ON THE CHANGE OF ROOT NUMBERS UNDER TWISTING AND APPLICATIONS

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ABSTRACT. The purpose of this article is to show how the root number of a modular form changes by twisting in terms of the local Weil-Deligne representation at each prime ideal. As an application, we show how one can for each odd prime p, determine whether a modular form (or a Hilbert modular form) with trivial nebentypus is Steinberg, Principal Series or Supercuspidal at p by analyzing the change of sign under a suitable twist. We also explain the case $p = 2$, where twisting is not enough in general.

1. INTRODUCTION

The theory of local root numbers for modular forms was developed in the well known work [AL70]. If $N = p_1^{r_1} \cdots p_t^{r_t}$, for each $i \in \{1, \ldots, t\}$ the authors prove the existence of an involution W_{p_i} in the space $S_k(\Gamma_0(N))$ which commutes with the Hecke operators T_p (for $p \nmid N$) and such that they commute with each other (see Lemma 8 and Lemma 10). Furthermore, the composition of all such involutions equal the canonical involution W_N (see Lemma 9).

Recall that if f is a modular form in $S_k(\Gamma_0(N))$ which is an eigenfunction for all Hecke operator, its L-series satisfies a functional equation relating its values at s and $1 - s$ with a sign factor called the "root number of f". In the subspace of new forms, the Hecke operators prime to the level diagonalize simultaneously into 1-dimensional eigenspaces (the so called multiplicity one result). Since the canonical involution W_N and the Atkin-Lehner involutions W_{p_i} commute with such operators, they must leave each eigenspace stable. If f is a newform, the "local factor of f at p" is the eigenvalue of the involution W_{p_i}. It can be shown that the root number equals minus the eigenvalue of W_N and, in particular, the root number can be read from the local factors.

Besides the definition and properties of the root numbers, the authors also proved some cases of the variance of the local root number under twisting. In particular, the results proven there imply the well known result that the variance of the global root number of a modular form of level N, by twisting by a quadratic field corresponding to a character χ with conductor prime to N, is given by $\chi(N)$. The case where the level and the conductor are not prime to each other is more subtle. Some partial results were proven in the

same work, and some extensions with a similar perspective was obtained in [AL78].

The existence of local factors of representations was proved by Deligne in [Del73]. Many authors used such description to compute explicitly the local root numbers in terms of the local representation of the Weil-Deligne group (as in [GK80], or [Li80]), but to our knowledge, although the variation of the local factor under twisting follows essentially from the properties of the local factor and is known to any expert in the area, it has not being written down explicitly in general. As will be showed in the article, it allows for example to easily compute for an elliptic curve the local type (i.e. whether it is Steinberg, Principal Series or Supercuspidal) at any odd prime (which of course can also be done by looking at the reduction of the elliptic curve, as explained in [Roh93], Section 1). For $p = 2$, a classification can be given in terms of the sign variation, but this does not completely determine the type (as is shown in Example 4.3 and Example 4.4). If one can have some other information, for example if one is able to compute the space of modular forms appearing in the quaternion algebra ramified at 2, then this extra information is enough to determine the type at 2 as well.

It should be mentioned the recent article of [LW10], were the authors give a method to compute explicitly the local data of a modular form (not just its type) for any nebentypus. They do not use any twisting argument, so their method is different from ours. However, they assume the modular form to be minimal between its twists (and it is not clear how to compute the minimal twist without computing some spaces of smaller levels with any nebentypus, see Example 5.2) and also to get the data they need to compute the whole space of modular forms instead of just the twist of a given one.

By the nature of the argument, the same results hold for Hilbert modular form with trivial nebentypus as well, but the problem is that in general the global characters to twist by, might not exist. Nevertheless, this problem can be overcome by adding some auxiliary prime to the twist, as is shown in the last section, so our method works for Hilbert modular forms as well.

The current article started some years ago while studying a characterization of the elliptic curves whose conductor is divisible by p but which do not show up in the quaternion algebra ramified at p and at ∞. After some numerical computations with Gonzalo Tornaría we conjectured most of the formulas proven here, and applied the formulas to find all elliptic curves in Cremona's tables not appearing in any quaternion algebra (answering a question raised by Professor Cremona to Gonzalo). The table can be found in `http://mate.dm.uba.ar/~apacetti/`. For a work in progress with Victor Rotger, we needed an exact formula giving the sign variation of a modular form under twisting in terms of the local type, which forced to reconsider the problem and by lack of references, write down a proof of the conjectures.

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Notation: Given an odd prime number p, by $\mathrm{val}_p(N)$ we denote the p-adic valuation of the integer N. By $p^\star$ we denote the element $\left(\frac{-1}{p}\right)p$, whose square root generates the quadratic extension of $\mathbb{Q}$ ramified only at p. For a positive integer r, we denote by U_r the usual filtration in the p-adic units, given by

$$U_r = \{x \in \mathbb{Z}_p^\times \ : \ x \equiv 1 \pmod{p^r}\}.$$

If E_p is a finite extension of $\mathbb{Q}_p$, Tr and Norm denote the trace and norm from E_p to $\mathbb{Q}_p$ respectively.

2. Some useful well known results

For K a local field, let $W(K)$ denote the Weil group of K; that is the preimage under the map

$$\mathrm{Gal}(\overline{K}/K) \mapsto \mathrm{Gal}(\overline{k}/k),$$

of the integer powers of Frobenius, where k denotes the residue field of K. Local class field theory gives an isomorphism between $W(K)^{\mathrm{ab}}$ and $K^\times$. Furthermore, if L/K is finite, the following diagrams are commutative:

(1)
$$
\begin{array}{ccccc}
W(L) & \longrightarrow & W(L)^{\mathrm{ab}} & \overset{\simeq}{\longrightarrow} & L^\times \\
\cup\downarrow & & \downarrow & & \downarrow{\scriptstyle N_{L/K}} \\
W(K) & \longrightarrow & W(K)^{\mathrm{ab}} & \overset{\simeq}{\longrightarrow} & K^\times,
\end{array}
$$

and

(2)
$$
\begin{array}{ccc}
W(K)^{\mathrm{ab}} & \overset{\simeq}{\longrightarrow} & K^\times \\
\downarrow{\scriptstyle t} & & \cup\uparrow \\
W(L)^{\mathrm{ab}} & \overset{\simeq}{\longrightarrow} & L^\times,
\end{array}
$$

where t denotes the transfer map.

Let $f \in S_k(\Gamma_0(N))$ be a non-zero weight k and level N newform, i.e. a new modular form which is an eigenfunction for all Hecke operators, and let $\rho_p(f)$ be the local representation of the Weil-Deligne group $W'(\mathbb{Q}_p)$ associated to f at the prime p. Although it is not so easy to give a description of the Weil-Deligne group (see [Tat79]), it is relatively easy to describe its representations. A complex 2-dimensional representation of $W'(\mathbb{Q}_p)$ is a pair (ρ, N), where:

(1) ρ is a representation $\rho : W(\mathbb{Q}_p) \mapsto \mathrm{GL}_2(\mathbb{C})$,
(2) N is a nilpotent endomorphism of $\mathbb{C}^2$ such that

$$wNw^{-1} = \omega_1(w)n, \text{ for all } w \in W(\mathbb{Q}_p),$$

where ω_1 is the unramified quasi-character giving the action of $W(\mathbb{Q}_p)$ on the roots of unity (and corresponds to the norm quasi-character $||\,.\,||_p$ under local class field theory).

Although representations of the Weil-Deligne group for any vector space V are defined in a similar way, the two dimensional complex case is enough for our purposes.

The correspondence between local components of automorphic representations for $\Gamma_0(N)$ and the local representations of the Weil-Deligne group, is given as follows (using the normalization given by Carayol in [Car86]):

(1) **Principal Series** (reducible case): the endomorphism $N = 0$ and
$$\rho_p(f) = \chi \oplus \chi^{-1}\omega_1^{1-k},$$
for some quasi-character $\chi : W(\mathbb{Q}_p)^{\mathrm{ab}} \mapsto \mathbb{C}^\times$.

(2) **Steinberg or Special Representation** (indecomposable but reducible as $W(\mathbb{Q}_p)$-representation): The endomorphism N is given by the matrix $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and the representation $\rho_p(f)$ is given by
$$\rho_p(f)(w) = \left(\begin{array}{cc} \chi(w)\omega_1(w) & 0 \\ 0 & \chi(w) \end{array} \right),$$
for some quasi-character $\chi : W(\mathbb{Q}_p) \mapsto \mathbb{C}^\times$ with $\chi^2|_{\mathbb{Z}_p^\times} = 1$.

(3) **Supercuspidal Representation I** (irreducible case, but inertia acts reducibly): the endomorphism $N = 0$ and
$$\rho_p(f) = \mathrm{Ind}_{W(E_p)}^{W(\mathbb{Q}_p)} \varkappa,$$
where E_p is a quadratic extension of $\mathbb{Q}_p$, and $\varkappa : W(E_p)^{\mathrm{ab}} \mapsto \mathbb{C}^\times$ is a quasi-character which does not factor through the norm map with a quasi-character of $W(\mathbb{Q}_p)^{\mathrm{ab}}$ (so that $\rho_p(f)$ is irreducible). Furthermore, if ϵ_p denotes the quadratic character corresponding to the extension $E_p/\mathbb{Q}_p$, then $\epsilon_p\varkappa = ||\,.\,||_p^{1-k}$ as quasi-characters of $\mathbb{Q}_p^\times$.

(4) **Supercuspidal Representation II** (inertia acts irreducibly): this only happens for $p = 2$. $N = 0$ and the image of $\rho_p(f)$ is an exceptional group.

Remark 1. In the n-dimensional case, the last case occurs only for $p \leq n$.

The previous description uses the assumption that the nebentypus of f is trivial, and the third case relies on the following two facts due to Henniart (see [Hen79] Theorem 8.2):

Theorem 2.1. *Let E/F be a finite separable extension of degree n of local fields, and ρ be a linear degree n representations of $W(E)$. If R denotes its induction to $W(F)$, then:*

(1) *Let ε be the character of $F^\times$ that corresponds to the determinant of the permutation representation of $W(F)$ acting on $W(F)/W(E)$. Then for $x \in F^\times$ one has*
$$\det(R)(x) = \varepsilon(x)^n \det(\rho)(x).$$

(2) *Assume that ρ is semi-simple, then R is semi-simple as well and one has*
$$\mathrm{cond}(R) = f(E/F)(n \cdot d(E/F) + \mathrm{cond}(\rho)),$$
where $\mathrm{cond}()$ denotes the exponent of the Artin conductor of the representation, $f(E/F)$ denotes the inertial degree and $d(E/F)$ is the discriminant.

3. The case of an odd prime

The previous description and the fact that the unique order two character of $\mathbb{Z}/p^r$ has conductor p, gives the well know condition on the exponents of the distinct types:

Corollary 3.1. *If $f \in S_k(\Gamma_0(N))$ and $p \neq 2$, we have:*

(1) *If $\rho_p(f)$ is principal series then $v_p(N)$ is even.*
(2) *If $\rho_p(f)$ is Steinberg then $\mathrm{val}_p(N) = 1$ or 2.*
(3) *If $\rho_p(f)$ is supercuspidal then $\mathrm{val}_p(N) \geq 2$.*

Furthermore, in the last case if $E_p/\mathbb{Q}_p$ is unramified then $\mathrm{val}_p(N)$ is even. If $E_p/\mathbb{Q}_p$ is ramified, then $\mathrm{val}_p(N)$ is odd unless $\varkappa$ has conductor 1, $p \equiv 3 \pmod 4$ and $\varkappa|_{\mathbb{Z}_p^\times} = \epsilon_p$, in which case the conductor is 2.

Proof. The first two statements are clear. For the last statement, we know that our representation is the induced representation from a quadratic extension E_p of $\mathbb{Q}_p$ of a character $\varkappa$. Then Theorem 2.1 implies that since f has trivial nebentypus, we must have

$$(3) \qquad \varkappa|_{\mathbb{Z}_p^\times} \epsilon_p = 1,$$

where ϵ_p is the quadratic character that corresponds via class field theory to the quadratic extension $E_p/\mathbb{Q}_p$. If such extension is unramified, the conductor of $\varkappa$ must be non-zero, since otherwise it will factor through the norm map. Hence the even condition in the exponent comes from the fact that the inertial degree is 2 in this case.

In the case where the extension is ramified, $d(E_p/\mathbb{Q}_p) = 1$, hence Theorem 2.1 implies that the conductor of the representation equals $1 + \mathrm{cond}(\varkappa)$.

The conductor of ϵ_p is 1, so condition (3) implies that $\mathrm{cond}(\varkappa) = 1$ and $\varkappa|_{\mathbb{Z}_p^\times} = \epsilon_p$ or its conductor is even, which implies the statement. Note that in the first case, $\varkappa|_{\mathbb{Z}_p^\times}$ is quadratic, and it factors through the norm map if and only if there is a character of order 4 in $\mathbb{Z}_p^\times$ with conductor p. This is indeed the case if and only if $p \equiv 1 \pmod 4$, so the representation is irreducible only when $p \equiv 3 \pmod 4$. $\qquad\square$

Remark 2. If the form f has CM by the extension $\mathbb{Q}[\sqrt{-p}]$, then its local component at p corresponds exactly to $E_p/\mathbb{Q}_p$ being ramified and $\varkappa$ of conductor 1.

Let χ be the quadratic character associated to the quadratic extension of $\mathbb{Q}$ ramified only at p. By class field theory, it can be identified with a character of the idèle group, i.e. characters $\{\chi_q\}_q$, with $\chi_q : \mathbb{Q}_q^\times \mapsto \mathbb{C}^\times$ satisfying the following conditions:

- If $q \neq p$, then χ_q is unramified, and $\chi_q(q) = \left(\frac{q}{p}\right)$.
- χ_p is ramified with conductor p, and its value in $\mathbb{Z}_p^\times$ factors through the unique quadratic character of $\mathbb{F}_p^\times$. Furthermore, $\chi_p(p) = 1$.

Given a modular form f in $S_k(\Gamma_0(N))$, we want to study how the local factors of f change while twisting by χ. Denote by ε_q the variation of the local factor of f at q while twisting by χ_q, where we choose the same additive character and Haar measure on both computations.

Remark 3. In the correspondence between automorphic forms and representations of the Weil-Deligne group, twisting an automorphic representation by a quasi-character, has the effect of twisting the Weil-Deligne representation by the inverse of the quasi-character, but since our character χ_p is quadratic, we can avoid this technical detail.

Theorem 3.2. *The number ε_q is given by:*

(1) *If $q \neq p$, then $\varepsilon_q = \left(\frac{q}{p}\right)^{\mathrm{val}_q(N)}$.*

(2) *If $\rho_p(f)$ is principal series, then*
$$\varepsilon_p = \begin{cases} \left(\frac{-1}{p}\right) & \text{if } \mathrm{val}_p(N) \neq 0, \\ \left(\frac{-1}{p}\right) p^k & \text{if } \mathrm{val}_p(N) = 0. \end{cases}$$

(3) *If $\rho_p(f)$ is supercuspidal and $E_p/\mathbb{Q}_p$ is unramified, then $\varepsilon_p = -\left(\frac{-1}{p}\right)$.*

(4) *If $\rho_p(f)$ is supercuspidal and $E_p/\mathbb{Q}_p$ is ramified, then*
$$\varepsilon_p = \begin{cases} 1 & \text{if } \mathrm{val}_p(N) = 2, \\ 1 & \text{if } E_p = \mathbb{Q}_p[\sqrt{p^\star}], \\ -1 & \text{elsewhere.} \end{cases}$$

(5) *If $\rho_p(f)$ is Steinberg with $\mathrm{val}_p(N) = 1$, choosing the additive character ψ unramified and the Haar measure normalized such that $\int_{\mathbb{Z}_p} dx = 1$, the local sign is given by*
$$\varepsilon(\rho_p(f), \psi, dx) = \frac{-1}{\chi(p)};$$
while the local sign of $\rho_p(f) \otimes \chi_p$ is given by
$$\varepsilon(\rho_p(f) \otimes \chi_p, \psi, dx) = \left(\frac{-1}{p}\right).$$

Remark 4. Although in the second case, the local root number is not just a sign, the power of the prime p appearing comes from the fact that the level of the form and its twists are different.

Remark 5. The result for the Steinberg representation and for the Principal series when $p \nmid N$ are well known, and can be found for example in [AL70] (Lemma 30 and Theorem 6), although the way to prove it uses the theory of the Atkin-Lehner involutions as global actions, while the proof we present is just of local nature.

The proof of the result is quite elementary, and is mainly based in the properties that the local constant satisfies, as explained in [Del73]. One of the main properties that determine the local constant is the following:

Property. *Let ρ be a virtual 0-dimensional representation of a finite extension $E_p/\mathbb{Q}_p$, then*
$$\varepsilon(\mathrm{Ind}_{W(E_p)}^{W(\mathbb{Q}_p)} \rho, \psi) = \varepsilon(\rho, \psi \circ \mathrm{Tr}_{E_p/\mathbb{Q}_p}).$$

See [Del73], Theorem 4.1 for a proof of the existence of local constants with the appropriate 4 conditions.

Proof of Theorem 3.2. We consider each case separately:

(1) If $q \neq p$, the character χ_q is unramified, hence by (5.5.1) of [Del73],

$$\varepsilon(\rho_q(f) \otimes \chi_q, \psi, dx) = \chi_q\left(q^{\text{val}_q(N) + q \dim(\rho_q(f))}\right) \varepsilon(\rho_q(f), \psi, dx).$$

Since $\chi_q(q) = \left(\frac{q}{p}\right)$ and $\dim(\rho_q(f)) = 2$, the statement follows.

(2) Since $\rho_p(f)$ is reducible in the principal series case, we need to see how the local constant of a quasi-character changes under twisting by χ_p. Let $a = \text{cond}(\chi_1)$ be the conductor of the quasi-character χ_1; chose ψ to be an additive character of $\mathbb{Q}_p$ with $\text{cond}(\psi) = 0$ (i.e. $\psi|_{\mathbb{Z}_p} = 1$ but $\psi|_{\frac{1}{p}\mathbb{Z}_p} \neq 1$) and the Haar measure dx such that $\int_{\mathbb{Z}_p} dx = 1$. The local epsilon factor is then given by

$$\int_{\mathbb{Z}_p^\times} \chi_1^{-1}\left(\frac{x}{p^a}\right) \psi\left(\frac{x}{p^a}\right) d\frac{x}{p^a} = \chi_1(p)^a p^a \sum_{b \in \mathbb{Z}_p^\times / U_a} \chi_1^{-1}(b) \psi\left(\frac{b}{p^a}\right) \int_{U_a} dx.$$

The normalization $\int_{\mathbb{Z}_p} dx = 1$ implies that $\int_{\mathbb{Z}_p^\times} dx = \frac{p-1}{p}$ and $\int_{U_a} dx = \frac{1}{p^a}$. Since the conductor of ψ is 0, $\psi\left(\frac{1}{p^a}\right) = \exp(2\pi i / p^a)^c$ for some c prime to p. Then

$$G(\chi_1^{-1}, c) = \chi_1^{-1}(c) \left(\sum_{b \in \mathbb{Z}_p^\times / U_a} \chi_1^{-1}(b) \exp\left(\frac{2\pi i b}{p^a}\right)\right),$$

is a Gauss sum. If we compute the product of the epsilon factor corresponding to χ_1 and the one corresponding to $\chi_1^{-1} || \cdot ||_p^{1-k}$, we get that the local factor is given by

$$||p^a||_p^{1-k} G(\chi_1^{-1}, c) G(\chi_1, c) = \frac{p^{ak}}{p^a} p^a \chi_1(-1) = p^{ak} \chi_1(-1).$$

The middle equality is a classical result of Gauss sums, see for example [Dav00] (Ex. 13(9), p. 295). If we replace χ_1 by $\chi_1 \chi_p$ in the previous computation, we get that the two numbers differ by $\chi_p(-1) = \left(\frac{-1}{p}\right)$ and a power of p if the level of f and that of $f \otimes \chi$ are not equal, as claimed.

(3) Since $(\text{Ind}_{W(E_p)}^{W(\mathbb{Q}_p)} \varkappa)\chi_p = \text{Ind}_{W(E_p)}^{W(\mathbb{Q}_p)}(\chi_p \varkappa)$ (where in the second term of the equality we are considering the restriction of χ_p to $W(E_p)$), we can apply the Property stated before to $\rho = \chi_p \varkappa - \varkappa$. Then

$$\frac{\varepsilon(\text{Ind}_{W(E_p)}^{W(\mathbb{Q}_p)} \chi_p \varkappa, \psi)}{\varepsilon(\text{Ind}_{W(E_p)}^{W(\mathbb{Q}_p)} \varkappa, \psi)} = \frac{\varepsilon(\chi_p \varkappa, \psi \circ \text{Tr}_{E_p/\mathbb{Q}_p})}{\varepsilon(\varkappa, \psi \circ \text{Tr}_{E_p/\mathbb{Q}_p})}.$$

This allows to restrict to the 1-dimensional case. Recall that since $E_p/\mathbb{Q}_p$ is unramified, $\varkappa$ is ramified (as was pointed out in the proof of Corollary 3.1). Let $\mathcal{O}_p$ denote the ring of integers of E_p, take p as a local uniformizer, and let $a = \text{cond}(\varkappa)$ be the conductor of $\varkappa$. Recall that since the nebentypus

of f is trivial, $\varkappa|_{\mathbb{Z}_p^\times} = 1$. Let ψ be an additive character with $\mathrm{cond}(\psi) = 0$ as before, and dx a Haar measure such that $\int_{\mathcal{O}_p} dx = 1$. To easy notation we will denote by $\tilde{\psi}$ the additive character $\psi \circ \mathrm{Tr}_{E_p/\mathbb{Q}_p}$. The local factor is given by

$$\varepsilon(\varkappa, \tilde{\psi}, dx) = \int_{\mathcal{O}_p^\times} \varkappa^{-1}\left(\frac{x}{p^a}\right) \tilde{\psi}\left(\frac{x}{p^a}\right) d\frac{x}{p^a} =$$

$$\varkappa(p)^a p^{2a} \left(\sum_{b \in \mathcal{O}_p^\times / U_a} \varkappa^{-1}(b) \tilde{\psi}\left(\frac{b}{p^a}\right) \right) \int_{U_a} dx.$$

The middle sum can be written as

$$\sum_{\alpha \in (\mathcal{O}_p/p^a)^\times / (\mathbb{Z}_p/p^a)^\times} \sum_{\beta \in (\mathbb{Z}_p/p^a)^\times} \varkappa^{-1}(\alpha\beta) \psi\left(\frac{\mathrm{Tr}(\alpha)\beta}{p^a}\right) =$$

$$\sum_{\alpha} \varkappa^{-1}(\alpha) \sum_{\beta} \psi\left(\frac{\mathrm{Tr}(\alpha)\beta}{p^a}\right).$$

If $p^{a-1} \nmid \mathrm{Tr}(\alpha)$, the last sum is zero, since it is a sum over all primitive roots of unity of order at least p^2. For elements where $p^{a-1} || \mathrm{Tr}(\alpha)$ (i.e. $p^{a-1} | \mathrm{Tr}(\alpha)$ but $p^a \nmid \mathrm{Tr}(\alpha)$), the last sum is -1. Such elements are of the form

$$r(p^{a-1} + \beta\sqrt{\delta}), \quad r \in (\mathbb{Z}_p/p^a)^\times \text{ and } p \nmid \beta.$$

Modulo multiplication by elements of $(\mathbb{Z}_p/p^a)^\times$ in $\mathcal{O}_p/p^a$, for $p \nmid s$, we have that

$$(p^{a-1} + \beta\sqrt{\delta})^s \sim (p^{a-1} + s^{-1}\beta\sqrt{\delta}),$$

so the sum with these terms is

$$(-1) \cdot \sum_{p^{a-1} || \mathrm{Tr}(\alpha)} \varkappa^{-1}(\alpha) = (-1) \cdot \sum_{s \in (\mathbb{Z}_p/p^a)^\times} \varkappa^{-1}(p^{a-1} + \sqrt{\delta})^s.$$

Since the conductor of $\varkappa$ is p^a, $\varkappa$ is non-trivial on such elements and the last sum is zero. Then the only remaining terms are the ones with $\mathrm{Tr}(\alpha) = 0$ (i.e. $\alpha = \sqrt{\delta}$ for a non-square element δ), and in this case all terms of the last sum are 1 so we get that

$$\varepsilon(\varkappa, \tilde{\psi}, dx) = p^{a-1}(p-1)\varkappa(\sqrt{\delta})^{-1}\varkappa(p)^a,$$

where δ is a non-square in $\mathbb{Q}_p$. If we make the same computation with $\chi_p \varkappa$, we get that

$$\varepsilon_p = \frac{p^{a-1}(p-1)\varkappa(\sqrt{\delta})^{-1}\chi_p(-\delta)\varkappa(p)^a}{p^{a-1}(p-1)\varkappa(\sqrt{\delta})^{-1}\varkappa(p)^a} = -\left(\frac{-1}{p}\right).$$

(4) This case is similar to the previous one, the main difference is that using the commutative diagram (1), we need to compose our character with the norm map. This gives another character (that abusing notation we also denote χ_p) which satisfies $\chi_p|_{\mathcal{O}_p^\times} = 1$, because the conductor of χ_p is p and the norm map from $\mathcal{O}_p^\times$ to $\mathbb{Z}_p^\times$ gives only squares modulo p. Then the terms in the sum are the same for $\varkappa$ and $\chi_p \varkappa$. Since $E_p/\mathbb{Q}_p$ is ramified, the

conductor of $\psi \circ \mathrm{Tr} = 1$, hence if we take $\pi = \sqrt{p\delta}$ as a local uniformizer, the local factor is given by

$$\varepsilon(\varkappa, \tilde{\psi}, dx) = \varkappa(\pi^{\mathrm{cond}(\varkappa)+1}) \int_{\mathcal{O}_p^\times} \varkappa^{-1}(x)\tilde{\psi}\left(\frac{x}{\pi^{2r+1}}\right) d\frac{x}{\pi^{2r+1}}.$$

The local factor of the twisted representation is given by

$$\varepsilon(\tilde{\chi}_p\varkappa, \tilde{\psi}, dx) = \chi_p(\mathrm{Norm}(\pi))^{\mathrm{cond}(\varkappa)+1}\varepsilon(\varkappa, \tilde{\psi}, dx).$$

Hence the quotient equals 1 if $\mathrm{cond}(\varkappa) = 1$ or

$$\varepsilon_p = \left(\frac{\mathrm{Norm}(\pi)/p}{p}\right) = \left(\frac{-\delta}{p}\right).$$

In particular, if $\delta = \left(\frac{-1}{p}\right)$ up to squares, $\varepsilon_p = 1$, while if it does not, then $\varepsilon_p = -1$ as claimed.

(5) Comes from the definition of the local epsilon factor attached to the representations where the nilpotent endomorphism is not trivial and how the local epsilon factors changes for the principal series. $\qquad\square$

Let f be in $S_k(\Gamma_0(N))$, where $N = p^r N'$, with $p \nmid N$, and let $\varepsilon(f)$ be its functional equation sign. Let χ_p be as before, and let $f \otimes \chi_p$ denote the newform obtained while twisting f by χ_p. Denote by $N(f \otimes \chi_p)$ its level. The previous statement allows the following classification:

Corollary 3.3. *With the previous notation, we have the following computational criteria to compute the local type:*

- $\pi_p(f)$ *is* Steinberg *if* $\mathrm{val}_p(N) = 1$ *or* $\mathrm{val}_p(N(f \otimes \chi_p)) = 1$.
- $\pi_p(f)$ *is* Principal Series *if it is not Steinberg,* $2 \mid \mathrm{val}_p(N)$ *and*

$$\varepsilon(f \otimes \chi_p) = \chi_p(N')\varepsilon(f)\left(\frac{-1}{p}\right).$$

- $\pi_p(f)$ *is* Supercuspidal *if it is not of the above type. Furthermore, if* $\mathrm{val}_p(N)$ *is even and greater than 2,* $\pi_p(f)$ *is induced from the unramified quadratic extension of* $\mathbb{Q}_p$, *while if* $\mathrm{val}_p(N)$ *is odd and greater than 2,*
    - $\pi_p(f)$ *is induced from the extension* $\mathbb{Q}_p[\sqrt{p^\star}]$ *if* $\varepsilon(f \otimes \chi_p) = \chi_p(N')\varepsilon(f)$.
    - $\pi_p(f)$ *is induced from the extension* $\mathbb{Q}_p[\delta\sqrt{p^\star}]$ *(for any non-square* δ*) if* $\varepsilon(f \otimes \chi_p) = -\chi_p(N')\varepsilon(f)$.

Remark 6. In the case $\mathrm{val}_p(N) = 2$, twisting only allows to determine the type, but does not distinguishes from which quadratic extension the representations is induced from.

Remark 7. We can replace the global functional equation sign in the last two corollaries by the local Atkin-Lehner involution W_p. Then the same statements are true replacing $\varepsilon()$ by the eigenvalue of W_p and removing the factor $\chi_p(N')$.

Remark 8. If $f \in S_k(\Gamma_1(N), \epsilon)$ and for $p \mid N$ the character $\epsilon_p = 1$, the same result holds.

4. THE CASE $p = 2$

When $p = 2$, there are more representations of the Weil group. It is a classical result that all subgroups of $\mathrm{PGL}_2(\mathbb{C})$ are isomorphic to: a cyclic group, a dihedral group, A_4, S_4 or A_5. The A_5 case cannot happen since the Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ is solvable.

The case $p \neq 2$ did not include exceptional cases since for odd characteristic all 2-dimensional representations of $W(\mathbb{Q}_p)$ are of dihedral type as mentioned in Remark 1. But for $p = 2$, the A_4 and S_4 case might occur. Weil proved in [Wei74] that over $\mathbb{Q}$, the A_4 case actually does not occur, so the only exceptional case has image S_4. Furthermore, there are only 8 cases with projective image S_4 and all cases it corresponds to the field extension of $\mathbb{Q}_2$ obtained by adding the coordinates of the 3-torsion points of the elliptic curves (see also [BR99], Section 8):

$$(4) \qquad E_1^{(r)} : ry^2 = x^3 + 3x + 2, \qquad r \in \{\pm 1, \pm 2\},$$

and

$$(5) \qquad E_2^{(r)} : ry^2 = x^3 - 3x + 1, \qquad r \in \{\pm 1, \pm 2\}.$$

Note that there are 3 quadratic extensions of $\mathbb{Q}$ which ramify only at 2. We will denote by χ_{-1}, χ_2 and χ_{-2} the quadratic character that corresponds to the such quadratic extensions, where χ_i corresponds to $\mathbb{Q}[\sqrt{i}]$ (of conductor 4 the first one and 8 the last two ones). Then the 4 curves of each type are twists of each other, where $E_i^{(r)} \otimes \chi_j = E_i^{(rj)}$ (abusing notation and considering the supra-indices modulo the equivalence relation given by squares). Furthermore, by [Rio06] (Section 6), the level of the modular form is 2^7 in the case of the curve $E_1^{(r)}$ (with $r \in \{\pm 1, \pm 2\}$), 2^4 for the curve $E_2^{(1)}$, 2^3 for the curve $E_2^{(-1)}$ and 2^6 for the curve $E_2^{(\pm 2)}$.

Before stating the equivalent of Corollary 3.1, recall that there are 7 quadratic extensions of $\mathbb{Q}_2$. One of them is unramified, two of them have discriminant with valuation 2 (corresponding to $\sqrt{3}$ and $\sqrt{7}$) and four of them have discriminant with valuation 3 (corresponding to $\sqrt{2}$, $\sqrt{10}$, $\sqrt{-2}$ and $\sqrt{-10}$). With the previous notations, we have

Corollary 4.1. *For $p = 2$, we have:*

- *If $\rho_2(f)$ is principal series then $v_2(N)$ is even (but not 2).*
- *If $\rho_2(f)$ is Steinberg then $\mathrm{val}_2(N) \in \{1, 4, 6\}$.*
- *If $\rho_2(f)$ is supercuspidal then $\mathrm{val}_2(N) \geq 2$. Furthermore, depending on the different extensions we have:*
  - *If $E_2/\mathbb{Q}_2$ is unramified then $\mathrm{val}_2(N)$ is even and greater or equal to 2.*
  - *If $E_2/\mathbb{Q}_2$ is ramified with valuation 2 then $\mathrm{val}_2(N) = 5$ or it is even and greater or equal to 6.*
  - *If $E_2/\mathbb{Q}_2$ is ramified with valuation 3, then $\mathrm{val}_2(N) = 8$ or it is odd and greater or equal to 9.*
- *If $\rho_2(f)$ is supercuspidal of type II, then $\mathrm{val}_2(N) \in \{3, 4, 6, 7\}$.*

Proof. The proof is the same as before for the first two cases. The supercuspidal dihedral case is also the same for the unramified extension, while

for the ramified extension, note that if $\varkappa|_{\mathbb{Z}_2^\times} = \epsilon_2$, then the conductor of $\varkappa$ is $2 \cdot \mathrm{cond}(\epsilon_2) - 1$ or it is even (and greater). Then Theorem 2.1 implies that the conductor of the induced representation is $3 \cdot \mathrm{cond}(\epsilon_2) - 1$ or congruent to $\mathrm{cond}(\epsilon_2)$ modulo 2. $\qquad\square$

Remark 9. Note that in the supercuspidal dihedral case induced from ramified quadratic extensions, all the representations are irreducible, since there are no characters of order 4 and conductor 4 nor characters of order 4 and conductor 8.

Remark 10. In the principal series case, the levels $1, 4$ and 6 are twists of each other, since all characters of conductor $1, 2$ or 3 are at most quadratic.

We summarize the previous corollary in Table 4.1 (where we used the notation: PS meaning principal series, ST meaning Steinberg, SCIa, SCIb, SCIc meaning Supercuspidal of dihedral type and the index a, b of c meaning induced from an extension with discriminant $0, 2$ and 3 respectively; and SCII meaning supercuspidal of second type).

$\mathrm{val}_2(N)$	Types	$\mathrm{val}_2(N)$	Types
0	PS	6	PS, ST, SCIa, SCIb, SCII
1	ST	7	SCII
2	SCIa	8	PS, SCIa, SCIb, SCIc
3	SCII	odd ≥ 9	SCIc
4	PS, ST, SCIa, SCII	even ≥ 10	PS, SCIa, SCIb
5	SCIb		

TABLE 4.1. Possible types for $p = 2$.

Let χ denote the character associated by class field theory to any of the characters χ_i, $i \in \{-1, \pm 2\}$ and denote by ε_q the change of the variation of the local factor at q under twisting by χ. Then ε_q it is given by:

Theorem 4.2. *The number ε_q (up to a power of 2) is given by:*

(1) *If $q \neq 2$, then $\varepsilon_q = \chi(q)^{\mathrm{val}_q(N)}$.*

(2) *If $\rho_2(f)$ is principal series, then $\varepsilon_2 = \chi(-1)$.*

(3) *If $\rho_2(f)$ is supercuspidal and $E_2/\mathbb{Q}_2$ is unramified, then $\varepsilon_2 = -\chi(-1)$.*

(4) *If $\rho_2(f)$ is supercuspidal and $E_2/\mathbb{Q}_2$ is ramified, then*

$$\varepsilon_2 = \begin{cases} 1 & \text{if } \mathrm{cond}(\varkappa) = 2 \cdot \mathrm{cond}(\epsilon_2) - 1, \\ 1 & \text{if } E_2 \text{ corresponds to } \chi_2, \\ -1 & \text{elsewhere.} \end{cases}$$

(5) *If $\rho_2(f)$ is Steinberg with $\mathrm{val}_2(N) = 1$, choosing the additive character ψ unramified and the Haar measure normalized such that $\int_{\mathbb{Z}_2} dx = 1$, the local sign is given by*

$$\varepsilon(\rho_2(f), \psi, dx) = \frac{-1}{\chi(2)};$$

while the local sign of $\rho_2(f) \otimes \chi_2$ is given by

$$\varepsilon(\rho_2(f) \otimes \chi_2, \psi, dx) = \chi_2(-1).$$

Proof. The proof is almost the same as the odd case, and can easily be checked. □

Remark 11. The computation in the non-dihedral supercuspidal case is straight forward, since one can compute for each elliptic curve the local root number and use the relations between the curves under twisting. In Table 4.2 we list the local root numbers of each curve at 2.

Curve	Root Number	Curve	Root Number
$E_1^{(1)}$	1	$E_2^{(1)}$	1
$E_1^{(-1)}$	1	$E_2^{(-1)}$	1
$E_1^{(2)}$	−1	$E_2^{(2)}$	−1
$E_1^{(-2)}$	1	$E_2^{(-2)}$	1

TABLE 4.2. Root numbers in the non-dihedral supercuspidal case.

Remark 12. Contrary to the odd case, where the variation of the sign under twisting allows to compute the exact type of a representation, for $p = 2$ this is no longer the case. The only cases that cannot be distinguished are those of a Principal Series representation and a dihedral supercuspidal representation induced from the quadratic extension $\mathbb{Q}_2(\sqrt{2})$, in the case they are not twists of lower level so in particular $\mathrm{val}_2(N)$ is even and greater than 8. This is so by Theorem 4.2 since, following the previous notation, $\chi_{-1}(-1) = -1$, $\chi_{-2}(-1) = -1$ and $\chi_2(-1) = 1$.

Example 4.3. Consider the curve $E768b$ in Cremona's notation. Its quadratic twists by χ_{-1}, χ_2 and χ_{-2} are the curves $E768h$, $E768d$ and $E768f$ respectively (an online table of the curves and their first twists can be found in [Tor04]). Looking at the local root number at 2 in such tables, we see that they change by −1, 1 and −1 respectively so we are in the condition of the last Remark. To see whether we are in the Principal Series case or in the Supercuspidal one, we can search for the curve in the quaternion algebra ramified at 2 and infinity. This can be done by choosing the correct order in such algebra (see [HPS89]) and constructing ideal representatives for it in order to compute the Brandt matrices (see [PS10] for an effective way to construct the ideals). It turns out that all four curves appear in such algebra (although it is clear that if one does the others do as well), hence the component at 2 of all of them is supercuspidal.

Example 4.4. Consider the elliptic curve $E3840c$ in Cremona's notation. Its quadratic twists by χ_{-1}, χ_2 and χ_{-2} are the curves $E3840w$, $E3840n$ and $E3840t$ respectively. Their local root numbers at 2 show that we are again in the condition of Remark 12. However, this curve does not show up in the quaternion algebra ramified at 2, hence it is Principal Series at 2.

The last two examples show that both cases actually do occur, as was expected, and in particular proves that by only considering the variation of the local root number under twisting is not enough to determine the local factor at the prime 2.

5. Some remarks on Hilbert modular forms

Although in all the previous sections we worked only with classical modular forms, the correspondence between Weil-Deligne representations and Hilbert modular forms works just as well. The properties/existence of the local root numbers do also, so we could just started with a Hilbert modular form over a totally positive number field K in the first case. All the local computations are the same but the problem is that there might be no global Hecke character χ to twist by. This comes from the fact that a totally positive number field K (other than $\mathbb{Q}$) does have totally positive units different from 1, which does not happen over $\mathbb{Q}$, so for a character $\chi_{\mathfrak{p}}$ to be well defined, it needs to be trivial at totally positive units.

To overcome this problem, starting from the character $\chi_{\mathfrak{p}}$, we chose an auxiliary prime $\mathfrak{q}$ which does not divide the level of the Hilbert modular form, and such that $\chi_{\mathfrak{p}}\chi_{\mathfrak{q}}$ is trivial on totally positive units. Such primes always exist, since if $\chi_{\mathfrak{p}}$ is non-trivial on units, we can chose a basis $\{\nu_1, \dots, \nu_r\}$ of the totally positive units such that $\chi_{\mathfrak{p}}(\nu_1) = -1$ and $\chi_{\mathfrak{p}}(\nu_i) = 1$ for all $2 \leq i \leq r$. This is equivalent to saying that our prime $\mathfrak{p}$ is inert in the (ring of integers of the) quadratic extension $K[\sqrt{\nu_1}]$ and splits in the extension $K[\sqrt{\nu_i}]$, for $2 \leq i \leq r$. Then any prime $\mathfrak{q}$ with the same splitting behavior satisfies our hypothesis (and they always exist by Tchebotarev).

The behavior of twisting by $\chi_{\mathfrak{q}}$ is computed using the first case of Theorem 3.2, where we need to replace the quadratic symbol by $\chi_{\mathfrak{q}}(\pi)$ for π a local uniformizer of $K_{\mathfrak{p}}$. In this way, we can extract the information needed to compute the local factor at $\mathfrak{p}$.

Example 5.1. Let $K = \mathbb{Q}[\sqrt{5}]$. The group of totally positive units is generated by the element $\langle \frac{3+\sqrt{5}}{2} \rangle$. Let $\mathfrak{P}_{31} = (6 + \sqrt{5})$ be a prime ideal of norm 31 in K. In this case, $\chi_{\mathfrak{P}_{31}}\left(\frac{3+\sqrt{5}}{2}\right) = \chi_{31}(14) = 1$, so no auxiliary prime is needed and everything works as over $\mathbb{Q}$. For example, consider the space of weight $(2, 2)$ and level $\mathfrak{P}_{31}^2$ Hilbert modular forms. This can be computed using Dembélé algorithm (see [Dem05]) which is implemented in Magma. It turns out that there are 3 forms having $\mathbb{Q}$ as coefficient field. One of them is the twist of the elliptic curve of conductor $\mathfrak{P}_{31}$ given in [Dem08], Example 1, hence both curves are Steinberg at the prime $\mathfrak{P}_{31}$.

The other two curves are one twist of the other, and have Weierstrass equation:

$$E : y^2 + y = x^3 - x^2 - \left(\frac{7 + 3\sqrt{5}}{2}\right) x,$$

(this equation was computed by Lassina Dembélé for us) and its twist has global minimal model:

$$E_{\mathfrak{P}_{31}} : y^2 + \sqrt{5}y = x^3 - \left(\frac{1 - \sqrt{5}}{2}\right) x^2 - (639 + 285\sqrt{5})x - \left(\frac{4733 + 2113\sqrt{5}}{2}\right).$$

If we compute the sign of their L-series, we see that E has sign -1 while $E_{\mathfrak{P}_{31}}$ has sign $+1$ (actually using SAGE, [sag], one can check that E has rank 1 while $E_{\mathfrak{P}_{31}}$ has rank 0). Since $\left(\frac{-1}{31}\right) = -1$, we conclude that E is principal series at $\mathfrak{P}_{31}$.

Example 5.2. Let K be the totally real field of discriminant 257 obtained by adding to $\mathbb{Q}$ a root of the polynomial $t^3 + 2t^2 - 3t - 1$ (it is not a Galois extension, since the discriminant is a non-square). The units in the ring of integer are generated by $\langle t, t - 1 \rangle$, with signatures $(-1, -1, 1)$ both of them. The class number of K is 1, but the ray class number is 2. The totally positive units are generated by $\langle t(t - 1), t^2 \rangle$. Let $\mathfrak{P}_3 = \langle t + 1 \rangle$. It is a non-principal ideal for the ray class group (the sign of $t + 1$ under the three embeddings are $(+, -, +)$). Also, $\chi_{\mathfrak{P}_3}(x(x - 1)) = \left(\frac{2}{3}\right) = -1$, so it does not define a global character. The space of parallel weight 2 forms of level $\mathfrak{P}_3^2$ has dimension 2, and it splits into two eigenforms with rational coefficients (this space was computed to us by John Voight). One form is a twist of the other one by the narrow class character. One of the forms correspond to the elliptic curve

$$E : y^2 + (t^2 + 3t + 3)xy + y = x^3 + (t^2 + t - 1)x^2 + (4t^2 + 19t + 4)x + (4t^2 + 9t + 2).$$

One way to prove that the curve is modular for the above modular form, is to notice that the curve has torsion $\mathbb{Z}/6\mathbb{Z}$, so it is modular and since there are no other forms in the space, it matches one of the two forms in our space (this argument and the equation for the elliptic curve is due to John Voight). Now we search for a prime ideal $\mathfrak{P}$ such that it has the same sign in the totally positive units as $\mathfrak{P}_3$. A small search reveals that the prime ideal $\mathfrak{P}_7 = \langle 2t + 1 \rangle$ satisfies the required property, since $\chi_{\mathfrak{P}_7}(t(t - 1)) = \left(\frac{-1}{7}\right) = -1$. So we can compute the twist of E by the ideal $\mathfrak{P}_3 \mathfrak{P}_7 = \langle 2t^2 + 3t + 1 \rangle$. It is given by the equation (in global minimal model)

$$E_{\mathfrak{P}_3 \mathfrak{P}_7} : y^2 + (t + 1)xy = x^3 + (t + 1)x^2 +$$
$$(-863t^2 - 1791t - 442)x + (18919t^2 + 40953t + 10179).$$

Its conductor has valuation 1 at the prime ideal $\mathfrak{P}_3$, hence both modular forms of level $\mathfrak{P}_3^2$ are Steinberg at $\mathfrak{P}_3$. Note that in this case, $\mathfrak{p}_3^2$ is the smallest conductor of any twist of the curve and has valuation 2 at $\mathfrak{p}_3$ (so it is not in any table precomputed before).

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