PREIMAGES FOR THE SHIMURA MAP ON HILBERT MODULAR FORMS

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ABSTRACT. In this article we explain a method to construct preimages of the Shimura correspondence for Hilbert modular forms of square-free level. The method relies in the ideas presented for the rational case in [PT07], which generalize the original method of Gross (see [Gro87]).

INTRODUCTION

First of all, we warn the reader that this is a survey article, so almost no proofs are given. Most of the general results are stated with a reference to the literature, while the details and missing proofs of the method itself will appear in an article of the second author. To our knowledge the only construction of preimages for the Shimura map on Hilbert modular forms is given in [Xue11], which only works when the level involved is a power of a prime ideal, and follows Gross' original construction.

We start this article by recalling the definition of a Hilbert modular form. We present the classical definition in the first section, where we state the basic facts we will need along the article.

In the second section, we give the adelic definition of Hilbert modular forms, and state the relation between Hilbert automorphic forms and r-tuples of classical Hilbert modular forms, where r is the ray class number of the base field (see Theorem 2.2).

Some good references for the theory of Hilbert modular forms are Garrett's book [Gar90] and Van Der Geer's book [vdG88] (see also Gebhardt's dissertation [Geb09]).

In the third section, we recall how given a totally definite quaternion algebra B and an Eichler order R in it, one can define Hecke operators acting on the space $M(R)$ generated by left ideal classes representatives for R. We state the main properties of these operators and show that, away from the discriminant of the order, they satisfy the same relations as the Hecke operators on Hilbert modular forms. We state a Jacquet-Langlands-type result that assures that the new subspace of Hilbert modular forms of a given level can be embedded into $M(R)$, if we choose B and R appropriately (see Theorem 3.2).

In the fourth section, we recall the definition of half-integral weight Hilbert modular forms, following [Shi87]. We also state the main properties of the Hecke operators acting on them, and at the end of the section we recall Shimura's Theorem relating the space of Hilbert modular forms of parallel weight 3/2 with the space of Hilbert modular forms of parallel weight 2 (see Theorem 4.2).

In the fifth section, we show how certain ternary theta series associated to the left ideal classes of a given order R can be used to produce Hilbert modular forms of parallel weight

$3/2$. This construction actually gives a Hecke linear map θ from the space $M(R)$ to the space of Hilbert modular forms of parallel weight $3/2$ (see Theorem 5.2).

In the sixth section we show how the results of the previous sections can be used to construct preimages of the Shimura map, at last in the case where the level of the modular form is square-free. We state some conjectures and remarks regarding this construction, which include a relation between the Fourier coefficients of the preimages and central values of twisted L-functions (see Conjecture 6.2).

In the final section we consider the space of Hilbert modular cusp forms over $F = \mathbb{Q}[\sqrt{5}]$, with level $(6 + \sqrt{5})$. This space is 1-dimensional, and it is spanned by a newform that corresponds to an elliptic E curve over F. We apply our method to this cusp form to construct a parallel weight $3/2$ modular form in Shimura correspondence with it, and compare its zero coefficients with the ranks of imaginary quadratic twists of E.

1. HILBERT MODULAR FORMS

Let F be a totally real number field of degree d over $\mathbb{Q}$. We let $\mathbf{a}$ denote the set of all embeddings $\tau : F \hookrightarrow \mathbb{R}$, and for $\xi \in F$ and $\tau \in \mathbf{a}$, we denote $\tau(\xi) = \xi_\tau$. We let F^+ denote the set of $\xi \in F$ such that $\xi_\tau > 0$ for all $\tau \in \mathbf{a}$.

Let G denote the group scheme SL_2 and $\tilde{G}$ the group scheme GL_2. Also, let

$$\tilde{G}^+ = \{\gamma \in \tilde{G}(F) : \det \gamma \in F^+\}.$$

Let $\mathcal{H}$ denote the Poincaré upper-half plane. Then $\mathrm{GL}_2^+(\mathbb{R})^\mathbf{a}$ acts on $\mathcal{H}^\mathbf{a}$ component-wise, and $\tilde{G}^+$ also acts on $\mathcal{H}^\mathbf{a}$ via the natural embedding $\tilde{G}^+(F) \hookrightarrow \mathrm{GL}_2^+(\mathbb{R})^\mathbf{a}$.

If $\gamma \in \mathrm{GL}_2^+(\mathbb{R})^\mathbf{a}$, with $\gamma_\tau = \left(\begin{smallmatrix} a_\tau & b_\tau \\ c_\tau & d_\tau \end{smallmatrix}\right)$, we let $j(\gamma, z)$ denote the cocycle

$$j(\gamma, z) = \prod_{\tau \in \mathbf{a}} (c_\tau z_\tau + d_\tau).$$

Again, this also makes sense for $\gamma \in \tilde{G}^+$. Given a function $g : \mathcal{H}^\mathbf{a} \to \mathbb{C}$ and $\gamma \in \tilde{G}^+$, we denote by $g|\gamma$ the function given by $(g|\gamma)(z) = N_{F/\mathbb{Q}}(\det \gamma) j(\gamma, z)^{-2} g(\gamma z)$. We will consider only forms of weight $\mathbf{2} = (2, \ldots, 2)$ (also called of parallel weight 2) for simplicity, but everything can be done for arbitrary weights.

Let $\tilde{\Gamma} \subseteq \tilde{G}^+$ be a congruence subgroup (we will consider only certain congruence subgroups defined below, see [Shi78, page 639] for a general definition). The space of Hilbert modular forms of weight $\mathbf{2}$ for $\tilde{\Gamma}$, which we denote by $M_\mathbf{2}(\tilde{\Gamma})$, is the space of holomorphic functions $g : \mathcal{H}^\mathbf{a} \to \mathbb{C}$ such that

- $g|\gamma = g \quad \forall \gamma \in \tilde{\Gamma}$.
- If $d = 1$, $g(z)$ is holomorphic at the cusps.

The holomorphicity condition at the cusps is automatic for totally real fields other than $\mathbb{Q}$. This is the so called Koecher principle (see [Gar90] for a proof).

Let $\mathcal{O}$ be the ring of integers of F. Given fractional ideals $\mathfrak{r}, \mathfrak{n}$ such that $\mathfrak{r}\mathfrak{n} \subseteq \mathcal{O}$, we will be mainly interested in the groups

$$\tilde{\Gamma}[\mathfrak{r}, \mathfrak{n}] = \left\{\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \tilde{G}^+ : a, d \in \mathcal{O}, b \in \mathfrak{r}^{-1}, c \in \mathfrak{r}\mathfrak{n}, \det \gamma \in \mathcal{O}^\times\right\},$$

$$\Gamma[\mathfrak{r}, \mathfrak{n}] = G(F) \cap \tilde{\Gamma}[\mathfrak{r}, \mathfrak{n}].$$

Note that the group $\tilde{\Gamma}[\mathfrak{r}, \mathfrak{n}]$ is conjugate over $\tilde{G}^+$ to $\tilde{\Gamma}[\mathfrak{r}, \xi\mathfrak{n}]$ for any $\xi \in F^+$.

Let $e_F : F \times \mathcal{H}^{\mathbf{a}} \to \mathbb{C}$ be the exponential function given by

$$e_F(\xi, z) = \exp\left(2\pi i \sum_{\tau \in \mathbf{a}} \xi_\tau z_\tau\right).$$

For a fractional ideal $\mathfrak{a}$, let $\mathfrak{a}^+ = \mathfrak{a} \cap F^+$, and denote by $\mathfrak{a}^\vee$ its dual with respect to the trace form. If $g \in M_{\mathbf{2}}(\tilde{\Gamma}[\mathfrak{r}, \mathfrak{n}])$, since $g(z + \xi) = g(z)$ for every $\xi \in \mathfrak{r}^{-1}$ (where $z + \xi = (z_\tau + \xi_\tau)_\tau \in \mathcal{H}^{\mathbf{a}}$), the form g has a Fourier series expansion

$$g(z) = \sum_{\xi \in ((\mathfrak{r}^{-1})^\vee)^+ \cup \{0\}} c(\xi, g) e_F(\xi, z).$$

We say that g is *cuspidal* if $c(0, g|\gamma) = 0$ for all $\gamma \in \tilde{G}^+$. The subspace of such g is denoted by $S_{\mathbf{2}}(\tilde{\Gamma}[\mathfrak{r}, \mathfrak{n}])$.

So far Hilbert modular forms are just a natural generalization of classical modular forms. This is indeed the case if F has narrow class number 1, but it becomes more technical in general due to the fact that the Hecke operators do not preserve "levels" defined in this way when the narrow class group $Cl^+(F)$ of F is not trivial, as we will see below.

2. AUTOMORPHIC HILBERT MODULAR FORMS

Let $F_{\mathbb{A}}$ be the ring of adeles of F, and denote by F_∞ and $F_{\mathbf{f}}$ its archimedean and non-archimedean parts respectively, so $F_{\mathbb{A}}^\times = F_\infty \times F_{\mathbf{f}}$. Let $F_{\mathbb{A}}^\times$ be the group of ideles of F.

For a fractional ideal $\mathfrak{a}$, denote by $[\mathfrak{a}]$ its class in $Cl^+(F)$. Take $\mathfrak{b}_1, \ldots, \mathfrak{b}_r \subseteq \mathcal{O}$ representatives for $Cl^+(F)$, which we fix from now on. Let $t_l \in F_{\mathbf{f}}^\times$ be such that the ideal corresponding to t_l is $\mathfrak{b}_l$. Denote by $\hat{\mathcal{O}}^\times = \prod_{\mathfrak{p}} \mathcal{O}_{\mathfrak{p}}^\times$, where the subscript $\mathfrak{p}$ denotes the completion at $\mathfrak{p}$, and let $F_\infty^+ \subseteq F_\infty^\times$ denote the connected component of the identity. We have a decomposition

$$(1) \qquad F_{\mathbb{A}}^\times = \bigsqcup_{l=1}^r F^\times t_l \, (F_\infty^+ \times \hat{\mathcal{O}}^\times).$$

Let $G(F_{\mathbf{f}})$ be the archimedean part of $G(F_{\mathbb{A}})$. Strong approximation for G asserts that $G(F) \, \mathrm{SL}_2(\mathbb{R})^{\mathbf{a}}$ is dense in $G(\mathbb{A}_F)$. This implies that if K is an open subgroup of $\tilde{G}(F_{\mathbf{f}})$, then the natural map

$$\tilde{G}(F) \backslash \tilde{G}(F_{\mathbb{A}}) / (\mathrm{GL}_2^+(\mathbb{R})^{\mathbf{a}} \times K) \longrightarrow F^\times \backslash F_{\mathbb{A}}^\times / (F_\infty^+ \times \det(K))$$

is a bijection. This fact together with decomposition (1) gives the following theorem.

Theorem 2.1. *Let K be an open subgroup of $\tilde{G}(F_{\mathbf{f}})$. If $\det(K) = \hat{\mathcal{O}}^\times$, then*

$$\tilde{G}(F_{\mathbb{A}}) = \bigsqcup_{l=1}^r \tilde{G}(F) \left(\begin{smallmatrix} 1 & 0 \\ 0 & t_l \end{smallmatrix}\right) (\mathrm{GL}_2^+(\mathbb{R})^{\mathbf{a}} \times K).$$

Example. Given $\mathfrak{c} \subseteq \mathcal{O}$, an important case is the open subgroup $K_0(\mathfrak{c}) \subseteq \tilde{G}(\hat{\mathcal{O}})$ given by

$$K_0(\mathfrak{c}) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \prod_{\mathfrak{p}} \mathrm{GL}_2(\mathcal{O}_{\mathfrak{p}}) : c_{\mathfrak{p}} \in \mathfrak{c}_{\mathfrak{p}} \, \forall \, \mathfrak{p} \right\}.$$

The following definition generalizes the notion of automorphic forms over $\mathbb{Q}$ to totally real fields.

Definition. *A map* $\phi : \tilde{G}(F_{\mathbb{A}}) \to \mathbb{C}$ *is a* Hilbert automorphic form *of weight* **2** *for* $K_0(\mathfrak{c})$ *if it satisfies*

(H.1) $\phi(\gamma x) = \phi(x)$ *for all* $\gamma \in \tilde{G}(F)$.

(H.2) *Consider the diagonal embedding* $F_{\infty}^{+} \hookrightarrow \mathrm{GL}_2^{+}(\mathbb{R})^{\mathbf{a}}$. *Then,* $\phi(tx) = \phi(x)$ *for all* $t \in F_{\infty}^{+}$.

(H.3) *For* $\theta \in \mathbb{R}^{\mathbf{a}}$, *let* $r(\theta) = \begin{pmatrix} \cos(\theta_{\tau}) & -\sin(\theta_{\tau}) \\ \sin(\theta_{\tau}) & \cos(\theta_{\tau}) \end{pmatrix}_{\tau} \in \mathrm{SO}_2(\mathbb{R})^{\mathbf{a}}$. *Then,*

$$\phi(xr(\theta)k) = e^{-2i \sum_{\tau \in \mathbf{a}} \theta_{\tau}} \phi(x), \quad \forall r(\theta) \in \mathrm{SO}_2(\mathbb{R})^{\mathbf{a}}, k \in K_0(\mathfrak{c}).$$

(H.4) ϕ *is "slowly increasing".*

(H.5) *As a function of* $\mathrm{GL}_2(\mathbb{R})^{\mathbf{a}}$, ϕ *is smooth.*

(H.6) ϕ *is an eigenfunction of the Casimir operator* Δ_{τ}, *with eigenvalue* 0, *for all* $\tau \in \mathbf{a}$.

We say that ϕ *is* cuspidal *if it also satisfies*

(H.7) $\int_{F_{\mathbb{A}}/F} \phi\left(\left(\begin{smallmatrix} 1 & y \\ 0 & 1 \end{smallmatrix}\right) x \, dy\right) = 0$ *for almost every* $y \in \tilde{G}(F_{\mathbb{A}})$.

Implicit in (H.2) and (H.3) lays the fact that we only consider forms with trivial character, which are enough for our purposes. For a precise statement of (H.4) and (H.6), we refer to [Gel75, Chapter 3]. We remark that if ϕ is cuspidal, then $|\phi| \in L^2(F_{\mathbb{A}}^{\times} \tilde{G}(F) \backslash \tilde{G}(F_{\mathbb{A}}))$.

Denote $\mathbf{i} = (i, \ldots, i) \in \mathcal{H}^{\mathbf{a}}$. Then $\mathrm{GL}_2^{+}(\mathbb{R})^{\mathbf{a}}$ acts transitively on $\mathcal{H}^{\mathbf{a}}$, with the stabilizer of $\mathbf{i}$ being $\mathrm{SO}_2(\mathbb{R})^{\mathbf{a}}$. Using this it is not hard to prove part of the following result (we refer to [Gel75, Proposition 3.1] or [Geb09, Theorem 2.3.7]).

Theorem 2.2. *Let* ϕ *be a Hilbert automorphic form of weight* **2** *for* $K_0(\mathfrak{c})$. *For* $l = 1, \ldots, r$ *let* $g_l : \mathcal{H}^{\mathbf{a}} \to \mathbb{C}$ *be given by*

$$g_l(z) = j(x_{\infty}, \mathbf{i})^2 \phi\left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & t_l \end{smallmatrix}\right) x_{\infty}\right),$$

where $x_{\infty} \in \mathrm{GL}_2^{+}(\mathbb{R})^{\mathbf{a}}$ *is any element satisfying* $x_{\infty}\mathbf{i} = z$. *Then* $g_l \in M_{\mathbf{2}}(\tilde{\Gamma}[\mathfrak{b}_l, \mathfrak{c}])$. *Furthermore,* g_l *is a cusp form if* ϕ *is a cusp form.*

Conversely, given $g_l \in M_{\mathbf{2}}(\tilde{\Gamma}[\mathfrak{b}_l, \mathfrak{c}])$ *for* $l = 1, \ldots, r$, *define* $\phi : \tilde{G}(F_{\mathbb{A}}) \to \mathbb{C}$ *by*

$$\phi(\gamma \left(\begin{smallmatrix} 1 & 0 \\ 0 & t_l \end{smallmatrix}\right) x_{\infty} k_0) = j(x_{\infty}, \mathbf{i})^{-2} g_l(x_{\infty}\mathbf{i}), \text{ for } \gamma \in \tilde{G}(F), x_{\infty} \in \mathrm{GL}_2^{+}(\mathbb{R})^{\mathbf{a}}, k_0 \in K_0(\mathfrak{c}).$$

Then ϕ *is an automorphic Hilbert modular form of weight* **2** *for* $K_0(\mathfrak{c})$. *Furthermore,* ϕ *is a cusp form if every* g_l *is a cusp form.*

If we denote

$$M_{\mathbf{2}}(\mathfrak{c}) = \prod_{l=1}^{r} M_{\mathbf{2}}(\tilde{\Gamma}[\mathfrak{b}_l, \mathfrak{c}]), \quad S_{\mathbf{2}}(\mathfrak{c}) = \prod_{l=1}^{r} S_{\mathbf{2}}(\tilde{\Gamma}[\mathfrak{b}_l, \mathfrak{c}]),$$

then this theorem says there is a bijection between $M_{\mathbf{2}}(\mathfrak{c})$ and the space of automorphic Hilbert modular forms for $K_0(\mathfrak{c})$ (which depends on the particular choice of representatives $\mathfrak{b}_l$), sending cusp forms to cusp forms. In particular, if $r = 1$ we have a bijection between Hilbert modular forms for $\tilde{\Gamma}[\mathcal{O}, \mathfrak{c}]$ and automorphic Hilbert modular forms for $K_0(\mathfrak{c})$, as in the rational case.

To an element g of $M_{\mathbf{2}}(\mathfrak{c})$ we can associate a "q-expansion" indexed by integral ideals. Letting $\left(\begin{smallmatrix} \epsilon & 0 \\ 0 & 1 \end{smallmatrix}\right)$ act on g_l, with $\epsilon \in \mathcal{O}_+^\times = \mathcal{O}^\times \cap F^+$, it is easy to see that $c(\xi, g_l)$ depends only on $\xi \mathcal{O}$. Then given $0 \neq \mathfrak{m} \subseteq \mathcal{O}$, we let

$$c(\mathfrak{m}, g) = c(\xi, g_l), \text{ for } \xi \in \mathfrak{b}_l^+ \text{ such that } \mathfrak{m} = \xi \mathfrak{b}_l^{-1}.$$

These Fourier coefficients can be obtained in terms of the automorphic form corresponding to g, and do not depend on the representatives $\mathfrak{b}_l$ chosen.

The action of the Hecke operators $T_{\mathfrak{p}}$ on $M_{\mathbf{2}}(\mathfrak{c})$ is naturally defined in the adelic setting, for which we refer to [Shi78]. This action is such that if $g_l \in M_{\mathbf{2}}(\tilde{\Gamma}[\mathfrak{b}_l, \mathfrak{c}])$, then $T_{\mathfrak{p}}(g_l) \in M_{\mathbf{2}}(\tilde{\Gamma}[\mathfrak{b}_{l'}, \mathfrak{c}])$, where l' is such that $[\mathfrak{p}\mathfrak{b}_l] = [\mathfrak{b}_{l'}]$. Note in particular that the Hecke operators do not preserve the spaces $M_{\mathbf{2}}(\tilde{\Gamma}[\mathfrak{b}_l, \mathfrak{c}])$, which explains why we need to consider r-tuples as above.

We denote by $\mathbb{T}_0$ the algebra of operators generated by the Hecke operators $T_{\mathfrak{p}}$ with $\mathfrak{p} \nmid \mathfrak{c}$. These are self-adjoint with respect to the Petersson inner product on $S_{\mathbf{2}}(\mathfrak{c})$, which in the automorphic setting is given by the inner product of $L^2(F_{\mathbb{A}}^\times \tilde{G}(F) \backslash \tilde{G}(F_{\mathbb{A}}))$.

The subspace of oldforms of $S_{\mathbf{2}}(\mathfrak{c})$, which we define in the adelic setting, is the space generated by the functions $x \mapsto \phi(x \left(\begin{smallmatrix} t^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right))$, with ϕ a cusp form of level $\mathfrak{b}$ with $\mathfrak{b} \mid \mathfrak{c}, \mathfrak{b} \neq \mathfrak{c}$, and $t \in F_{\mathbb{A}}^\times$ such that the ideal corresponding to t divides $\mathfrak{b}^{-1}\mathfrak{c}$. This space is stable under the action of $\mathbb{T}_0$, and hence the same property holds for its orthogonal complement, which we denote by $S_{\mathbf{2}}^{new}(\mathfrak{c})$. The forms in $S_{\mathbf{2}}^{new}(\mathfrak{c})$ which are eigenfunctions for all the operators in $\mathbb{T}_0$ are called *newforms*.

The following is the multiplicity one theorem for (Hilbert) automorphic forms, due to Miyake (see [Miy71]).

Theorem 2.3. *Let g be a newform in $S_{\mathbf{2}}^{new}(\mathfrak{c})$. If $h \in S_{\mathbf{2}}(\mathfrak{c})$ is an eigenfunction for all the operators in $\mathbb{T}_0$, with the same eigenvalues as g, then h is a multiple of g.*

3. Quaternionic modular forms

We refer to [Vig80] for the definitions and basic results concerning the arithmetic of quaternion algebras.

Let B a totally ramified quaternion algebra over F, i.e. such that $B_\tau = B \otimes_F F_\tau$ is a ramified quaternion algebra over F_τ for every $\tau \in \mathbf{a}$. We fix an Eichler order R in B of discriminant $\mathfrak{D}$, and we let $\mathfrak{I}(R)$ denote the set of invertible left R-ideals.

Two ideals $I, J \in \mathfrak{I}(R)$ are equivalent if there exists $x \in B^\times$ such that $I = Jx$. We denote by we denote $[I]$ the equivalence class of I under this relation. We fix $I_1, \ldots, I_n \in \mathfrak{I}(R)$ representing the left ideals equivalence classes. The problem of calculating such representatives has been considered in [DV10] and in [PS12].

Let $M(R)$ denote the vector space over $\mathbb{C}$ spanned by the ideal classes $[I_1], \ldots, [I_n]$. On $M(R)$ we consider the inner product defined by

$$\langle [I_i], [I_j] \rangle = \begin{cases} 0, & i \neq j, \\ [R_r(I_i)^\times : \mathcal{O}^\times], & i = j. \end{cases}$$

We let $e_0 = \sum_{i=1}^n \frac{1}{\langle [I_i], [I_i] \rangle} [I_i] \in M(R)$, and we denote by $S(R)$ the orthogonal complement of $\mathbb{C}e_0$ in $M(R)$. Then $S(R) = \{v \in M(R) : \deg v = 0\}$, where $\deg : M(R) \to \mathbb{C}$ is the linear map defined by $\deg([I_i]) = 1$.

Let $\mathfrak{m} \subseteq \mathcal{O}$ be a non-zero ideal. For $I \in \mathfrak{I}(R)$ denote

$$t_{\mathfrak{m}}(I) = \{J \in \mathfrak{I}(R) : J \subseteq I, [I : J] = \mathfrak{m}^2\},$$

where $[I : J]$ denotes the index of J in I. Let $T_{\mathfrak{m}}$ be the $\mathfrak{m}$-th Hecke operator acting on $M(R)$, defined by

$$T_{\mathfrak{m}}([I]) = \sum_{J \in t_{\mathfrak{m}}(I)} [J].$$

This definition agrees with the action of Hecke operators on quaternionic modular forms given in [DV10].

There is an action of the group of fractional ideals on $M(R)$. Given a fractional ideal $\mathfrak{n}$ and $I \in \mathfrak{I}(R)$, the ideal $\mathfrak{n}I \in \mathfrak{I}(R)$ is locally given by $(\mathfrak{n}I)_{\mathfrak{p}} = R_{\mathfrak{p}}(x_{\mathfrak{p}}\xi_{\mathfrak{p}})$, if $\mathfrak{n}$ and I are given locally by $\mathfrak{n}_{\mathfrak{p}} = \mathcal{O}_{\mathfrak{p}}\xi_{\mathfrak{p}}$, and $I_{\mathfrak{p}} = R_{\mathfrak{p}}x_{\mathfrak{p}}$, respectively. This action, which is trivial if $Cl(F)$ is trivial, commutes with the action of the Hecke operators.

The Hecke operators are normal with respect to $\langle\,,\rangle$, but not necessarily self-adjoint if $Cl(F)$ is non trivial; more precisely, the adjoint of $T_{\mathfrak{m}}$ is $\mathfrak{m}^{-1}T_{\mathfrak{m}}$. Since e_0 is an eigenvector for $T_{\mathfrak{m}}$ and for $Cl(F)$ (it corresponds to the Eisenstein series), the Hecke operators preserve the space $S(R)$.

The Hecke operators on $M(R)$ satisfy the following equalities, which are also satisfied by the Hecke operators on Hilbert modular forms.

Proposition 3.1. *Let $k \geq 0$, and let $\mathfrak{p}$ be a prime ideal in $\mathcal{O}$ such that $\mathfrak{p} \nmid \mathfrak{D}$. The Hecke operators on $M(R)$ satisfy*

(1) $T_{\mathfrak{m}}T_{\mathfrak{n}} = T_{\mathfrak{mn}}$, *if* $(\mathfrak{m} : \mathfrak{n}) = 1$.
(2) $T_{\mathfrak{p}^{k+2}} = T_{\mathfrak{p}^{k+1}}T_{\mathfrak{p}} - N(\mathfrak{p})T_{\mathfrak{p}^k}$.
(3) $T_{\mathfrak{mp}} + N(\mathfrak{p})T_{\mathfrak{m}/\mathfrak{p}} = T_{\mathfrak{m}}T_{\mathfrak{p}}$, *if* $\mathfrak{p} \mid \mathfrak{m}$.

Proof. The proof is the same as the one given in [PT07, Proposition 1.3], where it is proved in the case $F = \mathbb{Q}$. $\qquad\square$

Since the Hecke operators are normal operators, $M(R)$ (and also $S(R)$) has a basis of simultaneous eigenvectors for the whole Hecke algebra. However, since the operators $T_{\mathfrak{p}^k}$ with $\mathfrak{p} \mid \mathfrak{D}$ do not satisfy the same relations as the Hecke operators on Hilbert modular forms, we will be interested only in the algebra of operators $\mathbb{T}_0$ generated by the $T_{\mathfrak{p}}$ for $\mathfrak{p} \nmid \mathfrak{D}$.

The following result is a generalization of the solution to the basis problem studied by Eichler, vastly generalized by Jacquet-Langlands (see for example [Hid81, Proposition 2.12]).

Theorem 3.2. *Let B be a quaternion algebra ramified at all the infinite places and at an even number of primes of $\mathcal{O}$, and let R be an Eichler order in B of discriminant $\mathfrak{c}$. Then there is an injective map of $\mathbb{T}_0$-modules $S_2^{new}(\mathfrak{c}) \hookrightarrow S(R)$.*

4. Hilbert modular forms of half-integral weight

We follow [Shi87] closely, though omitting and avoiding most of the technical details.

As in the rational case, half-integral weight Hilbert modular forms are defined in terms of the theta function

$$\theta(z) = \sum_{\xi \in \mathcal{O}} e_F(\xi^2, z/2).$$

We let $J(\gamma, z) = \left(\dfrac{\theta(\gamma z)}{\theta(z)} \right) j(\gamma, z)$ for $\gamma \in G(F)$.

Given an ideal $\mathfrak{b} \subseteq \mathcal{O}$ divisible by 4, let ψ be a Hecke character of F with conductor dividing $\mathfrak{b}$, and denote by ψ^* the character on ideals prime to $\mathfrak{b}$ induced by ψ. Denote by $\mathfrak{d}$ the different ideal of F. A Hilbert modular form of weight $\mathbf{3/2} = (3/2, \ldots, 3/2)$ (also called of parallel weight $3/2$), level $\mathfrak{b}$ and character ψ, is an holomorphic function f on $\mathcal{H}^{\mathbf{a}}$ satisfying

$$f(\gamma z) = \psi_{\mathfrak{b}}(a) J(\gamma, z) f(z)$$

for all $\gamma \in \Gamma[2^{-1}\mathfrak{d}, \mathfrak{b}]$, where we denote $\psi_{\mathfrak{b}} = \prod_{\mathfrak{p}|\mathfrak{b}} \psi_{\mathfrak{p}}$. The space of such f is denoted by $M_{\mathbf{3/2}}(\mathfrak{b}, \psi)$. It is trivial unless $\psi_{\infty}(-1) = (-1)^d$, where we denote $\psi_{\infty} = \prod_{\tau \in \mathbf{a}} \psi_{\tau}$.

This definition is slightly different from the definition used in the rational case, in which we used the cocycle given by $\left(\dfrac{\theta(\gamma z)}{\theta(z)} \right)^3$. However both definitions are equivalent. If $F = \mathbb{Q}$ and $f \in M_{\mathbf{3/2}}(N\mathbb{Z}, \psi)$, where ψ is the Hecke character induced by the Dirichlet character $\tilde{\psi}$, then $\tilde{f}(z) = f(2z)$ is a classical modular form of weight $3/2$, level N and character $\tilde{\psi} \cdot \left(\frac{-1}{*} \right)$.

Hecke operators in this setting are defined in [Shi87]. We use the classical notation, so that in this survey $T_{\mathfrak{p}^2}$ equals the operator $T_{\mathfrak{p}}$ considered in that article. In particular, as in the classical case, we have that $T_{\mathfrak{p}} = 0$.

The adelic counterpart of half-integral weight Hilbert modular forms is more involved than in the integral weight case, since the former correspond to functions on the metaplectic covering of $G(F_{\mathbb{A}})$. Note that working with unimodular matrices is enough, as opposed the integral weight case. This is due to the following two facts, which allow to define an action of the Hecke operators in the metaplectic covering of $G(F_{\mathbb{A}})$:

- The Hecke operators $T_{\mathfrak{n}}$ are zero if $\mathfrak{n}$ is not a perfect square.
- Instead of using the matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & \pi_{\mathfrak{p}}^2 \end{smallmatrix} \right)$ for defining $T_{\mathfrak{p}^2}$, we can use the unimodular matrix $\left(\begin{smallmatrix} 1/\pi_{\mathfrak{p}} & 0 \\ 0 & \pi_{\mathfrak{p}} \end{smallmatrix} \right)$, since these matrices are conjugate.

In particular, using strong approximation over $G(F_{\mathbb{A}})$ we get that that an automorphic form of half-integral weight corresponds to a single function on $\mathcal{H}^{\mathbf{a}}$, instead of the r-tuple of functions that we need to consider in the integral weight case.

Given $f \in M_{\mathbf{3/2}}(\mathfrak{b}, \psi)$, there is a Fourier series attached to each cusp in F. More precisely, for every $\xi \in F$ and every fractional ideal $\mathfrak{m}$ there is a complex number $\lambda(\xi, \mathfrak{m}, f)$, such that

$$f(z) = \sum_{\xi \in F} \lambda(\xi, \mathcal{O}, f) e_F(\xi, z/2) \qquad \text{(the } q\text{-expansion at } \mathcal{O}),$$

and satisfying

$$\lambda(\xi b^2, \mathfrak{m}, f) = N_{F/\mathbb{Q}}(b)\psi_{\infty}(b)\lambda(\xi, b\mathfrak{m}, f) \quad \forall b \in F^{\times},$$

$$\lambda(\xi, \mathfrak{m}, f) = 0, \quad \text{unless } \xi \in (\mathfrak{m}^{-2})^+ \cup \{0\}.$$

The Fourier coefficients $\lambda(\xi, \mathfrak{m}, f)$ for non-principal $\mathfrak{m}$ are harder to describe, but this can be done explicitly in the case of forms given by theta series, which we will consider below.

Proposition 4.1. *If $\mathfrak{p} \nmid \mathfrak{b}$, then for every $f \in M_{\mathbf{3/2}}(\mathfrak{b}, \psi)$,*

$$\lambda(\xi, \mathcal{O}, T_{\mathfrak{p}^2}(f)) = \lambda(\xi, \mathfrak{p}, f) + \psi^*(\mathfrak{p})N(\mathfrak{p})^{-1}\left(\frac{\xi}{\mathfrak{p}} \right)\lambda(\xi, \mathcal{O}, f) + \psi^*(\mathfrak{p}^2)N(\mathfrak{p})^{-1}\lambda(\xi, \mathfrak{p}^{-1}, f),$$

where $\left(\frac{*}{\mathfrak{p}}\right)$ denotes the quadratic residue symbol modulo $\mathfrak{p}$.

The following result is based on the generalization of the Shimura correspondence for Hilbert modular forms. It follows from [Shi87, Theorems 5.5 and 6.1].

For $\mathfrak{n} \subseteq \mathcal{O}$, we introduce a formal symbol $M(\mathfrak{n})$ such that $M(\mathfrak{nm}) = M(\mathfrak{n})M(\mathfrak{m})$ for all $\mathfrak{n}, \mathfrak{m} \subseteq \mathcal{O}$. These will play the role of $N(\mathfrak{n})^{-s}$ with complex s in the following formula.

Theorem 4.2. *For each $\xi \in \mathcal{O}^+$, there is a $\mathbb{T}_0$-linear map $\mathrm{Shim}_\xi : M_{\mathbf{3/2}}(\mathfrak{b}, \psi) \to M_{\mathbf{2}}(\mathfrak{b}/2, \psi^2)$, characterized by the following property. Write $\xi\mathcal{O} = \mathfrak{q}^2\mathfrak{r}$ with $\mathfrak{q}, \mathfrak{r} \subseteq \mathcal{O}$ and $\mathfrak{r}$ square-free, and let ϵ_ξ be the Hecke character corresponding to $F(\sqrt{\xi})/F$. Let $\mathfrak{P}$ be a set of primes of $\mathcal{O}$. If $0 \neq f \in M_{\mathbf{3/2}}(\mathfrak{b}, \psi)$ is a cusp form such that $T_{\mathfrak{p}^2}(f) = \omega_\mathfrak{p} f$ for all $\mathfrak{p} \in \mathfrak{P}$, then (formally)*

$$\sum_{\mathfrak{m} \subseteq \mathcal{O}} c(\mathfrak{m}, \mathrm{Shim}_\xi(f))M(\mathfrak{m})$$
$$= \prod_{\mathfrak{p} \notin \mathfrak{P}} \left(1 - \omega_\mathfrak{p} M(\mathfrak{p}) + \psi^*(\mathfrak{p})^2 N(\mathfrak{p})^{-1} M(\mathfrak{p}^2)\right)^{-1} \cdot \prod_{\mathfrak{p} \in \mathfrak{P}} \left(1 - (\psi^*\epsilon_\xi^*)(\mathfrak{p})N(\mathfrak{p})^{-1}M(\mathfrak{p})\right)$$
$$\cdot \sum_{(\mathfrak{m}:\mathfrak{P})=\mathcal{O}} \lambda(\xi, \mathfrak{q}^{-1}\mathfrak{m}, f)M(\mathfrak{m}).$$

Note that the formula implies that $\mathrm{Shim}_\xi(f) \neq 0$ if ξ is such that $\lambda(\xi, \mathfrak{q}^{-1}\mathcal{O}, f) \neq 0$.

5. Ternary theta series

Theta series of totally definite ternary quadratic forms can be used to construct Hilbert modular forms of weight $\mathbf{3/2}$. Since the number of variables of these quadratic forms is not even, they are not considered in the classical literature. Transformation formulas in this (and much more) generality are studied in [Shi87, Section 11], where the results needed to prove Proposition 5.1 below can be found.

Let B be as before. For $x \in B$, denote $\Delta(x) = \mathrm{Tr}(x)^2 - 4N(x)$. Then Δ determines an integral, totally negative definite quadratic form on $V = B/F$. For $I \in \mathfrak{J}(R)$, we consider $R_r(I)/\mathcal{O}$ as a lattice in V, which we denote by L_I.

¿From here on, let ψ be the Hecke character corresponding to the quadratic extension $F(\sqrt{-1})/F$. This quadratic character has conductor dividing $4\mathcal{O}$, and the corresponding ideal character satisfies $\psi^*(\mathfrak{p}) = (\frac{-1}{\mathfrak{p}})$ for $\mathfrak{p} \nmid 2$. By local class field theory, ψ satisfies the equality $\psi_\infty(-1) = (-1)^d$. Hence, the space $M_{\mathbf{3/2}}(4\mathfrak{D}, \psi)$ is not trivially zero.

Proposition 5.1. *Given $I \in \mathfrak{J}(R)$, let*

$$\vartheta_I(z) = \sum_{x \in L_I} e_F\left(-\Delta(x) \cdot \tfrac{z}{2}\right).$$

Then $\vartheta_I \in M_{\mathbf{3/2}}(4\mathfrak{D}, \psi)$. Furthermore, the Fourier coefficients of ϑ_I are given by

$$\lambda(\xi, \mathfrak{a}, \vartheta_I) = N(\mathfrak{a})^{-1} \cdot \#\{[x] \in \mathfrak{a}^{-1}L_I : -\Delta(x) = \xi\}.$$

Remark. The last equality follows from the definition of ϑ_I for $\mathfrak{a} = \mathcal{O}$, but it is not trivial for non-principal $\mathfrak{a}$. It is crucial for proving the Hecke linearity in Theorem 5.2 below.

For $\xi \in F^+ \cup \{0\}$, a fractional ideal $\mathfrak{a}$ and $I \in \mathfrak{I}(R)$, denote $a(\xi, \mathfrak{a}, [I]) = \#\{[x] \in \mathfrak{a}L_I : -\Delta(x) = \xi\}$. Let

$$e_\xi = \sum_{[J] \in Cl(R)} \frac{a(\xi, \mathcal{O}, [J])}{\langle [J], [J] \rangle} \cdot [J] \quad \in M(R).$$

This agrees with our previous definition of e_0.

Theorem 5.2. *Given $v \in M(R)$, let*

$$\theta(v)(z) = \sum_{\xi \in \mathcal{O}^+ \cup \{0\}} \langle e_\xi, v \rangle e_F\left(\xi, \tfrac{z}{2}\right) = \deg(v) + \sum_{\xi \in \mathcal{O}^+} \langle e_\xi, v \rangle e_F\left(\xi, \tfrac{z}{2}\right), \quad z \in \mathcal{H}^{\mathbf{a}}.$$

Then, $\theta(v) \in M_{\mathbf{3/2}}(4\mathfrak{D}, \psi)$. Furthermore, for every ideal $\mathfrak{m}$ prime to $4\mathfrak{D}$,
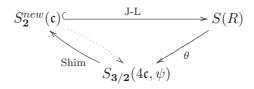
$$T_\mathfrak{m}(\theta(v)) = \theta(T_\mathfrak{m}(v)).$$

Proof. $\theta([I]) = \vartheta_I$, which implies the first claim. Hecke linearity follows as in [PT07], using Propositions 4.1 and 5.1. $\qquad\square$

6. Computing preimages

The main application of what has been explained in the previous sections is to construct preimages of the Shimura map; that is given a newform $g \in S_{\mathbf{2}}^{new}(\mathfrak{c})$, how can we construct $f \in S_{\mathbf{3/2}}(4\mathfrak{c}, \psi)$ such that $\mathrm{Shim}(f) = g$?

Here is a naive answer to the question: given a level $\mathfrak{c}$, let B be a quaternion algebra ramified at the infinite places of F and at an even number of primes dividing $\mathfrak{c}$, and let R be an Eichler order in B of discriminant $\mathfrak{c}$. We have the following diagram of $\mathbb{T}_0$-linear maps.

$$
\begin{array}{ccc}
S_{\mathbf{2}}^{new}(\mathfrak{c}) & \xhookrightarrow{\quad\text{J-L}\quad} & S(R) \\
& \searrow\!\!\!\!\!\!\cdots & \downarrow{\scriptstyle\theta} \\
{\scriptstyle\text{Shim}}\quad & S_{\mathbf{3/2}}(4\mathfrak{c}, \psi) &
\end{array}
$$

So given $g \in S_{\mathbf{2}}^{new}(\mathfrak{c})$, there is a $\mathbb{T}_0$-eigenvector $v_g \in S(R)$ with the same eigenvalues as g. Then *if $\theta(v_g) \neq 0$*, then $\theta(v_g)$ is a weight $\mathbf{3/2}$ form whose image under the Shimura map has the same eigenvalues as g, and so by multiplicity one (Theorem 2.3) it corresponds to g itself. The main issue is to know whether there exists a quaternion algebra B and an Eichler order R such that $\theta(v_g) \neq 0$.

The following two conjectures are just naive generalizations to Hilbert modular forms of results due to Böcherer and Schulze-Pillot (see [BSP90]) in the case $\mathfrak{c}$ is odd and square free.

Conjecture 6.1. *The form $\theta(v_g)$ is non zero if and only if $L(g, 1) \neq 0$ and the quaternion algebra B ramifies at all primes $\mathfrak{p}$ dividing $\mathfrak{c}$ where the Atkin-Lehner involution $w_\mathfrak{p}$ acts on g with eigenvalue -1.*

If this conjecture is true, then it implies that the choice of the quaternion algebra B and the Eichler order R is unique, up to isomorphism. The relation between Fourier coefficients and central values of twisted L-series is given by the following.

Conjecture 6.2. *Let $g \in S_{\mathbf{2}}(\mathfrak{c}, \psi^2)$ be a newform, and $f = \theta(v_g) \in M_{\mathbf{3/2}}(4\mathfrak{c}, \psi)$ be a non-zero cusp form. Let $\xi \in \mathcal{O}^+$ be such that $-\xi$ is a fundamental discriminant. Let ϵ_ξ be the Hecke character corresponding to $F(\sqrt{-\xi})/F$. Then*

$$|\lambda(\xi, \mathcal{O}, f)|^2 = \kappa L(g, \epsilon_\xi, 1) \prod_{\mathfrak{p}|\mathfrak{c}} (c(\mathfrak{p}, g) - \epsilon_\xi(\mathfrak{p})),$$

where κ is a non-zero constant, and $L(g, \epsilon_\xi, s)$ is the twist of the L-series of g by ϵ_ξ.

Remark. This conjecture is true for classical forms (see [BSP90, page 377]), and is true for Hilbert modular forms when the level $\mathfrak{c}$ is a prime power (see [Xue11]), or for forms which have multiplicity one for the Hecke algebra $\mathbb{T}_0$ (see [Shi93, Theorem 3.6]). Unfortunately, our construction does not satisfy this last hypothesis.

In particular, under the above assumptions, this conjecture states that $L(g, \epsilon_\xi, 1) = 0$ if and only if $\lambda(\xi, \mathcal{O}, f) = 0$, if we choose ξ such that the product on the right hand side is non-zero. This sort of results are important for obtaining (under the Birch and Swinnerton-Dyer conjecture) information about the rank of twists of elliptic curves, as in the congruent number problem.

Remark. The hypothesis on $\mathfrak{c}$ being square-free was used in two different contexts. First of all, not all modular forms appear in definite quaternion algebras (one needs the automorphic representation to be non principal series at the primes ramified in B). So for using Jacquet-Langlands, one needs some information of the parallel weight 2 form g one starts with.

The second place where we used the hypothesis was when we chose R such that $\theta(v_g) \neq 0$. If the primes where the Atkin-Lehner involutions acting on g have eigenvalue -1 appear to the first power in $\mathfrak{c}$, then one can take R to be an Eichler order in the definite quaternion algebra ramified at such primes, and the same results should hold. For a general form g, the theory is more involved. See for example [PT07] for the case $F = \mathbb{Q}$ and $\mathfrak{c} = p^2\mathbb{Z}$.

7. An example

We let $F = \mathbb{Q}(\sqrt{5})$, which has trivial narrow class group, and we denote $\omega = \frac{1+\sqrt{5}}{2}$. We let E be the elliptic curve over F given by

$$E : \quad y^2 + xy + \omega y = x^3 - (1 + \omega)x^2.$$

This curve has prime conductor, equal to $\mathfrak{c} = (5 + 2\omega)$, so in particular Conjecture 6.1 and Conjecture 6.2 are known in this case. The space $M_{\mathbf{2}}(\mathfrak{c})$ has dimension 2, and it is generated by an Eisenstein series and a newform g which corresponds to E. Its first eigenvalues are given in [Dem05]; we only state that $c(\mathfrak{c}, g) = -1$. We choose B to be the unramified totally definite algebra over F, i.e. the algebra generated by $1, i, j, k$, where $i^2 = j^2 = -1, ij = k = -ji$. If R is an Eichler order of discriminant $\mathfrak{c}$ in B, then Theorem 3.2 asserts that there exists $v \in S(R)$ which is an eigenvector for $\mathbb{T}_0$ with the same eigenvalues as g.

Using the algorithm presented in [PS12], with the aid of SAGE ([S+11]), we obtain the desired order, which is given by

$$R = \left\langle \frac{1 - (\omega + 1)j - (\omega + 10)k}{2}, \frac{i - \omega j + -(\omega + 21)k}{2}, j - 5k, (5\omega - 3)k \right\rangle_{\mathcal{O}}.$$

This order has class number equal to 2, and hence there is no need to compute the Brandt matrices in this example, since $S(R)$ is 1-dimensional. A set of representatives for the set of R-ideal classes is given by R and the ideal I given by

$$I = \left\langle \frac{1 - (\omega + 1)j - (\omega + 38)k}{2}, \frac{i - \omega j + -(\omega + 49)k}{2}, j + 3k, (5\omega - 3)k \right\rangle_{\mathcal{O}}.$$

We have that $v = [R] - [I]$ must be an eigenvector for the whole Hecke algebra, since $\deg(v) = 0$. In particular, the cusp eigenform $f = \theta(v)$ maps to g under the Shimura map. It is not zero, which was already known since $L(E, 1) \neq 0$ and the involution $w_{\mathfrak{c}}$ has eigenvalue equal to 1.

We consider L_R and L_I as lattices of dimension 6 over $\mathbb{Z}$, and use LLL on the integral, positive definite quadratic form $\mathrm{Tr}_{F/\mathbb{Q}} \circ (-\Delta)$ to compute the Fourier coefficients $\lambda(\xi, \mathcal{O}, f)$, with $\mathrm{Tr}_{F/\mathbb{Q}}(\xi) \leq 100$ and $-\xi$ a fundamental discriminant. The zero coefficients split into two families, which we consider below.

- **The trivial zeros** are the ones such that $\lambda(\xi, \mathcal{O}, \theta([R])) = \lambda(\xi, \mathcal{O}, \theta([I])) = 0$. For this zeros Conjecture 6.2 is easy to prove. The local-global principle for quadratic forms implies that the non existence of points $x \in L_R \cup L_I$ with $-\Delta(x) = \xi$ is equivalent to the equality $\epsilon_\xi(\mathfrak{c}) = -1$, so in this case both sides of the formula vanish trivially.
- **The non-trivial zeros** are the ones such that $\lambda(\xi, \mathcal{O}, \theta([R])) = \lambda(\xi, \mathcal{O}, \theta([I])) \neq 0$. The totally positive elements ξ with $-\xi$ being a fundamental discriminant and $\mathrm{Tr}_{F/\mathbb{Q}}(\xi) \leq 100$ are

  $$35 + 8w, 39 + 15w, 47 - 9w, 51 - 5w, 62 - 27w.$$

  For such ξ, Conjecture 6.2 (together with the Birch and Swinnerton-Dyer conjecture) claims that the rank of the quadratic twist of E by $-\xi$ should be positive (and even, because the sign of the functional equation equals 1). We verified using 2-descent that all these curves have rank equal to 2.

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