type matches that of an unramified one. When $p = 2$, the same phenomenon occurs in many cases. We refer to [BH06, Section 41.3] for a detailed description. Following their notation, all supercuspidal representations are imprimitive (see Definition in page 255 and Lemma 41.3 of [BH06]) and the way to test whether a local type appears for different quadratic extensions is by computing the number of quadratic twists that give isomorphic representations (denoted by $I(\rho)$). In particular, if the form is triply imprimitive (i.e. it comes from more than one quadratic extension), it must be the case that $\frac{\theta}{\theta^\sigma}$ is a quadratic character, where σ generates $\mathrm{Gal}(E/\mathbb{Q}_2)$. With this criterion, the following types are simply imprimitive:

- representations induced from $E/\mathbb{Q}_2$ ramified with discriminant valuation 2 and $\mathrm{cond}(\theta) \geq 7$.
- representations induced from $E/\mathbb{Q}_2$ ramified with discriminant valuation 3 and $\mathrm{cond}(\theta) \geq 6$.

The case $E/\mathbb{Q}_2$ with discriminant 3 and $\mathrm{cond}(\theta) = 5$ is of particular interest. For any E, $\epsilon_E(5) = -1$. Each field has 3 different local Galois orbits (by Lemma 12) two of order 2 and one of order 4; if $E = \mathbb{Q}_2(\sqrt{2})$ or $\mathbb{Q}_2(\sqrt{-6})$, then by Lemma 13 two characters factor through the norm map for each of them (when θ has order 2) which we discard.

Let θ be quadratic and let ϕ be any order 4 character of $(\mathbb{Z}/16)^\times$. In particular, $\phi(9) = -1$, so $\theta \cdot (\phi \circ \mathrm{Norm})$ is trivial at 5 hence gives a character of conductor 3 of E (or the trivial character in the discarded cases). In particular, the twist $\mathrm{Ind}_{\mathrm{W(E)}}^{\mathrm{W(\mathbb{Q}_p)}}(\theta) \otimes \phi = \mathrm{Ind}_{W(E)}^{W(\mathbb{Q}_2)}(\theta \cdot (\phi \circ \mathrm{Norm}))$ has conductor 3 (with non-trivial Nebentypus) so by [BH06, Proposition 41.4] it matches the supercuspidal unramified type, which was counted before.

If θ has order 4, the representation $\mathrm{Ind}_{W(E)}^{W(\mathbb{Q}_2)} \theta$ is triply imprimitive. An easy computation proves that the set $I(\mathrm{Ind}_{W(E)}^{W(\mathbb{Q}_2)} \theta)$ equals:

- $\{1, \chi_3, \chi_2, \chi_6\}$ if $E = \mathbb{Q}_2(\sqrt{2})$,
- $\{1, \chi_3, \chi_{-2}, \chi_{-6}\}$ if $E = \mathbb{Q}_2(\sqrt{-6})$,
- $\{1, \chi_3, \chi_{-2}, \chi_{-6}\}$ if $E = \mathbb{Q}_2(\sqrt{-2})$,
- $\{1, \chi_3, \chi_2, \chi_6\}$ if $E = \mathbb{Q}_2(\sqrt{6})$,

where χ_i denotes the quadratic character of the extension $\mathbb{Q}_2(\sqrt{i})$. In particular, all such local types match those from $\mathbb{Q}_2(\sqrt{3})$, hence we do not need to count them again.

• **Sporadic supercuspidal representations:** the assertion follows from Weil's results stated before. □

2. Types from modular forms

Let $f = \sum_{n \geq 1} a_n q^n \in S_k(\Gamma_0(N))$ be a newform, and let π_f be the automorphic representation of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ attached to it. It is well known that π_f is a restricted tensor product ${\bigotimes_p}' \pi_{f,p} \otimes \pi_{f,\infty}$, where $\pi_{f,p} \in \mathcal{A}_p$ is a representation of $\mathrm{GL}_2(\mathbb{Q}_p)$. Then for each prime p, the form f has attached a local type (that of $\pi_{f,p}$). Let $K_f = \mathbb{Q}(a_n)$ denote the coefficient field of f. If N is a positive integer, let ξ_N denote an N-th primitive root of unity and $\mathbb{Q}(\xi_N)^+$ the maximal totally real subextension of $\mathbb{Q}(\xi_N)$.

Lemma 15. *Let $f \in S_k(\Gamma_0(N))$ and let p be a prime number. If $\pi_{f,p}$ is isomorphic to a Principal Series $\pi(\chi_1, \chi_2)$, where $\chi_1|_{\mathbb{Z}_p^\times}$ has order d, then $\mathbb{Q}(\xi_d)^+ \subset K_f$.*

Proof. Let $L = K_f \cap \mathbb{Q}(\xi_d)$. Suppose that $L \subsetneq \mathbb{Q}(\xi_d)^+$ and let $\ell \neq p$ be a prime such that there exists a prime ideal λ of $\mathscr{O}_L$ (the ring of integers of L) whose inertial degree in $\mathbb{Q}(\xi_d)^+$ is not 1. Let $\rho_{f,\lambda} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(K_{f,\lambda})$ be the Galois representation attached to f (by [Del71]). The restriction to the decomposition group at p matches (up to isomorphism) the representation $\chi_1 \oplus \chi_1^{-1}\chi_\ell^{k-1}$ (where χ_ℓ denotes the ℓ-th cyclotomic character). Evaluating at elements of $\mathbb{Z}_p^\times$ (corresponding via Local-Langlands to elements in the inertia group) we see that $K_{f,\lambda}$ contains $\xi_d + \xi_d^{-1}$, which generates $\mathbb{Q}(\xi_d)^+$. But our assumption on λ implies that the completion of $\mathbb{Q}(\xi_d)^+$ (at a prime dividing λ) and $K_{f,\lambda}$ (at λ) are different, giving a contradiction. □

Lemma 16. *Let $f \in S_k(\Gamma_0(N))$ and let p be a prime number. If $\pi_{f,p}$ is isomorphic to a Supercuspidal Representations, say $\pi_{f,p} = \mathrm{Ind}_{W(\mathbb{Q}_p)}^{W(E)} \theta$, where θ has order d, then $\mathbb{Q}(\xi_d)^+ \subset K_f$.*

Proof. The restriction of $\rho_{f,\lambda}$ to $W(E)$ equals $\theta \oplus \theta'$, where if $\sigma \in W(\mathbb{Q}_p) \setminus W(E)$, then $\theta'(\mu) = \theta(\sigma\mu\sigma^{-1})$. The result follows from the same argument as the principal series case, via evaluating at elements of $\mathbb{Z}_p^\times$; note that the trivial Nebentypus condition implies that on such elements $\theta' = \theta^{-1}$. □

Lemmas 15 and 16 imply that the coefficient field contains many automorphisms to conjugate the form f. If we fix a prime p dividing the level, the global Galois orbit of the modular form f contains representatives for all elements of the same local type Galois orbit of $\pi_{f,p}$.

Theorem 17. *Let $f \in S_k(\Gamma_0(N))$ be a newform and $p \mid N$ a prime number. Then the set $\{\pi_{\sigma(f),p} : \sigma \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q})\}$ of local types at p of the Galois conjugates of f equals the local type Galois orbit of $\pi_{f,p}$.*

Proof. Clearly $\{\pi_{\sigma(f),p} : \sigma \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q})\}$ is contained in the local type Galois orbit of $\pi_{f,p}$, hence we need to prove the other containment.

The result is clear when the local type of $\pi_{f,p}$ is Steinberg, as there is a unique element in the class. In the Principal Series case, note that $\pi(\chi_1, \chi_2)$ and $\pi(\chi_2, \chi_1)$ are isomorphic. Furthermore, the trivial Nebentypus hypothesis implies that $\chi_2|_{\mathbb{Z}_p^\times} = \chi_1^{-1}|_{\mathbb{Z}_p^\times}$. Suppose $\pi_{f,p} = \pi(\chi_1, \chi_2)$, where χ_1 is a primitive character of order d and conductor $n/2$. The Galois orbit of $\pi(\chi_1, \chi_2)$ has $\varphi(d)$ elements. Among such conjugates, $\varphi(d)/2$ are non-isomorphic when $d \neq 2$ and contains a unique element when $d = 2$. Lemma 15 implies that K_f contains $\mathbb{Q}(\xi_d)^+$.

Claim: let $\sigma \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q})$. Then the local type of $\pi_{\sigma(f),p}$ is isomorphic to that of $\pi(\sigma(\chi_1), \sigma(\chi_2))$.

Let μ be the restriction of σ to $\mathbb{Q}(\xi_d)^+$ and let $\pi_{\mu(f),p} = \pi(\psi_1, \psi_2)$. The characters (ψ_1, ψ_2) are determined by $\mu(f)$: the values $\{\psi_1(x), \psi_2(x)\}$ are roots of the polynomial $x^2 - \mu(\chi_1 + \chi_2)x + \chi_1\chi_2 \in \mathbb{Q}(\xi_d)^+$ (which is the characteristic polynomial of the image of x under the Galois representations attached to $\mu(f)$). In particular they match the values $\{\sigma(\chi_1)(x), \sigma(\chi_2)(x)\}$. If $p \neq 2$, $(\mathbb{Z}/p^k)^\times$ is cyclic, so taking x to be a generator, the characters are uniquely determined by their values on x. In particular, $\psi_1 = \sigma(\chi_1)$ or $\psi_1 = \sigma(\chi_2)$. For $p = 2$, $(\mathbb{Z}/2^k)^\times = \mathbb{Z}/2 \times \mathbb{Z}/2^{k-2}$, and the trivial Nebentypus hypothesis imply that both characters ψ_1 and ψ_2 take the same value at the generator of the $\mathbb{Z}/2$-part. Then again, $\psi_1 = \sigma(\chi_1)$ or $\psi_1 = \sigma(\chi_2)$. Note that the two choices of (ψ_1, ψ_2) are conjugate to each other, and give isomorphic local types (which explains the discrepancy between the action on characters of the group $\mathrm{Gal}(\mathbb{Q}(\xi_d)/\mathbb{Q})$ and $\mathrm{Gal}(\mathbb{Q}(\xi_d)^+/\mathbb{Q})$).

The supercuspidal case follows from a similar computation using Lemma 16 to get the different conjugates of the character θ. □

While working with Galois orbits of modular forms, there is another natural invariant to consider, namely the Atkin-Lehner eigenvalue at each prime $p \mid N$. By the theory of Atkin and Lehner (see[AL70]), if $f \in S_k(\Gamma_0(N))$ is a newform and $p \mid N$, then f is an eigenform for the A-L involution W_p, i.e. $W_p(f) = \lambda_p f$, with $\lambda_p = \pm 1$.

Lemma 18. *Let $f \in S_k(\Gamma_0(N))$ and let $p \mid N$ a prime number such that $W_p(f) = \lambda_p f$. If $\sigma \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ then $W_p(\sigma(f)) = \lambda_p \sigma(f)$.*

Proof. It is immediate from the fact that W_p is an involution and commutes with the Hecke operators. In particular, the space $S_k(\Gamma_0(N), \mathbb{Q}) = S_k(\Gamma_0(N), \mathbb{Q})^+ \oplus S_k(\Gamma_0(N), \mathbb{Q})^-$, and the Hecke operators preserve both spaces. □

There is a delicate situation while computing A-L operators. If $f \in S_k(\Gamma_0(N))$, it needs not be minimal among twists with trivial Nebentypus. For example, if $f \in S_k(\Gamma_0(p))$, and we look at forms in its Galois orbits $\{\sigma(f)\}$, we can twist them by χ_p (the quadratic character unramified outside p) and get a Galois orbit of new forms $\{\sigma(f) \otimes \chi_p\}$ in $S_k(\Gamma_0(p^2))$. All such forms will have a predetermined A-L sign, namely $\chi_p(-1)$ (see [AL70, Theorem 6]), while the A-L eigenvalue of f at p might take any value ± 1, so we "lost" the invariant. Our final goal is to determine invariants of Galois orbits of eigenforms, so we can either look at forms which have minimal level up to (quadratic) twists, or to add the A-L sign of a minimal twist.

Definition 19. Let $f \in S_k(\Gamma_0(N))$ be a newform, and let $p \mid N$. Consider the set of quadratic twists $T_f = \{f \otimes \psi\}$ where ψ ranges over all quadratic characters unramified outside p. Then either:

(1) all elements in T_f have level greater or equal than that of f or
(2) there exists a unique form $g \in T_f$ of minimal level smaller than N.

We define the *minimal Atkin-Lehner sign* of f at p to be that of f in the first case, and that of g in the second one.

The minimal A-L sign at p of a newform f is sometimes determined by the local type $\tilde{\pi}_{f,p}$ of f at p.

Theorem 20. *Let p be a prime number and $\tau \in \mathcal{A}_p$ be such that $\tilde{\tau} = \pi_{\tilde{f},p}$ for $f \in S_k(\Gamma_0(N))$ a newform. Then:*

(1) If $\tilde{\tau}$ is principal series, a ramified twist of Steinberg or a supercuspidal unramified representation (i.e. induced from an unramified quadratic extension of $\mathbb{Q}_p$), then the eigenvalue of the Atkin-Lehner involution W_p is the same for all modular form f with local type at p $\tilde{\tau}$.

(2) If $\tilde{\tau}$ is Steinberg, or if $p \neq 2$ and $\tilde{\tau}$ is a ramified Supercuspidal representations induced from a character with even conductor, then there are two possible signs for the Atkin-Lehner involution for modular forms with local type at p $\tilde{\tau}$. Furthermore, the two values are interchanged while twisting by the quadratic unramified character (which clearly preserves types).

(3) If $p = 2$ and $\tilde{\tau}$ is a ramified supercuspidal representations induced from a character θ of a quadratic extension $E/\mathbb{Q}_2$ with discriminant valuation 2, then:
   - *there are two possible signs (interchanged by the unramified quadratic twist) when $\operatorname{cond}(\theta) = 3$,*
   - *there is a unique possible sign when $\operatorname{cond}(\theta)$ is even.*

   If $E/\mathbb{Q}_2$ has discriminant valuation 3, then
   - *there is a unique sign for $\operatorname{cond}(\theta) = 5$,*
   - *there are two possible signs for even conductors.*

(4) If $p = 2$ and $\tilde{\tau}$ is a sporadic supercuspidal representations, then the situation is as follows:
   - *If $\tilde{\tau}$ has level 2^7 or 2^3, then both Atkin-Lehner signs appear, and they are exchanged by the quadratic unramified twist.*
   - *If $\tilde{\tau}$ has level 2^4 or 2^6, then the quadratic unramified twist preserves the local sign, but these types are not-minimal, they are twists of the level 2^3 one.*

Proof. The result is well known to experts, and follows from the characterization of the local sign of automorphic forms given by Deligne (see [Del73] and also [**?**]). In the 1-dimensional case it is clear that the local root number is determined by the restriction to inertia of the character as well as its value at a local uniformizer (see for example (3.4.3.2) of [Del73]).

- Suppose that $\tilde{\tau}$ is principal series, and $\tilde{\pi}(\chi_1, \chi_2) \in \tilde{\tau}$. Then the local sign of $\pi(\chi_1, \chi_2)$ equals the product of the two local signs. But the trivial Nebentypus hypothesis implies that the product of the two characters evaluated at p is uniquely determined, hence their restriction to inertia determines uniquely the sign of $\pi(\chi_1, \chi_2)$.
- The Steinberg case is well understood. In this case the Atkin-Lehner involution at p is related to the p-th Fourier coefficient $\lambda_p(f)p^{\frac{k-2}{2}} = -a_p(f)$. Note that the Weil-representation equals $\omega^{\frac{k}{2}-1}(\psi \oplus \psi\omega)$, where ω is the unramified quasi-character giving the action of $W(\mathbb{Q}_p)$ on the roots of unity and ψ is a quadratic unramified

character. Then $\lambda_p(f) = -\psi(p)$. Clearly twisting by the character corresponding to the unramified quadratic extension of $\mathbb{Q}_p$ changes the A-L eigenvalue.

The ramified twist of Steinberg case is well known (see for example [AL70, Theorem 6]). It can also be recovered by studying the local sign variation under twisting (see (3.4.3.5) and Theorem 4.1 (1) of [Del73])

- If $\tilde{\tau}$ is supercuspidal representations, the local factor can be explicitly computed (following [Del73]). Recall that one of the local sign properties (see [Del73, 3.12 (C)]) is

$$\varepsilon\left(\mathrm{Ind}_{\mathrm{W(E)}}^{\mathrm{W}(\mathbb{Q}_\mathrm{p})}(\theta), \psi, dx\right) = \varepsilon\left(\theta, \psi\circ \mathrm{Tr}, dx\right).$$

The Swan conductor of θ, denoted $sw(\theta)$, equals 0 if θ is unramified and $\mathrm{cond}(\theta)-1$ otherwise. Let $s = \mathrm{cond}(\psi\circ\mathrm{Tr}) + sw(\theta) + 1$ and let π be a local uniformizer. By [Del73, page 528],

$$\varepsilon\left(\mathrm{Ind}_{\mathrm{W(E)}}^{\mathrm{W}(\mathbb{Q}_\mathrm{p})}(\theta), \psi, dx\right) = \theta(\pi)^s \int_{\mathscr{O}^\times} \theta^{-1}(x)\psi\circ\mathrm{Tr}\left(\frac{x}{\pi^s}\right) d\frac{x}{\pi^s}. \tag{4}$$

In particular, the local sign depends on the restriction of θ to $\mathscr{O}^\times$ and its value in a local uniformizer. Recall that the determinant of the representation equals $\epsilon_E\theta|_{\mathbb{Q}_p^\times}$, hence the value of $\theta(p)$ is uniquely determined. If $E/\mathbb{Q}_p$ is unramified, p is a local uniformizer, hence the local sign only depends on the Weil-Deligne type.

If $E/\mathbb{Q}_p$ is ramified and π is a local uniformizer, the trivial Nebentypus condition determines the value of $\theta(p) = \theta(\pi^2)$, but not that of $\theta(\pi)$. Chose ψ to be an additive character with conductor 0 (i.e. it is trivial on $\mathbb{Z}_p$ but non-trivial on $\frac{1}{p}\mathbb{Z}_p$). Then clearly $\mathrm{cond}(\psi\circ\mathrm{Tr}) \equiv v_p(\mathrm{Disc}(E)) \pmod 2$.

  - If $p \neq 2$, $v_p(\mathrm{Disc}(E)) \equiv 1 \pmod 2$ hence if $\mathrm{cond}(\theta) = 1$, s is even and the local sign is uniquely determined (recall that such type matches the unramified one!). If $\mathrm{cond}(\theta)$ is even, s is odd (by Lemma 12) hence there are two possible signs. Furthermore, we can move from one sign to the other twisting by the unramified quadratic character (which changes the sign of $\theta(\pi)$).
  - If $p = 2$ and $v_2(\mathrm{Disc}(E/\mathbb{Q}_2)) = 2$, $s \equiv \mathrm{cond}(\theta) \pmod 2$, hence the sign is uniquely determined for all θ of even conductor. If $\mathrm{cond}(\theta) = 3$, there are two possibilities (corresponding to modular forms of level 2^5). The forms of level 2^6 are quadratic twists of these ones, hence although the local sign is uniquely determined, they have two possible minimal Atkin-Lehner signs at 2.
    At last, if $v_2(\mathrm{Disc}(E/\mathbb{Q}_2) = 3$, $s \equiv \mathrm{cond}(\theta)+1 \pmod 2$, so the sign is uniquely determined for $\mathrm{cond}(\theta) = 5$ (recall that this case matches the unramified one) while there are two possible values for even conductors (and both signs change by a local twist).

- Suppose $p = 2$ and τ is sporadic supercuspidal representations, so the Weil representation ρ attached to it has image isomorphic to $\tilde{S}_4$, i.e. there exists $E/\mathbb{Q}_2$ with $\mathrm{Gal}(E/\mathbb{Q}_2) \simeq \tilde{S}_4$.

The character table of $\tilde{S_4} \simeq \mathrm{GL}_2(\mathbb{F}_3)$ is recalled in Table 5. The representations Sg, St_2 and St_3 are the representations obtained from quotients of $\mathrm{PGL}_2(\mathbb{F}_3) \simeq S_4$, and they are the sign representation, the 2-dimensional standard representation obtained from the isomorphism $S_4/\langle(12)(34),(13)(24)\rangle \simeq S_3$, and the 3-dimensional representation of S_4. The representation V is the alluded in Section 1.2.1.

Another description of such representations come from the group $\mathrm{GL}_2(\mathbb{F}_3)$: the two 1-dimensional ones are the ones factoring through the determinant. The last 3 representations come from "principal series": if χ is the non-trivial character of $\mathbb{F}_3^\times$, $\pi(\chi,1)$ gives the irreducible four dimensional representation; $\pi(1,1)$ and $\pi(\chi,\chi)$ have an irreducible quotient/subspace of dimension three (the "Steinberg" ones). Finally, the two dimensional ones, can be constructed as follows: identify $\mathbb{F}_9^\times$ with the non-split Cartan $\mathcal{C}_{ns} = \left\{\left(\begin{smallmatrix} a & -b \\ b & a\end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{F}_3)\right\}$; pick ψ a non-trivial additive character of $\mathbb{F}_3$ and let $\theta : \mathbb{F}_9^\times \to \mathbb{C}^\times$ be a character. Let θ_ψ be the character in $M = \left\{Z(\mathrm{GL}_2(\mathbb{F}_3)) \cdot \left(\begin{smallmatrix} 1 & \mathbb{F}_3 \\ 0 & 1\end{smallmatrix}\right)\right\}$ given by $\theta_\psi\left(\left(\begin{smallmatrix} a & 0 \\ 0 & a\end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & u \\ 0 & 1\end{smallmatrix}\right)\right) = \theta(a)\psi(u)$. Then if θ is not trivial nor quadratic, the virtual representation $\mathrm{Ind}_M^{\mathrm{GL}_2(\mathbb{F}_3)} \theta_\psi - \mathrm{Ind}_{\mathcal{C}_{ns}}^{\mathrm{GL}_2(\mathbb{F}_3)} \theta$ is an irreducible representation independent of ψ (see [BH06, Theorem 6.4]). If θ has order 8, we get the representation V and its twist, while θ of order 4 gives the representation St_2.

	$\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix} 2 & 0 \\ 0 & 2\end{smallmatrix}\right)$	$\left(\begin{smallmatrix} 0 & 1 \\ 2 & 0\end{smallmatrix}\right)$	$\left(\begin{smallmatrix} 0 & 1 \\ 1 & 2\end{smallmatrix}\right)$	$\left(\begin{smallmatrix} 0 & 1 \\ 1 & 1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix} 2 & 1 \\ 0 & 2\end{smallmatrix}\right)$	$\left(\begin{smallmatrix} 1 & 0 \\ 0 & 2\end{smallmatrix}\right)$
$\mathbb{1}$	1	1	1	1	1	1	1	1
Sg	1	1	1	-1	-1	1	1	-1
St_2	2	2	2	0	0	-1	-1	0
V	2	-2	0	$\sqrt{-2}$	$-\sqrt{-2}$	-1	1	0
$V \otimes \mathrm{Sg}$	2	-2	0	$-\sqrt{-2}$	$\sqrt{-2}$	-1	1	0
St_3	3	3	-1	-1	-1	0	0	1
$\mathrm{St}_3 \otimes \mathrm{Sg}$	3	3	-1	1	1	0	0	-1
W	4	-4	0	0	0	1	-1	0

TABLE 5. Character table for $\mathrm{GL}_2(\mathbb{F}_3)$.

Consider the following subgroups of $\mathrm{GL}_2(\mathbb{F}_3)$: $C_4 = \langle\left(\begin{smallmatrix} 0 & 1 \\ 2 & 0\end{smallmatrix}\right)\rangle$, $C_6 = \langle\left(\begin{smallmatrix} 2 & 1 \\ 0 & 2\end{smallmatrix}\right)\rangle$ and $C_8 = \langle\left(\begin{smallmatrix} 0 & 1 \\ 1 & 2\end{smallmatrix}\right)\rangle$. Using the character table and Frobenius reciprocity, it is easy to verify the following formulas

$$\mathrm{Ind}_{C_4}^{\mathrm{GL}_2(\mathbb{F}_3)} \chi_4 \simeq V \oplus (V \otimes \mathrm{Sg}) \oplus 2W \tag{5}$$

$$\mathrm{Ind}_{C_6}^{\mathrm{GL}_2(\mathbb{F}_3)} \chi_6 \simeq V \oplus (V \otimes \mathrm{Sg}) \oplus W \tag{6}$$

$$\mathrm{Ind}_{C_8}^{\mathrm{GL}_2(\mathbb{F}_3)} \chi_8 \simeq (V \otimes \mathrm{Sg}) \oplus W, \tag{7}$$

where χ_j is a character of order j in the corresponding group and we chose $\chi_8\left(\left(\begin{smallmatrix} 0 & 1 \\ 1 & 2 \end{smallmatrix}\right)\right) = \exp\left(-\frac{\pi i}{4}\right)$. Then

$$V \simeq \operatorname{Ind}_{C_8}^{\mathrm{GL}_2(\mathbb{F}_3)} \chi_8 - \operatorname{Ind}_{C_4}^{\mathrm{GL}_2(\mathbb{F}_3)} \chi_4 + \operatorname{Ind}_{C_6}^{\mathrm{GL}_2(\mathbb{F}_3)} \chi_6. \tag{8}$$

To compute the sign variation, we can consider the formal representation $\kappa V - V$, where κ is the quadratic unramified character of $\mathbb{Q}_2$. Using (8) and the local sign formalism ([Del73, Theorem 4.1]) we obtain

$$\varepsilon(\kappa V - V, \psi, dx) = \frac{\varepsilon(\kappa\chi_8, \psi \circ \operatorname{Tr}_{K_{C_8}}, dx)}{\varepsilon(\chi_8, \psi \circ \operatorname{Tr}_{K_{C_8}}, dx)} \frac{\varepsilon(\chi_4, \psi \circ \operatorname{Tr}_{K_{C_4}}, dx)}{\varepsilon(\kappa\chi_4, \psi \circ \operatorname{Tr}_{K_{C_4}}, dx)} \frac{\varepsilon(\kappa\chi_6, \psi \circ \operatorname{Tr}_{K_{C_6}}, dx)}{\varepsilon(\chi_6, \psi \circ \operatorname{Tr}_{K_{C_6}}, dx)} \tag{9}$$

The characters in (9) are understood as class field characters, giving the corresponding field extension. Recall that each sign variation depends only on the value $\kappa(\operatorname{Norm}(\pi_2))^s$, where π_2 is a local uniformizer and $s = \operatorname{val}_2(\operatorname{Disc}(K_i)) + sw(\chi_i) + 1$. In particular, we need to compute the inertial degree of K_i and s for each field. The extensions K_{C_4}, K_{C_6} and K_{C_8} are contained in the fixed field of $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, a Galois extension with Galois group isomorphic to S_4. The ramification indices are $f(K_{C_4}/\mathbb{Q}_2) = 2$, $f(K_{C_6}/\mathbb{Q}_2) = 2$ and $f(K_{C_8}/\mathbb{Q}_2) = 1$. Then the sign contribution is trivial for the first two ones (as $\operatorname{Norm}(\pi_2)$ is a square), and only depends on the first term of (9). Furthermore, $2 \mid \operatorname{val}_2(\operatorname{Disc}(K_{C_8}))$, hence $s \equiv sw(\chi_8) + 1 \pmod 2$.

To compute $sw(\chi_8)$, we consider the field extensions $\mathbb{Q}_2 \subset K_{C_8} \subset K_{C_4} \subset K_{-1} \subset E$. The group C_8 has characters of orders: 1, 2 (both unramified), 4 and 8. The conductor discriminant formula gives the equality

$$\operatorname{Disc}(E/K_{C_8}) = \prod_\theta \operatorname{cond}(\theta).$$

Let θ_i denote the corresponding character of order i (so $\theta_8 = \chi_8$). The relative discriminant formula provides the equations:

$$\operatorname{Disc}(E/K_{C_8}) = \operatorname{cond}(\theta_8)^2 \operatorname{cond}(\theta_4)^2. \tag{10}$$

$$\operatorname{Disc}(K_{-1}/K_{C_8}) = \operatorname{cond}(\theta_4)^2. \tag{11}$$

$$\operatorname{Disc}(K_{-1}/\mathbb{Q}_2) = \operatorname{Norm}(\operatorname{Disc}(K_{-1}/K_{C_8})) \operatorname{Disc}(K_{C_8}/\mathbb{Q}_2)^4. \tag{12}$$

$$\operatorname{Disc}(E/\mathbb{Q}_2) = \operatorname{Norm}(\operatorname{Disc}(E/K_{C_8})) \operatorname{Disc}(K_{C_8}/\mathbb{Q}_2)^8. \tag{13}$$

Then computing for each of the 8 fields the values $\operatorname{Disc}(E/\mathbb{Q}_2)$, $\operatorname{Disc}(K_{-1}/\mathbb{Q}_2)$ and $\operatorname{Disc}(K_{C_8}/\mathbb{Q}_2)$, a simple manipulation determines $sw(\chi_8)$.

Equations for the 8 extensions appear in the online tables of [JR06]. Note that in $\mathrm{GL}_2(\mathbb{F}_3)$ there are two non-conjugate subgroups of order 8, hence each extension can be obtained by two different degree 8 polynomials. The extensions are obtained as the Galois closure of the polynomials:

– $x^8 + 20x^2 + 20$, $x^8 + 28x^2 + 20$, $x^8 + 6x^6 + 20$ and $x^8 + 2x^6 + 20$.

– $x^8+4x^7+4x^2+14$, $x^8+4x^7+12x^2+2$, $x^8+4x^7+12x^2+14$ and $x^8+4x^7+12x^2+10$. The values of $\mathrm{Disc}(E/\mathbb{Q}_2)$ (for each extension) already appeared in [Rio06, Table 10]), and equal: 64, 76, 100 and 100 for the first four fields and 136 for all fields in the second list. The other discriminants as well as the value of $\mathrm{sw}(\chi_8)$ are given in Table 6, which proves the stated result.

Polynomial	$\mathrm{val}_2(\mathrm{Disc}(E/\mathbb{Q}_2))$	$\mathrm{val}_2(\mathrm{Disc}(K_{-1}/\mathbb{Q}_2))$	$\mathrm{val}_2(\mathrm{Disc}(K_{C_8}/\mathbb{Q}_2))$	$\mathrm{val}_2(\mathrm{cond}(\chi_8))$
x^8+20x^2+20	64	28	6	1
x^8+28x^2+20	76	28	6	18
x^8+6x^6+20	100	28	6	2
x^8+2x^6+20	100	28	6	2
$x^8+4x^7+4x^2+14$	136	52	10	11
$x^8+4x^7+12x^2+2$	136	52	10	11
$x^8+4x^7+12x^2+14$	136	52	10	11
$x^8+4x^7+12x^2+10$	136	52	10	11

TABLE 6. Discriminant and conductor table.

□

Remark 21. The Atkin-Lehner sign of sporadic supercuspidal representations of level 2^4 and 2^6 should be the same for all of them (and equal to $+1$). The proof should follow the Steinberg case proved before using the trivial Nebentypus hypothesis.

Remark 22. The local Atkin-Lehner sign statement in [Pac13, Remark 11] is not correct. In the case of a supercuspidal representation, the correct local computation is the one done in the previous proof.

To a newform $f \in S_k(\Gamma_0(N))$ and $p \mid N$ a prime, we can attach the pair $(\tilde{\pi}_{f,p}, \lambda_p)$ consisting of the local type of f at p, and its minimal Atkin-Lehner sign.

Definition 23. Let $\mathbf{LO}(p^n)$ denote the number of pairs $(\tilde{\tau}, \epsilon)$ where $\tilde{\tau}$ is a local type Galois orbit of level p^n and ϵ is a compatible minimal Atkin-Lehner eigenvalue (i.e. the existence of a newform $f \in S_k(\Gamma_0(p^n))$ for some k with $(\tilde{\pi}_{f,p}, \lambda_p) = (\tilde{\tau}, \epsilon)$ does not contradict Theorem 20).

Theorem 24. *The values of* $\mathbf{LO}(p^n)$ *are given in Table 7.*

Proof. The result comes from Theorem 8, Theorem 14 and Theorem 20. □

3. Existence of local types with compatible Atkin-Lehner sign

Theorem 25. *Let N be a positive integer such that N is a prime power or N is square-free. For each prime $q \mid N$, let $\tilde{\tau}_q$ be a local type of level $q^{\mathrm{val}_q(N)}$ and let $\epsilon_q \in \{\pm 1\}$ be a compatible Atkin-Lehner sign for $\tilde{\tau}_q$. Then there exists a positive integer k_0 such that for any $k \geq k_0$, there exists a newform $f \in S_k(\Gamma_0(N)$ such that:*

(1) $\tilde{\pi}_{f,q} \simeq \tilde{\tau}_q$ for all primes q,
(2) the Atkin-Lehner eigenvalue of f at q equals ϵ_q,

n	$\gcd(p,6)=1$	$p=3$	$p=2$
0	1	1	1
1	2	2	2
2	$\sigma_0(p+1)+\sigma_0(p-1)-1$	9	1
3	4	8	2
4	$\sigma_0(p+1)+\sigma_0(p-1)$	10	6
5	4	8	4
6	$\sigma_0(p+1)+\sigma_0(p-1)$	10	16
≥ 7 *odd*	4	8	8
≥ 8 *even*	$\sigma_0(p+1)+\sigma_0(p-1)$	10	10

TABLE 7. The values of $\mathbf{LO}(p^n)$.

(3) f does not have complex multiplication.

Proof. A very similar result in this direction is Theorem 1.1 of [Wei09] (see also Theorem 4.3), where an asymptotic formula for the number of types in the space of cusp forms of level N is given for k large enough. An important feature of its proof is that such number grows linearly in the weight k (for k big enough). Unfortunately, the result only counts types, not the whole local representation (so we do not get any information on the Atkin-Lehner signs); still, in the cases where the is a unique Atkin-Lehner sign at each local type, for example the case of modular forms whose local types are all principal series (see Theorem 20) Weinstein result is indeed enough for our purposes.

In the work [Mar18] (Theorem 3.3) the existence of forms with any combination of local Atkin-Lehner signs is proven for N square-free (i.e. only Steinberg local types). A different approach is given in [Gro11] (Section 10), were using the trace formula, the existence of automorphic forms for the group PGL_2 with any supercuspidal local representations at a finite set of primes (of $\mathrm{PGL}_2(\mathbb{Q}_p)$) is proven. Gross' result is generalized in [KST16]. Using the trace formula ideas (as in Gross' article), they prove (Theorem 1.2) that if G is any connected reductive group over a totally real field, the number of automorphic forms of weight k and level N with prescribed local representations (which are supercuspidal at ramified primes) grows linearly with k (recall that $\dim(\xi)=k-1$ if ξ is the discrete series of weight k, which gives the linear growth). Furthermore, the result can be extended to include Steinberg types as done in Section 6 loc. cit. where a similar result is proven in Theorem 6.4.

Therefore the results in the aforementioned articles prove the first two claims of the theorem, and in most situations this is also enough to get the last one (as complex multiplication forms are supercuspidal at all primes). In the general setting, the number of complex multiplication forms with a fixed level N is bounded as a function on the weight (see for example [Tsa14, Corollary 4.5]). On the other hand, the existence results stated above imply that the forms satisfying the first two conditions grows linearly in k, hence for k large enough the space always contains a non-CM modular form (of any given local inertial type). □

Remark 26. The constant k_0 in the last theorem can be made explicit by computing all the constants involved in the cited articles; we did not pursue this objective. We expect the previous result to hold in general, but we did not find a suitable reference for it. Note that the proof given looks stronger than the Theorem itself, as it involves a control on the whole local representation. Such control does not hold in general, namely we cannot fix a principal series representation and expect it to appear in a modular form. The reason is that fixing a principal series "involves" fixing the value of the p-th eigenvalue as well (if the representation is unramified, it implies fixing the Hecke eigenvalue, while in the ramified case it implies fixing the Hecke eigenvalue of a base change of the form), which is a very strong condition. However, once we know that local types do exist (by Weinstein result) we are only asking for unramified twists of a type that appears in the space of modular forms to appear as well. This weaker statement should be easier to prove, but we do not have a direct proof of it.

Theorem 27. *Let N be a prime power or square free. Then there exists k_0 such that for $k \geq k_0$,*

$$\prod_{p|N} \mathbf{LO}(p^{v_p(N)}) \leq \mathbf{NCM}(N,k). \tag{14}$$

Proof. By Theorem 25 we know that there exists k_0 such that for $k \geq k_0$ and for each local type with a compatible A-L sign, a modular form f of weight k and level N exists with the specified local type and Atkin-Lehner eigenvalue. Theorem 17 implies that Galois conjugate local types appear in the same Galois orbit of f, which gives the desired inequality. □

Remark 28. If Theorem 25 holds in general as explained in Remark 26, then for any positive integer N we get the inequality

$$\prod_{p|N} \mathbf{LO}(p^{\mathrm{val}_p(N)}) \leq \mathbf{NCM}(N,k), \tag{15}$$

for k large enough.

A natural question is to study how sharp is the inequality in (15) for general N. It is not true that the first inequality is an equality in general! The reason is that when N is a prime power (or a prime power times a square-free integer), there are enough automorphisms in the coefficient field to conjugate each of the local types so as to get the whole local Galois orbit for each of them. The problem arises when the automorphisms needed for two different primes correspond to the same extension (see Lemmas 15 and 16). Here is a concrete example: suppose that $N = 11^2 \cdot 31^2$. Let τ_{11} be a principal series corresponding to an order 5 character, and τ_{31} be a principal series of an order 5 character as well. Let $f \in S_k(\Gamma_0(11^2 \cdot 31^2))$ be a newform with the chosen local types at 11 and 31. Lemma 15 implies that $\mathbb{Q}(\xi_5)^+$ is contained in the coefficient field of f, so conjugating we can fix a local type at 11 in the orbit. Once we fixed such type at 11, we cannot conjugate the type at 31 (globally), so we get 2 different types at 31 in the Galois orbit of f. In this case, using Theorem 25 we get 2 as a lower bound for $\mathbf{NCM}(11^2 \cdot 31^2, k)$ (for k large enough) instead of 1.

With this example in mind, and the techniques developed before, one can give a better but more involved lower bound formula for the number of Galois orbits of modular forms of general level N and large enough weight k, assuming that Theorem 25 holds in general. However, in many instances (for example when N has a unique prime whose square divides it, or if it holds that whenever $p^r \mid N$ and $q^s \mid N$, $\gcd((p-1)p, (q-1)q) = 1$) the product of local Galois types is the best possible bound with our method. This is precisely the case for the data gathered in [Tsa14].

Another natural question is the existence of other Galois orbit invariants. Based on numerical computations done by the third author (see [Tsa14]) it seems that the answer should be negative, hence we propose the following problem.

Question 29. *If N is a prime power or square-free, is (14) an equality? I.e. is it true that that for k large enough the number of Galois orbits of modular forms of level N equals the number of Galois conjugate local types with compatible Atkin-Lehner signs?*

Remark 30. Due to the existence result (Theorem 26) an affirmative answer to Question 29 is equivalent to a uniqueness result (for k large enough) for the Galois orbits of newforms with given Galois conjugate local type and compatible Atkin-Lehner signs.

Clearly such statement is in the spirit of Maeda's original conjecture, hence it seems natural to expect that if there is no reason for forms to be non-conjugate, then they should be conjugate. We do not claim that we are convinced on the veracity of the problem, but we want to stress that numerical experiments suggest that the answer might be positive (see [Tsa14]) as the values of $\mathbf{LO}(p^n)$ seem to match the number of orbits of non-CM newforms in the respective space of modular forms of weight k starting at very small values of k.

However, for $p = 2$ and $n \geq 8$ even, there is a discrepancy that we cannot explain.

Example 2. Let $N = 2^8$ and $k = 12$. The space $S_{12}(\Gamma_0(256))$ contains 17 Galois orbits. Five of them corresponds to CM forms (four with rational coefficients, and one whose coefficient field is quadratic). The remaining 12 orbits have dimensions: $2, 2, 4, 4, 6, 6, 8, 8, 8, 10, 10, 12$. Computing a few Hecke operators, it can be checked:

- The 2-dimensional ones are twists of each other (via χ_{-1}) and that each orbit is stable under twisting by χ_{-2}. It corresponds to the unramified supercuspidal representations.
- The 4-dimensional orbits are stable under twisting by χ_{-1} hence are induced from $E = \mathbb{Q}_2(\sqrt{-1})$ and are twist of each other by χ_2.
- The same is true for the 6-dimensional ones.
- Two of the 8-dimensional ones have Galois orbits invariant under twisting by χ_{-2}. They are induced from the unramified quadratic extension.
- The other 8-dimensional one is induced from $\mathbb{Q}_2(\sqrt{3})$.
- The two 10-dimensional ones are principal series at 2.

- The 12 dimensional one is also induced from $\mathbb{Q}_2(\sqrt{3})$.

Note that we obtain four Galois orbits of newforms from the field $\mathbb{Q}_2(\sqrt{-1})$, while we expect only two of them. This phenomenon seems to persist for higher weights. The value $\mathbf{NCM}(2^8)$ seems to be 12 (we have computed up to weight 28), while our lower bound equals 10.

It would be interesting to have some statistical data on the size of the smallest k for equality to hold (which in particular is related to an effective proof of Theorem 25).

Note that a suitable variant of Question 29 makes sense for general N. Giving a more involved formula (as the example explained for level $N = 11^2 \cdot 31^2$) obtained by a detailed study of the local types for primes dividing N (and the coefficient fields of such modular forms) one can ask whether the obtained inequality is best possible. The cases not covered by Theorem 27 involve very large levels, so we could not gather any computational data which might suggest a positive or negative answer for the generalized Maeda's problem on general levels N.

4. Possible generalizations

There are many similar situations to study. The first natural question is what happens when working with modular forms with non-trivial Nebentypus. The situation is more subtle, and there are two different problems to be considered. One is that we are forced to look at minimal twists (and we only considered minimal quadratic twists in the trivial Nebentypus situation). The second one, is that there are no Atkin-Lehner involutions! One needs to replace them by the operators defined by Atkin and Li in [AL78]. We will consider this situation in a sequel of the present article. There is an obstacle in studying the number of orbits of modular forms with Nebentypus coming from its computational complexity. Still, it is true in this situation that the number of CM modular forms is bounded in the weight.

A second reasonable generalization is to study the case of Hilbert modular forms, i.e. changing the base field $\mathbb{Q}$ by a totally real one F. To study CM modular forms, the same ideas in [Tsa14] give a bound of their number independent of their weight. The same techniques developed in this article can be used to compute the number of local types of level $\mathfrak{p}^n$, for $\mathfrak{p}$ a prime ideal. Still, the formula is more involved in each case, as it depends on the degree $[F : \mathbb{Q}]$, on the inertial degree of $\mathfrak{p}$ over $\mathfrak{p} \cap \mathbb{Z}$ and its ramification degree. Then there are other invariants appearing related to the class number of F. An interesting question to study is if there are other type of Galois invariants besides the ones described in this article and the ones coming from the class group (it also seems natural, in the same way that CM forms were treated separately in the present article, to treat separately special cases of Hilbert modular forms such as those coming from base change, or from base change up to twist, from a smaller field). The toy example should be that of a real quadratic field, where base change forms are easy to handle by the results of the present article.

At last, it is natural to consider a similar question for other algebraic reductive groups G over $\mathbb{Q}$ to see if there are more invariants than those appearing for GL_2. For example if G is the group obtained from a rational quaternion algebra ramified at an even number of finite places, by the Jacquet-Langlands correspondence, automorphic forms for G correspond to (some particular) automorphic forms on GL_2. In particular, all the results of this article work for such algebraic groups, and we do not expect new invariants for such groups (as we do not expect them for GL_2). As suggested to the first author by M. Harris, it would also be interesting to test other groups like $\mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}})$ (or over a totally real number field) or $\mathrm{GSp}_n(\mathbb{A}_{\mathbb{Q}})$ to see if these phenomena persist. Again, in such a context it seems natural to exclude all "special" forms (i.e., those coming from automorphic forms from a smaller reductive group via Langlands functoriality) before checking if there is uniqueness for orbits with given local constraints, for sufficiently large weight (existence results are known in this generality, as was mentioned in section 4). We must admit that we did not consider any of these problems from a theoretical point of view nor gathered any computational evidence but it is our hope that this article may spark some research interest towards this direction.

4.1. **Applications of Question 29.** It is well known that Maeda's conjecture has many applications to different problems in number theory. The veracity of Problem 29 has as many applications as the original conjecture. Let us recall some of them.

4.1.1. *Inner twists.* The truth of Question 29 implies that the existence of inner twists for a newform is a purely local property, depending on the local types of the form.

Proposition 31. *Assume that Question 29 has an affirmative answer. Let $f \in S_k(\Gamma_0(p^r))$ be a newform of prime power level whose local type and Atkin-Lehner sign equals (τ, ϵ). Let μ be an inner twist of f (i.e. a finite order character such that $f \otimes \mu$ is Galois conjugate to f). Then for any $k' \geq k_0$ (where k_0 is the weight after which the conjecture becomes effective) and any newform $g \in S_{k'}(\Gamma_0(p^r))$ whose local type equals τ and whose Atkin-Lehner sign equals ϵ, μ is an inner twist of g.*

Furthermore, if μ is any finite order character ramified only at one prime p, for any pair (τ, ϵ) as before invariant (up to Galois conjugation) under twisting by μ and every $k' \geq k_0$, all newforms $g \in S_{k'}(\Gamma_0(p^r))$ with local data (τ, ϵ) have inner twist given by μ.

Proof. The proof is automatic due to the uniqueness result implied by Question 29 (see Remark 30). If we assume that f has an inner twist by μ this implies that μ ramifies only at p and that the local type and Atkin-Lehner signs of f are invariant (up to Galois conjugation) under twisting by μ. Therefore the same is true for any g with the same local data. Since Maeda's conjecture implies uniqueness of the Galois orbit with a fixed local data (at prime power level and weight greater or equal to k_0) the twist $g \otimes \mu$ must lie in the same orbit as g. The last claim follows from the same argument with the existence result given by Theorem 25. ☐

4.1.2. *Base Change.* The proof of (non-solvable) Base Change for classical modular forms and other cases of Langlands functoriality given in [Die15] relies on the construction of a "safe" chain of congruences linking arbitrary pairs of modular Galois representations. For the construction of such a chain in the aforementioned article, it is crucial to pass through a space of newforms having a unique Galois orbit: the space used in loc. cit. is a space of forms of prime level with non-trivial Nebentypus of fixed order and relatively large weight, a space that was computed in order to check that it contains indeed a unique Galois orbit of newforms. The conjecture proposed in Question 29 (i.e., the truth of the claim stated therein), gives an alternative and more theoretical way to complete the proof of Base Change (a proof not requiring computations): in fact, for the construction of the safe chain instead of a space with a unique Galois orbit (which is an option, but requires non-trivial Nebentypus) it is enough to know that in certain spaces of newforms of sufficiently large weight (and prime power level) there is a unique orbit with a specific supercuspidal local inertial type at the prime in the level, a fact that is implied by our conjecture.

This strategy for the construction of safe chains is explained in [DP15], in the more general context of Hilbert modular forms over a given totally real number field F. The construction of a safe chain connecting the Galois representations attached to any pair of Hilbert newforms over F, from whose existence relative non-solvable Base Change would follow immediately, can be reduced following the strategy described in loc. cit. to a case where the two Hilbert newforms have the same level, the same (large) parallel weight, and common inertial types at primes in their common level, thus a suitable generalization to Hilbert modular forms of the uniqueness claim proposed in Question 29 gives a way of completing the safe chain described in loc. cit. thus completing the proof of relative Base Change.

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