Biblioteca de la Revista Matemática Iberoamericana
Proceedings of the "Segundas Jornadas de Teoría de Números" (Madrid, 2007), 1–21

Shimura Correspondence and Central values of twisted L-series

Ariel Pacetti

Abstract

This article is a survey of developments on constructing preimages of the Shimura map and its relation to central values of twisted L-series. We start with some motivation and definitions of the classical modular forms and extend the ideas to half integral weight ones. After a short exposition of the Hecke algebra and its action on the space of modular forms, we relate the two spaces via the Shimura map. On the last section we state the known results and some recent ideas and developments on preimage constructions and central values.

1. Modular Forms

1.1. Motivation and definition of classical modular forms

Modular forms are certain holomorphic functions on the Poincaré upper half plane $\mathfrak{h} = \{z \in \mathbb{C} : \Im(z) > 0\}$ that are invariant under congruence subgroups. If we denote by $GL_2(\mathbb{R})^+$ the multiplicative group of two by two matrices with real coefficients and positive determinant, it has a natural action on $\mathfrak{h}$ by Möbius transformations. It is given for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2(\mathbb{R})^+$ by

$$\gamma.z = \frac{az+b}{cz+d}.$$

Since diagonal matrices act trivially, we can consider the quotient group. Let $SL_2(\mathbb{R})$ be the subgroup of $GL_2(\mathbb{R})$ of determinant 1 matrices. Then the quotient is isomorphic to $PSL_2(\mathbb{R})$. It acts faithful and transitively on

2000 Mathematics Subject Classification: Primary: 11F37; Secondary: 11F67
Keywords: Shimura Correspondence, L-series, Quadratic Twists

$\mathfrak{h}$, hence for getting curves on quotients of $\mathfrak{h}$ we need to work with "discrete" subgroups of $SL_2(\mathbb{R})$. The notion of "discrete" can be given in terms of its action on $\mathfrak{h}$, $\Gamma \subset SL_2(\mathbb{R})$ is discrete if and only if it acts discontinuously on $\mathfrak{h}$; that is for any compact subsets $K_1, K_2 \subset \mathfrak{h}$, the set

$$\{\gamma \in \Gamma \ : \ K_2 \cap \gamma.K_1 \neq \emptyset\}$$

is finite (see [3], Proposition 1.6). If N denotes a positive integer, the most important discrete subgroups are the congruence groups $\Gamma(N)$, and the subgroups $\Gamma_1(N)$ and $\Gamma_0(N)$ given by

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \ : \ a \equiv d \equiv 1 \mod N,\, b \equiv c \equiv 0 \mod N \right\},$$

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \ : \ a \equiv d \equiv 1 \mod N,\, c \equiv 0 \mod N \right\}$$

and

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \ : \ c \equiv 0 \mod N \right\}.$$

Note that $\Gamma(N) \triangleleft \Gamma_1(N) \triangleleft \Gamma_0(N)$ and also $\Gamma(N) \triangleleft SL_2(\mathbb{Z})$. A congruence subgroup of $SL_2(\mathbb{Z})$ is a subgroup Γ such that it contains $\Gamma(N)$ for some N. These families of groups are discrete subgroups of $SL_2(\mathbb{R})$ hence the quotient space is a Hausdorff space. It can be given a complex structure in a natural way, although care must be taken around the elliptic points, that is points where the stabilizer is different from $\pm\,\mathrm{Id}$ (see [1], Chapter XI, §2 pp. 311). In the case of $SL_2(\mathbb{Z})$, the only elliptic points of $\mathfrak{h}$ are the classes of i and $\frac{-1+\sqrt{-3}}{2}$ whose stabilizers (modulo $-\,\mathrm{Id}$) are $\{\mathrm{Id}, \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\}$ and $\{\mathrm{Id}, \left(\begin{smallmatrix} -1 & -1 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right)\}$ respectively.

Although $\Gamma\backslash\mathfrak{h}$ is not compact it can be compactified by adding a finite number of points. A compactification can be achieved by considering the extended upper half plane $\mathfrak{h}^* = \mathfrak{h} \cup \mathbb{P}^1(\mathbb{Q})$ endowed with a suitable topology and an action of $SL_2(\mathbb{Z})$.

For Γ a congruence subgroup, it is natural to consider the space of holomorphic (respectively meromorphic) differentials of the compact curve $\Gamma\backslash\mathfrak{h}^*$. Since $\mathfrak{h}$ is a cover of $\Gamma\backslash\mathfrak{h}$ via the projection map

$$\pi : \mathfrak{h} \to \Gamma\backslash\mathfrak{h},$$

for ω a meromorphic differential form on $\Gamma\backslash\mathfrak{h}$ its pullback $\pi^*(\omega)$ is a differential form on $\mathfrak{h}$ hence $\pi^*(\omega) = f(z)dz$ for some meromorphic function $f : \mathfrak{h} \to \mathbb{C}$.

For k a positive integer, $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2(\mathbb{R})^+$ and $f : \mathfrak{h} \to \mathbb{C}$ a meromorphic function, we define

$$f|_k[\gamma](z) = f(\gamma.z)(cz+d)^{-k}\,\mathrm{Det}(\gamma)^{k/2}.$$

This gives a group action on such functions, i.e. $f|_k[\alpha\beta] = (f|_k[\alpha])|_k[\beta]$.

Definition 1.1. If Γ is a congruence subgroup, a modular form f of weight k for Γ is an holomorphic function $f : \mathfrak{h} \to \mathbb{C}$ such that:

- $f|_k[\gamma](z) = f(z)$ for all $\gamma \in \Gamma$.
- f is holomorphic at all cusps.

The holomorphicity condition at the ∞ cusp is as follows: since the matrix $\left(\begin{smallmatrix} 1 & N \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma$ for some N, the function $f(z)$ is invariant under translation by N, hence it admits a Fourier expansion of the form:

$$f(z) = \sum_{n=-\infty}^{+\infty} a_n e^{2\pi i \frac{n}{N} z}.$$

We define f to be holomorphic at ∞ if $a_n = 0$ for all $n < 0$. If $\frac{p}{q}$ is another cusp, there exists an element $\alpha \in SL_2(\mathbb{Z})$ such that $\alpha.\infty = \frac{p}{q}$. We say that f is holomorphic at $\frac{p}{q}$ if the function $g(z) = f|_k[\alpha](z)$ is holomorphic at ∞. $f(z)$ is called a *cusp form* if it vanished (i.e. $a_0 = 0$) at all cusps. We denote by $M_k(\Gamma)$ and $S_k(\Gamma)$ the space of holomorphic and cuspidal forms respectively. We have ([1] Proposition 11.6):

Proposition 1.1. *The map $\omega \to f_\omega$, where $\pi^*(\omega) = f_\omega dz$ is an isomorphism between $\Omega_{hol}(\Gamma\backslash\mathfrak{h}^*)$ and $S_2(\Gamma)$.*

1.2. Half integral weight modular forms

For reasons that will become clear later, we would like to extend the notions of integral weight modular forms to that of half integral weight. This means we are looking for holomorphic functions f such that for an odd positive integer k they satisfy

$$f(\gamma.z) = \left(\sqrt{cz+d}\right)^k f(z) \tag{1.1}$$

for all elements $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ on a congruence subgroup.

Let us consider the square root with argument in $(-\pi/2, \pi/2]$, which is holomorphic on $\mathbb{C}\backslash\mathbb{R}_{\leq 0}$. Equation (1.1) does not give a group action. Suppose it did. Let Γ be a group containing $\Gamma(N)$, and consider the elements

$$\alpha = \begin{pmatrix} N+1 & N \\ -N & 1-N \end{pmatrix} \qquad \text{and} \qquad \beta = \begin{pmatrix} 1 & 0 \\ -N & 1 \end{pmatrix}.$$

The invariant conditions for $f|_{k/2}[\alpha\beta]$, and $(f|_{k/2}[\alpha])|_{k/2}[\beta]$ would imply

$$(\sqrt{-zN+1})^k \left(\sqrt{\frac{-Nz}{-Nz+1}} + 1 - N\right)^k = (\sqrt{z(N^2-N)+1-N})^k.$$

When k is odd, both terms coincide when squared. Thus, they differ by at most a sign. Since $z \in \mathfrak{h}$, the terms inside the square root on the left are in $-\mathfrak{h}$ hence their square roots are in the fourth quadrant and their product is on the third or forth quadrant. The term on the right is the square root of an element in $\mathfrak{h}$ hence it lies on the first quadrant.

To fix this, let $\psi : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}^\times$ be a Dirichlet character. We define the space $M_k(\Gamma_0(N), \psi)$ as the subspace of $M_k(\Gamma_1(N))$ consisting of those modular forms satisfying

$$f|_k[\gamma](z) = \psi(d)f(z) \quad \forall \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N).$$

Since $\Gamma_1(N) \lhd \Gamma_0(N)$ and $\Gamma_0(N)/\Gamma_1(N) \simeq (\mathbb{Z}/N\mathbb{Z})^\times$ we get a decomposition

$$M_k(\Gamma_1(N)) = \oplus_\psi M_k(\Gamma_0(N), \psi)$$

where the sum is over all Dirichlet characters ψ of $(\mathbb{Z}/N\mathbb{Z})^\times$. If we look for modular forms such that their squares are modular forms with a character, we find the well known theta series

$$\theta(z) = \sum_{n=-\infty}^{+\infty} e^{2\pi i n^2 z}.$$

Proposition 1.2. *For all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(4)$ the theta function θ satisfies the equation*

$$\theta(\gamma z) = \left(\frac{c}{d}\right) \varepsilon_d^{-1} \sqrt{cz+d}\,\theta(z)$$

where

$$\varepsilon_d = \begin{cases} 1 & \text{if } d \equiv 1 \pmod 4 \\ i & \text{if } d \equiv 3 \pmod 4 \end{cases}$$

In particular $\theta(z)^2 \in S_1(\Gamma_0(4), \chi)$ where $\chi(d) = \left(\frac{-1}{d}\right)$.

Proof. See [2] Chapter III, §4. □

Definition 1.2. Let k be an odd integer and Γ a congruence subgroup contained in $\Gamma_0(4)$. A modular form of weight $k/2$ for Γ is an holomorphic function $f : \mathfrak{h} \to \mathbb{C}$ such that

$$f(\gamma z) = \left(\frac{\theta(\gamma z)}{\theta(z)}\right)^k f(z) \quad \forall \gamma \in \Gamma$$

and f is holomorphic at the cusps.

We denote by $M_{k/2}(\Gamma)$ such space and by $S_{k/2}(\Gamma)$ the subspace of cuspidal forms. As in the classical case, if N is a positive integer divisible by 4 and ψ is a character modulo N we define $M_{k/2}(\Gamma_0(N),\psi)$ (respectively $S_{k/2}(\Gamma_0(N),\psi)$) the space of modular forms (respectively cuspidal modular forms) for $\Gamma_1(N)$ such that

$$f(\gamma z) = \left(\frac{\theta(\gamma z)}{\theta(z)}\right)^k \psi(d) f(z) \quad \forall \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N).$$

2. Hecke Algebra and Hecke Operators

2.1. Hecke Operators on classical modular forms

Hecke operators can be defined in several ways for classical modular forms. We review the definition in terms of double cosets since it generalizes to modular forms of half integral weight easily.

For Γ, Γ' congruence subgroups of $SL_2(\mathbb{Z})$ we say that Γ' and Γ are commensurable (and denote it by $\Gamma' \sim \Gamma$) if $\Gamma' \cap \Gamma$ has finite index in both Γ and Γ'. By Lemma 3.10 of [3] one has

Lemma 2.1. *If Γ is a congruence subgroup, $\Gamma \sim \alpha^{-1}\Gamma\alpha$ for all $\alpha \in GL_2(\mathbb{Q})$.*

Note that $\Gamma \sim \alpha^{-1}\Gamma\alpha$ implies that $\Gamma\alpha\Gamma$ contains $\Gamma\alpha$ with finite index, since if $\{\delta_1, \dots, \delta_r\}$ is a set of coset representatives of $(\Gamma \cap \alpha^{-1}\Gamma\alpha)\backslash\Gamma$ then $\Gamma\alpha\Gamma = \cup\Gamma\alpha\delta_i$ (where the union is disjoint).

Let R_Γ be the $\mathbb{Z}$-module spanned by the classes $[\Gamma\alpha\Gamma]$ where $\alpha \in GL_2(\mathbb{Q})$. Let $\deg([\Gamma\alpha\Gamma]) := \#((\Gamma \cap \alpha^{-1}\Gamma\alpha)\backslash\Gamma)$. Extend it additively to R_Γ, i.e. $\deg(\sum_{i=1}^n c_i[\Gamma\alpha_i\Gamma]) = \sum_{i=1}^n c_i \deg([\Gamma\alpha_i\Gamma])$. A multiplication law can be defined on R_Γ as follows: if $\Gamma\alpha\Gamma = \cup_{i=1}^r \Gamma\alpha_i$ and $\Gamma\beta\Gamma = \cup_{j=1}^s \Gamma\beta_i$,

$$[\Gamma\alpha\Gamma].[\Gamma\beta\Gamma] = \sum_{\{[\Gamma\delta\Gamma]\,:\,\delta\in GL_2(\mathbb{Q})^+\}} \eta_\delta[\Gamma\delta\Gamma]$$

where $\eta_\delta = \#\{(i,j) \ : \ \Gamma\alpha_i\beta_j = \Gamma\delta\}$. One has the following (see [3] Proposition 3.3 and Proposition 3.4).

Proposition 2.2. *The product is associative and* deg *is a morphism of algebras, i.e.* $\deg(X.Y) = \deg(X)\deg(Y)$ *for* $X, Y \in R_\Gamma$.

For simplicity we restrict to the cases where $\Gamma = \Gamma_0(N)$, $\Gamma = \Gamma_1(N)$ or $\Gamma = \Gamma(N)$ for some positive integer N (see [3] Section 3.3 for the general case).

Definition 2.1. For n be a positive integer, define the element in R_Γ

$$T(n) = \sum_{\alpha \in \Gamma\backslash\Delta_n/\Gamma} [\Gamma\alpha\Gamma]$$

where $\Delta_n \subset \{\gamma \in M_2(\mathbb{Z}) \,:\, \mathrm{Det}(\gamma) = n\}$ is given by those elements γ that satisfy

$$\gamma \equiv \begin{cases} \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} & \bmod N \quad \text{if } \Gamma = \Gamma_0(N) \\ \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} & \bmod N \quad \text{if } \Gamma = \Gamma_1(N) \\ \begin{pmatrix} 1 & 0 \\ 0 & * \end{pmatrix} & \bmod N \quad \text{if } \Gamma = \Gamma(N). \end{cases}$$

Consider the Hecke algebra $\mathbb{T}$ (over $\mathbb{Q}$) spanned by all $T(n)$ with n a positive integer. It satisfies the following properties:

Theorem 2.3. $\mathbb{T}$ *is a commutative algebra. The elements* $T(n)$ *satisfy:*

- *If* $\gcd(n, m) = 1$, $T(nm) = T(n)T(m)$.
- *If* $p \mid N$, $T(p^r) = T(p)^r$ *and* $\deg(T(p)) = p$.
- *If* $\gcd(n, N) = 1$, $\deg(T(n)) = \sigma(n) = \sum_{d|n} d$.

See Theorem 3.34 and Theorem 3.35 of [3] for proofs of these facts.

2.1.1. Hecke Operators

Let Γ be a congruence subgroup and $f \in M_k(\Gamma)$. For $\alpha \in GL_2(\mathbb{Q})^+$ let

$$f|_k[\Gamma\alpha\Gamma] = \mathrm{Det}(\alpha)^{k/2-1} \sum_{n=1}^{d} f|_k[\alpha_n]$$

where $\Gamma\alpha\Gamma = \cup_{n=1}^{d} \Gamma\alpha_n$. The operator $|_k[\Gamma\alpha\Gamma]$ maps $M_k(\Gamma)$ (respectively $S_k(\Gamma)$) into itself (since it averages over representativess). Extending by linearity we get an action of R_Γ on $M_k(\Gamma)$. This gives an action of $\mathbb{T}$ on modular forms. Let $T(m)$ (the m-th Hecke operator) be the operator obtained by the action of $T(m)$ on $M_k(\Gamma)$.

Define on $S_k(\Gamma)$ the *Petersson inner product*, given by

$$\langle f, g \rangle = \frac{1}{[SL_2(\mathbb{Z}) : \Gamma]} \int_{\Gamma\backslash\mathfrak{h}} f(z)\overline{g(z)} y^{k-2} dx dy$$

where $z = x + iy$. It is not hard to see that this integral is well defined and converges. Furthermore, it satisfies (see [3] Proposition 3.39) the following

Theorem 2.4. *Let* $\alpha \in GL_2(\mathbb{Q})^+$ *and* $f, g \in S_k(\Gamma)$. *Then*

$$\langle f|_k[\alpha], g\rangle = \langle f, g|_k[\mathrm{Adj}(\alpha)]\rangle$$

where $\mathrm{Adj}(\alpha)$ *is the adjoint matrix of* α.

If f is an eigenform for all Hecke operators, its Fourier coefficients are uniquely determined in terms of the eigenvalues (see [3] Theorem 3.43). The problem of considering the whole Hecke algebra is that its action on $M_k(\Gamma)$ needs not be semisimple (see [3] Remark 3.59). Let us restrict to the case $\Gamma = \Gamma_1(N)$.

Theorem 2.5. *Let* $f \in S_k(\Gamma_0(N), \psi)$ *and* p *be a prime number. If* $f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$ *then* $T(p)(f) = \sum_{n=1}^{\infty} b_n e^{2\pi i n z}$ *where*

$$b_n = a_{pn} + \psi(p)p^{k-1}a_{n/p}$$

and $a_{n/p} = 0$ *if* $p \nmid n$.

Theorem 2.6. *Let* n *be a positive integer prime to* N *and* $\sigma_n \in SL_2(\mathbb{Z})$ *satisfy* $\sigma_n \equiv \left(\begin{smallmatrix} 1/n & 0 \\ 0 & n \end{smallmatrix}\right) \mod N$. *Then for any* $f, g \in M_k(\Gamma_1(N))$

$$\langle T_n(f), g\rangle = \langle f|_k[\sigma_n], T_n(g)\rangle.$$

In particular for any Dirichlet character ψ *of* $(\mathbb{Z}/N\mathbb{Z})^\times$, *if* c_n *denotes the complex conjugate of any square root of* $\psi(n)$, *the operator* $c_n T_n$ *is self adjoint on* $S_k(\Gamma_0(N), \psi)$.

Proof. Let α be a left class representative of a double coset in Δ_n. Then

$$\langle f|_k[\alpha], g\rangle = \left\langle (f|_k[\sigma_n])|_k[\sigma_n^{-1}\alpha], g\right\rangle = \langle f|_k[\sigma_n], g|_k[\mathrm{Adj}(\alpha)\sigma_n]\rangle.$$

Since α has the form $\left(\begin{smallmatrix} 1 & b \\ 0 & d \end{smallmatrix}\right) \mod N$ with $d \equiv n \pmod N$, $\mathrm{Adj}(\alpha)\sigma_n \equiv \left(\begin{smallmatrix} 1 & -bn \\ 0 & n \end{smallmatrix}\right) \mod N$ so the map $\alpha \mapsto \mathrm{Adj}(\alpha)\sigma_n$ is a bijection of Δ_n into itself. Since $T(n)$ is sums over all such representatives, the assertion follows. $\square$

This implies that the Hecke Algebra $\mathbb{T}^0 = \langle T(n), \gcd(n, N) = 1\rangle_{\mathbb{Q}}$ does act semisimply on $M_k(\Gamma_1(N))$ so a basis of common eigenforms for all operators on $\mathbb{T}^0$ exists.

2.2. Hecke Operators on half integral weight modular forms

The theory of Hecke operators on half integral weight modular forms is the same as the one for classical modular forms working with groups similar (and isomorphic) to the congruence ones (see [Sh] for a complete exposition of the subject). We just mention the basic results and the analogy.

If $\alpha \in \Gamma_0(4)$, let $j(\alpha, z)$ be the holomorphic function on $\mathfrak{h}$

$$j(\alpha, z) = \theta(\alpha z)/\theta(z)$$

Consider the group

$$\widetilde{G} = \left\{(\alpha, \phi(z)) \;:\; \alpha \in GL_2(\mathbb{Q})^+, \phi(z)^2 = \pm(cz+d)/\sqrt{\det(\alpha)}\right\}$$

and

$$\widetilde{\Gamma_0(4)} = \{(\alpha, j(\alpha, z)) \;:\; \alpha \in \Gamma_0(4)\}$$

where multiplication is given by

$$(\alpha, \phi(z))(\beta, \psi(z)) = (\alpha\beta, \phi(\beta.z)\psi(z)).$$

By Proposition 1.2, $\widetilde{\Gamma_0(4)} \subset \widetilde{G}$. Consider the natural map

$$\Psi : \Gamma_0(4) \to \widetilde{\Gamma_0(4)}, \quad \Psi(\alpha) = (\alpha, j(\alpha, z))$$

and define the congruence subgroups (for N divisible by 4) as the image of the classical ones. As before, we restrict to the case where $\widetilde{\Gamma}$ is the image of $\Gamma(N)$, $\Gamma_1(N)$ or $\Gamma_0(N)$. For $(\alpha, \phi) \in \widetilde{G}$ and $f \in M_{k/2}(\widetilde{\Gamma})$ let

$$f|_k[(\alpha, \phi(z))](z) = f(\alpha.z)\phi(z)^{-k}.$$

The notions of commensurable groups, double cosets, products of double cosets, etc. are the same as in the classical case. For a positive integer n consider the element $\xi_n \in \widetilde{G}$ given by

$$\xi_n = (\alpha_n, n^{1/4}) \quad \text{where} \quad \alpha_n = \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix}.$$

It can be seen that $\widetilde{\Gamma}\xi_n\widetilde{\Gamma}$ is commensurable with $\widetilde{\Gamma}$ hence $\widetilde{\Gamma}\xi_n\widetilde{\Gamma} = \cup_{i=1}^m \widetilde{\Gamma}\xi_i$. Define the Hecke operators $T_k(n)$ acting on $M_{k/2}(\Gamma)$ by

$$T_k(n)f = n^{(k/4)-1} \sum_{i=1}^m f|_k[\xi_i].$$

If $\gcd(n, N) = 1$ and n is not a square, the Hecke operators $T_k(n)$ are zero, so we can restrict to square ones. The Hecke operators satisfy:

- If $\gcd(n, m) = 1$ then $T_k(nm) = T_k(n)T_k(m)$.
- They commute with each other.

- If $\gcd(n, N) = 1$, the operators $T_k(n^2)$ are hermitian on $S_{k/2}(\Gamma_0(N), \psi)$ with respect to the *Petersson inner product* given by

$$\langle f, g\rangle = \frac{1}{[\Gamma_0(4) : \Gamma_1(N)]} \int_{\Gamma_1(N)\backslash\mathfrak{h}} f(x)\overline{g(z)}y^{\frac{k}{2}-2}dxdy.$$

- Let $f \in M_{k/2}(\Gamma_0(N), \psi)$. If $f(z) = \sum_{n\geq 0} a_n e^{e\pi inz}$ then $T_k(p^2)(f) = \sum_{n\geq 0} b_n e^{e\pi inz}$ where

$$b_n = a_{p^2n} + \psi_1(p)\left(\frac{n}{p}\right)p^{(k-3)/2}a_n + \psi(p^2)p^{k-2}a_{n/p^2},$$

$\psi_1(m) = \psi(m)\left(\frac{-1}{m}\right)^{(k-1)/2}$ and $a_{n/p^2} = 0$ if $p^2 \nmid n$.

(see [Sh] Proposition 1.6, Theorem 1.7 and [Sh2] Lemma 5). We can now give a definition of the Shimura map (the main Theorem of [Sh]):

Theorem 2.7. *Let $f \in M_{k/2}(\Gamma_0(N), \psi)$ be an eigenform of all Hecke operators $T_k(p^2)$ for all primes p, say $T_k(p^2)(f) = \lambda_p f$. Define the function* $\mathrm{Shim}(f) = \sum_{n\geq 0} a_n e^{2\pi inz}$ *where the $\{a_n\}$ satisfy*

$$\sum_{n\geq 0} a_n n^{-s} = \prod_p (1 - \lambda_p p^{-s} + \psi(p)^2 p^{k-2-2s})^{-1}.$$

Then $\mathrm{Shim}(f) \in M_{k-1}(\Gamma_0(M), \psi^2)$ *for some M. Furthermore, if $k \geq 5$ and $f \in S_{k/2}(\Gamma_0(N), \psi)$ then* $\mathrm{Shim}(f) \in S_{k-1}(\Gamma_0(M), \psi^2)$.

Remark. From Theorem 2.5 it is easy to see that $\mathrm{Shim}(f)$ is an eigenform for $\mathbb{T}$ and $T_n(\mathrm{Shim}(f)) = \lambda_n \mathrm{Shim}(f)$ for all n.

By a Theorem of Niwa (see [Ni]) M can be taken to be $N/2$. For $k = 3$ one needs to consider a subspace of $S_{3/2}(\Gamma_0(N), \psi)$ to get cusp forms on the image. Let r, t be positive integers such that $4tr^2 \mid N$. Let χ be a primitive character modulo r such that $\chi(-1) = -1$ and $\psi(n) = \left(\frac{-t}{n}\right)\chi(n)$ for all $n \in \mathbb{Z}/N\mathbb{Z}$. Define $U(\Gamma_0(N), \psi)$ to be the subspace of $S_{3/2}(\Gamma_0(N), \psi)$ spanned by $\left\{\sum_{m\in\mathbb{Z}} \chi(m)me^{2\pi im^2z}\right\}$ and $V(\Gamma_0(N), \psi)$ to be its orthogonal complement. If $f \in V(\Gamma_0(N), \psi)$, $\mathrm{Shim}(f) \in S_2(\Gamma_0(M), \psi^2)$.

At this point the Shimura map is only defined on eigenforms for all Hecke operators. Since $\mathbb{T}$ is commutative we can decompose $S_{k/2}(\Gamma_0(N), \psi)$ as a direct sum of irreducible subspaces for its action.

Definition 2.2. Let $\mathcal{V}$ be an irreducible subspace of $S_{k/2}(\Gamma_0(N), \psi)$ for the action of $\mathbb{T}$. Let $f \in \mathcal{V}$ be the unique (up to a scalar factor) non-zero eigenform for $\mathbb{T}$. We define $\mathrm{Shim}(\mathcal{V}) := \mathrm{Shim}(f)$.

Remark. Since the Hecke algebra $\mathbb{T}^0$ is hermitian on $S_{k/2}(\Gamma_0(N),\psi)$, all the modular forms on $\mathcal{V}$ have the same eigenvalues for all Hecke operators $T_k(p^2)$ for $p \nmid N$.

We will focus on the following three questions:

1. Is this map surjective?
2. The Hecke operators relates Fourier coefficients that differ by squares. What is the meaning of the square free Fourier coefficients of half integral weight modular forms?
3. How can we construct preimages of the Shimura map?

The first question can be answered using automorphic representations. Denote by $\pi = \pi_\infty \otimes \bigotimes_p \pi_p$ the automorphic representation associated to F. We have the following (see [Fl]):

Proposition 2.8. *$F \in S_{k-1}(N,\psi^2)$ is in the image of the Shimura map if and only if at the primes $p \mid N$ where π_p is a principal series (say $\pi_p = \pi_p(\mu_{1,p},\mu_{2,p})$) the characters satisfy $\mu_{1,p}(-1) = \mu_{2,p}(-1) = 1$.*

In particular the map is surjective into $S_{k-1}(N)$ when N is odd and square free. In this case, if ψ is a quadratic character, consider

Definition 2.3. The Kohnen space (denoted by $S^+_{k/2}(\Gamma_0(4N),\psi)$) is the subspace of $S_{k/2}(\Gamma_0(4N),\psi)$ given by forms whose Fourier coefficients satisfy

$$a_n = 0 \text{ for all } n \text{ with } \psi(-1)(-1)^k n \equiv 2,3 \pmod 4.$$

The main results of [Ko] are summarized in the next

Theorem 2.9. *If N is odd and square free and ψ is a quadratic Dirichlet character, the Hecke operators preserve the Kohnen space $S^+_{k/2}(\Gamma_0(4N),\psi)$. If we restrict to the subspace of new forms, then the eigenforms of $\mathbb{T}^0$ are uniquely determined up to multiplication by non-zero complex numbers (i.e. there is multiplicity one). Furthermore, the Shimura map is an isomorphism between $S^{+,new}_{k/2}(4N,\psi)$ and $S^{new}_{k-1}(N)$.*

The first answer to the second question is the work of Waldspurger.

Notation. By χ_n we will denote the quadratic character corresponding to the field $\mathbb{Q}[\sqrt{n}]$. If $F \in M_k(\Gamma_1(N))$ and χ is any character modulo N, $L(F,\chi,s)$ will denote the L-series of the newform corresponding to the twist of F by χ.

The main Theorem of ([Wa]) is complicated to state, but the heart of his work is that up to local factors one has the following (see Theorem 3.5 below for an explicit version)

Theorem 2.10. *Let $F \in S_{k-1}(M, \psi^2)$ and $f \in S_{k/2}(N, \psi)$ mapping to F under the Shimura map. Let n_1, n_2 be square free positive integers such that $n_1/n_2 \in (\mathbb{Q}_p^\times)^2$ for all $p \mid N$. If $f(z) = \sum_{n\geq 1} a_n e^{2\pi i n z}$ then*

$$a_{n_1}^2 L(F, \psi_0^{-1}\chi_{n_2}, \tfrac{k-1}{2})\psi(n_2/n_1)n_2^{k/2-1} = a_{n_2}^2 L(F, \psi_0^{-1}\chi_{n_1}, \tfrac{k-1}{2})n_1^{k/2-1}$$

where $\psi_0(n) = \psi(n)\left(\frac{-1}{n}\right)^{(k-1)/2}$.

Thus fixing n_1, for all n which differ from n_1 locally at the primes dividing N by a square, we get the formula

$$a_n^2 = \kappa L(F, \psi_0^{-1}\chi_n, \tfrac{k-1}{2})\psi(n)n^{k/2-1}$$

where κ is a global constant. An explicit version of this Theorem for N square free was given in [Ko2] (Corollary 1), it states

Theorem 2.11. *Let $f(z) = \sum_{n\geq 1} a_n e^{2\pi i n z} \in S_{k/2}^{+,new}(4N)$ and $F \in S_{k-1}^{new}(N)$ such that f maps to F under the Shimura map. For each prime p dividing N, let w_p denote the Atkin-Lehner sign of F at p. Let d be a discriminant with $(-1)^k d > 0$ and such that $\left(\frac{d}{p}\right) = w_p$ for all primes $p \mid N$. Then*

$$\frac{|a_d|^2}{\langle f, f\rangle} = 2^{\nu(N)}\frac{(k-1)!}{\pi^k}|d|^{k/2-1}\frac{L(F, \chi_d, \frac{k-1}{2})}{\langle F, F\rangle}$$

where $\nu(N)$ is the number of prime divisors of N.

3. Constructing preimages of the Shimura map

Constructing preimages of the Shimura map allows to compute $L(F, d, \frac{k-1}{2})$ for a lot of discriminants d in an efficient way (as saw on Theorem 2.11) if one can compute the Fourier expansion of the weight 3/2 modular forms (see [RV] for an exposition and [Pa-To2] for examples). In [Shi] a method to construct preimages (in some cases) is given by computing integrals over geodesics of the weight $k-1$ (even) modular form. The disadvantage of this method is that, numerically speaking, it is not efficient. The first approach using theta series was the work of Tunnell on the congruent number problem (see [Tu]). It is not hard to see that d is a congruent number if and only if the elliptic curve

$$E_d : y^2 = x^3 - d^2 x$$

has infinitely many rational points. Since this curve is the twist by χ_d of the elliptic curve $E : y^2 = x^3 - x$, we are asking which quadratic twists of this curve have positive rank. The elliptic curve E corresponds to the unique newform of $S_2(\Gamma_0(32))$. If ψ is any quadratic character modulo 128, the space $S_{3/2}(\Gamma_0(128), \psi)$ has dimension 3. So does the space $S_{1/2}(\Gamma_0(128), \psi)$. By considering the theta series of a binary quadratic form, Tunnell constructed a weight 1 modular form g of level 128 and character χ_{-2}. Multiplying by g a basis of $S_{1/2}(\Gamma_0(128), \psi\chi_{-2})$ we obtain a basis of $S_{3/2}(\Gamma_0(128), \psi)$. Furthermore, since $S_{1/2}(\Gamma_0(128), \psi\chi_{-2})$ is spanned by theta series of unary quadratic forms, the basis for $S_{3/2}(\Gamma_0(128), \psi)$ constructed consists of theta series of ternary quadratic forms. Computing the action of $T(p^2)$ on Fourier coefficients (which can be done in this case because the dimension of the space of modular forms is small) he gets a basis of eigenforms maping under Shimura to E. Using an explicit version of Waldspurger's Theorem he proves

Theorem 3.1. *The forms*

$$f_1 = \sum_{(x,y,z) \in \mathbb{Z}^3} e^{2\pi i(2x^2+y^2+32z^2)} - \frac{1}{2} \sum_{(x,y,z) \in \mathbb{Z}^3} e^{2\pi i(2x^2+y^2+8z^2)}$$

and

$$f_2 = \sum_{(x,y,z) \in \mathbb{Z}^3} e^{2\pi i(4x^2+y^2+32z^2)} - \frac{1}{2} \sum_{(x,y,z) \in \mathbb{Z}^3} e^{2\pi i(4x^2+y^2+8z^2)}$$

map to E under the Shimura map. Furthermore if d is odd, $L(E_d, 1) = \kappa_1 a_d(f_1)$ (respectively if d is even, $L(E_d, 1) = \kappa_2 a_d(f_2)$) for some non-zero constants κ_i where $a_n(f)$ denotes the n-th Fourier coefficient of f.

By a Theorem of Coates and Wiles (see [Co-Wi]) if d is odd (respectively even) and $a_d(f_1) \neq 0$ (respectively $a_d(f_2) \neq 0$) then d is not a congruent number. The converse is true assuming the Birch and Swinnerton-Dyer conjecture.

Following [Pa-To], we present a method to construct preimages of the Shimura map for any level N. This method is based (and includes) the case of prime level studied in [Gr] and the case of square free level studied in [Bö-SP] although in both works the authors prove stronger results using different tools (see section Further Developments).

3.1. Quaternion Algebras

A be a quaternion algebra B over $\mathbb{Q}$ is a central simple algebra of dimension 4 over $\mathbb{Q}$. It can always be presented as $B = \mathbb{Q} + \mathbb{Q}i + \mathbb{Q}j + \mathbb{Q}k$ where the product satisfies $ij = k = -ji$ and $i^2 = a$, $j^2 = b$ for $a, b \in \mathbb{Q}^\times$

square free. It has an antiautomorphism given by "conjugation". For $x \in B$ we denote $\mathcal{N}\,x$ and $\operatorname{Tr} x$ the *reduced norm* and *reduced trace* of x, with respect to it. A definite quaternion algebra is a quaternion algebra such that the quadratic form $\mathcal{N}\,x$ is positive definite. We will only consider definite quaternion algebras.

The *norm* of a lattice $\mathfrak{a} \in B$ is defined as $\mathcal{N}\,\mathfrak{a} := \gcd\{\mathcal{N}\,x \;:\; x \in \mathfrak{a}\}$. We equip $\mathfrak{a}$ with the quadratic form $\mathcal{N}_{\mathfrak{a}}(x) := \mathcal{N}\,x/\mathcal{N}\,\mathfrak{a}$. It is primitive, its determinant is a square, and we denote the positive square root by $\mathrm{D}(\mathfrak{a})$ (the *reduced discriminant*).

An order $R \subseteq B$ is a lattice which is also a ring with unity. If $\mathfrak{a}$ is a lattice, we define $R_l(\mathfrak{a})$, its *left order* (respectively $R_r(\mathfrak{a})$, its *right order*) by

$$R_l(\mathfrak{a}) := \{x \in B \;:\; x\mathfrak{a} \subseteq \mathfrak{a}\}.$$

If R is an order in B, we let $\widetilde{\mathcal{I}}(R)$ be the set of *left R-ideals*, that is the set of lattices $\mathfrak{a} \subseteq B$ such that for every prime p, $\mathfrak{a}_p = R_p x_p$ for some $x_p \in B_p^\times$ (subscript p denotes localization at p, namely $\mathfrak{a}_p := \mathfrak{a} \otimes \mathbb{Z}_p$). For $\mathfrak{a}, \mathfrak{b} \in \widetilde{\mathcal{I}}(R)$ define the relation $\mathfrak{a} \sim \mathfrak{b}$ if $\mathfrak{a} = \mathfrak{b}x$ for some $x \in B^\times$. This gives an equivalence relation on $\widetilde{\mathcal{I}}(R)$ and by $[\mathfrak{a}]$ we denote the class of $\mathfrak{a}$. The set of all left R-ideal classes (denoted by $\mathcal{I}(R)$) is known to be finite.

3.2. Hecke operators and the height pairing

Let $\mathcal{M}(R)$ be the free $\mathbb{Z}$-module with basis $\mathcal{I}(R)$. Define $\deg : \mathcal{M}(R) \to \mathbb{Z}$ as the linear map satisfying $\deg(\mathfrak{a}) = 1$. On $\mathcal{M}(R)$ we define *the height paring* by

$$\langle [\mathfrak{a}], [\mathfrak{b}] \rangle := \tfrac{1}{2}\#\big\{x \in B^\times \;:\; \mathfrak{a}x = \mathfrak{b}\big\} = \begin{cases} \tfrac{1}{2}\#R_r(\mathfrak{a})^\times & \text{if } [\mathfrak{a}] = [\mathfrak{b}], \\ 0 & \text{otherwise,} \end{cases}$$

The height pairing induces an inner product on $\mathcal{M}_{\mathbb{R}}(R) := \mathcal{M}(R) \otimes \mathbb{R}$ and $\mathcal{I}(R)$ is an orthogonal basis of this space. The *dual lattice* $\mathcal{M}^\vee(R) := \{\mathbf{v} \in \mathcal{M}_{\mathbb{R}}(R) \;:\; \langle \mathbf{v}, \mathcal{M}(R) \rangle \subseteq \mathbb{Z}\}$ is spanned by $[\mathfrak{a}]^\vee := \frac{1}{\langle \mathfrak{a}, \mathfrak{a} \rangle}[\mathfrak{a}]$ with $[\mathfrak{a}] \in \mathcal{I}(R)$. Let $\mathfrak{a} \in \widetilde{\mathcal{I}}(R)$, and $m \geq 1$ an integer. We set

$$\mathcal{T}_m(\mathfrak{a}) := \Big\{\mathfrak{b} \in \widetilde{\mathcal{I}}(R) \;:\; \mathfrak{b} \subseteq \mathfrak{a}, \quad \mathcal{N}\,\mathfrak{b} = m\,\mathcal{N}\,\mathfrak{a}\Big\}.$$

The *Hecke operators* $t_m : \mathcal{M}(R) \to \mathcal{M}(R)$ are then defined by

$$t_m[\mathfrak{a}] := \sum_{\mathfrak{b} \in \mathcal{T}_m(\mathfrak{a})} [\mathfrak{b}]$$

for $m \geq 1$ and $[\mathfrak{a}] \in \mathcal{I}(R)$ (they correspond to the classical Brandt matrices). In addition, we define $t_0 : \mathcal{M}(R) \to \mathcal{M}(R)$ by $t_0[\mathfrak{b}] := \frac{1}{2}\Big(\sum_{[\mathfrak{a}] \in \mathcal{I}(R)} [\mathfrak{a}]^\vee\Big)$.

Proposition 3.2. *The Hecke operators have the following properties:*

1. t_n *is self-adjoint for the height pairing.*
2. *If* $(n,m)=1$*, then* $t_{nm}=t_nt_m$.
3. *The operators* t_n *with* $\gcd(n,\mathrm{D}(R))=1$ *commute with each other.*

Since the Hecke operators t_n with $\gcd(n,\mathrm{D}(R))=1$ generate a commutative ring $\mathbb{T}^0$ of self-adjoint operators, $\mathcal{M}_{\mathbb{R}}(R)$ has an orthogonal basis of eigenvectors for $\mathbb{T}^0$.

If R is an order, let $R^* := \{x \in B : \mathrm{Tr}(xR) \subseteq \mathbb{Z}\}$ and define $\mathrm{L}(R) := \mathcal{N}(R^*)^{-1}$. There is an injective (but in general not surjective, see [J-L]) $\mathbb{T}^0$-linear map $\phi : \mathcal{M}^\vee(R) \otimes_{\mathbb{T}^0} \mathcal{M}(R) \longrightarrow M_2(\Gamma_0(\mathrm{L}(R)))$ given by

$$\phi(\mathbf{v},\mathbf{w}) := \sum_{m\geq 0} \langle \mathbf{v}, t_m\mathbf{w}\rangle q^m$$

(see Proposition 3.3 of [Pa-To]). In particular if $\mathbf{v} \in \mathcal{M}(R)$ is an eigenvector for $\mathbb{T}^0$, so is the modular form $\phi(\mathbf{v},\mathbf{v})$ with the same eigenvalues.

We would like to associate weight 3/2 modular forms to elements in $\mathcal{M}(R)$. For $x \in B$, define $\Delta\, x := (\mathrm{Tr}\, x)^2 - 4\,\mathcal{N}\, x$ (the discriminant of the characteristic polynomial of x). Let

$$\Omega(R) := \gcd\{\Delta\, x \ : \ x \in R\}.$$

Note that $\Delta\, x \leq 0$, and $\Delta\, x = 0$ if and only if $x \in \mathbb{Q}$; thus $-\Delta\, x/\,\Omega(R)$ defines a *primitive* positive definite ternary quadratic form on the lattice $R/\mathbb{Z}$. Define the *level* and the *character* of an order R to be

$$\mathrm{N}(R) := 4\,\mathrm{L}(R)/\,\Omega(R), \qquad \text{and} \qquad \chi_R(n) := \left(\frac{\Omega(R)}{n}\right).$$

Theorem 3.3. *The map* $\Theta : \mathcal{M}(R) \to M_{3/2}\left(\mathrm{N}(R),\chi_R\right)$ *which maps* $[\mathfrak{a}] \in \mathcal{M}(R)$ *to*

$$\Theta(\mathfrak{a}) = \frac{1}{2} \sum_{x \in R_r(\mathfrak{a})/\mathbb{Z}} e^{-2\pi i z\,\Delta\, x/\,\Omega(R)},$$

is $\mathbb{T}^0$*-linear. Furthermore, if* $\deg(\mathbf{v})=0$ *then* $\Theta(\mathbf{v})$ *is a cusp form.*

Remark. The map Θ needs not be injective.

Proof. see Proposition 1.11 of [Pa-To]. The second assertion of Theorem 3.3 comes from the fact that the ternary quadratic form associated to $\mathfrak{a}$ lies in the same genus as the one associated to R. Hence, the linear combination vanishes at all cusps if and only if it vanishes at infinity. $\square$

In particular if $\mathbf{v}$ is an eigenvector for $\mathbb{T}^0$, and $\Theta(\mathbf{v}) \neq 0$, then $\Theta(\mathbf{v})$ corresponds to $\phi(\mathbf{v},\mathbf{v})$ under the Shimura map.

3.3. Orders in quaternion algebras

For the method in the previous section to work, given $F \in S_2(\Gamma_0(N))$ we need to chose a quaternion algebra B and an order R in it such that $\mathrm{L}(R) = N$ and $F \in \mathcal{M}(R)$. For a given N, which quatgernion algebra and which orders should be considered?

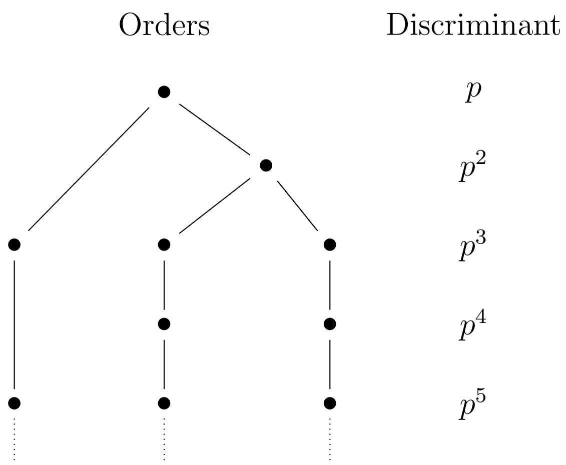
In case of prime level N the only possibility is to take B to be the quaternion algebra ramified at N and ∞ and R a maximal order. Then for all forms in $S_2(\Gamma_0(N))$ whose corresponding eigenvector in $\mathcal{M}(R)$ is not in the kernel of Θ, the previous section construction works. Furthermore, the so called "Gross Formula" (Theorem 3.4) implies that $F \in S_2(\Gamma_0(N))$ gives the zero weight $3/2$ modular form if and only if $L(F, 1) = 0$.

Now assume that N is square free. Let $F \in S_2(\Gamma_0(N))$ with positive functional equation sign. Define $\mathcal{P}_F = \{p \mid N \,:\, w_p = -1\}$ and take B to be the quaternion algebra ramified at $p \in \mathcal{P}_F$ and ∞. Let R be an order of level N in B such that R_p is an Eichler order for all $p \mid N$. In [Bö-SP] they prove that for this choice, the previous section construction works and that it gives the zero weight $3/2$ modular form if and only if $L(f, 1) = 0$ (see Theorem 3.5).

For a general level N and $F \in S_2(\Gamma_0(N))$, the choice of the quaternion algebra depending on the Atkin-Lehner involutions is quite standard. Which orders should be considered on such algebra? By Eichler's work orders R of B are in one-to-one correspondence with collection of orders $\{R_p\}$ in B_p such that R_p is maximal for almost all p. If we consider a non-ramified prime q, the natural candidate for an order of level q^r is an Eichler order (the intersection of two maximal ones). At a ramified prime p, there is a single maximal order, hence the unique Eicher order in this case has level p. To get "well-behaved" orders of level p^r the natural candidates (studied on [Hi-Pi-Sh]) correspond to some "primitive ternary quadratic forms" (see [Jo] for a survey of the relation between orders and ternary quadratic forms).

Definition 3.1. An order R is *Gorenstein* if all lattices L with $R_l(L) = R$ are left R-ideals. An order is *Bass* if every order containing it is Gorenstein.

Given an order R there exists a Gorenstein order $G(R)$ and an integer $b(R)$ such that $R = \mathbb{Z} + b(R)G(R)$. Since the image by Θ of R has the same level as the image of $G(R)$, we can restrict to work with Gorenstein orders. The Bass (respectively Gorenstein) condition is a local condition, hence we need to define the local Bass orders (and we will restrict to these ones). For a ramified prime $p \neq 2$ of B, all the Bass orders in B_p lie in the following tree (see [Jo] for the tree in the case $p = 2$).



The properties of such orders depend on three cases:

- If B_p is a division algebra and R_p is one order from the left column of the previous tree (including the maximal one), it satisfies $\Omega(R_p) = 1$ hence $\mathrm{N}(R_p) = \mathrm{L}(R_p)$. Furthermore, if $x \in R_p$ then $\left(\frac{\Delta\, x}{p}\right) = 0$ or -1.
- If B_p is a division algebra and R_p is any other order, $\Omega(R_p) = p$ hence $\mathrm{N}(R_p) = \mathrm{L}(R_p)/p$. If R_p has discriminant grater than p^2, there are two orders with the same discriminant. If $x \in R_p$, $\left(\frac{\Delta\, x/p}{p}\right) = 0$ or -1 for one order and $\left(\frac{\Delta\, x/p}{p}\right) = 0$ or 1 for the other.
- If B_q is isomorphic to a matrix algebra and R_q is an Eichler order on B_q, $\Omega(R_q) = 1$ hence $\mathrm{N}(R_q) = \mathrm{L}(R_q)$. Furthermore, if R_q is not a maximal order, it can be seen that for $x \in R_q$, $\left(\frac{\Delta\, x}{q}\right) = 0, 1$.

This justifies the choice of the quaternion algebra in Theorem 3.5 so as to be consistent with Theorem 2.11.

A complete description of the modular forms appearing in $\mathcal{M}(R)$ for orders R which are locally one of the above can be found on [Hi-Pi-Sh]. These orders are the ones to be considered for constructing weight $3/2$ modular forms in the general case, although the conditions for the construction of the previos section being zero in this general setting still needs to be done.

As an example, let us consider the case of level $N = p^2$ (with p a prime number). In [Pa-To] the authors considered the two orders of discriminant p^3 from the right part of the tree. The ternary quadratic forms attached to them lie in two different genera. Theorem 3.3 for one order gives a weight $3/2$ modular form whose Fourier coefficients are related to the twists by discriminants of the form p times a square modulo p (in the sense of Theorem 2.10). The other order gives a modular form whose Fourier coefficients are related to the twists by discriminants of the form p times a non-square modulo p. Furthermore, if $p \equiv 3 \pmod 4$, $-p$ is a fundamental discriminant and twisting by χ_{-p} is an involution on $S_2(\Gamma_0(p^2))$. Then if $F \in S_2(\Gamma_0(p^2))$ there exists a form $g \in S_{3/2}(\Gamma_0(4p^2), \chi_{-p})$ whose Fourier coefficients are related to quadratic twists of F by positive discriminants.

3.4. Gross Formula

The construction of Section 3.2 in some cases provides a formula like Theorem 3.1. For example, let $p \neq 2$ a prime, B be the quaternion algebra ramified at p and ∞ and $\mathcal{O}$ a maximal order in B.

Theorem 3.4 (Gross Formula)**.** *Let $F \in S_2(\Gamma_0(p))$ and $v_F \in \mathcal{M}(\mathcal{O})$ be a Hecke eigenvector corresponding to F. If $-d$ is a negative fundamental discriminant, then*

$$L(F,1)L(F,-d,1) = \kappa \frac{\langle F, F\rangle}{\sqrt{d}} \frac{c_{d^2}(f)}{\langle v_F, v_F\rangle}$$

where $\kappa = 1$ if $p \nmid d$, $\kappa = 2$ if $p \mid d$ and $c_m(f)$ is the m-th Fourier coefficient of the weight $3/2$ modular form $f = \Theta(v_F)$.

See [Gr] for a proof. In particular, if $L(F,1) = 0$ then $\Theta(v_F) = 0$. Since there exists a fundamental discriminant $-d$ with $L(F,-d,1) \neq 0$ (see [Bu-Fr-Ho]), the converse is also true, i.e. $\Theta(v_F) = 0$ if and only if $L(F,1) = 0$. There is a generalization of this result for square free levels N on [Bö-SP], where they prove

Theorem 3.5. *Let $F \in S_2(\Gamma_0(N))$, let B be the quaternion algebra ramified exactly at the primes where $w_p = -1$ and R an Eichler order of level N. If $v_F \in \mathcal{M}(\mathcal{O})$ is a Hecke eigenvector corresponding to F, for $-d$ a negative fundamental discriminant*

$$\prod_{p \mid \frac{N}{\gcd(N,d)}} \left(1 + \left(\frac{-d}{p}\right)\right) L(F,1)L(F,-d,1) = 2^{\nu(N)} \frac{\langle F, F\rangle}{\sqrt{d}} \frac{c_{d^2}(f)}{\langle v_F, v_F\rangle}$$

where $\nu(N)$ is the number of prime divisors of N.

As in Gross' case, the weight $3/2$ modular form is zero if and only if $L(F,1) = 0$. In the level p^2 case a similar formula is conjectured (but not proved) in [Pa-To] with the same implications.

3.5. Further Developments

We would like to end this survey saying a few words about weight functions (they are introduced in [Ma-RV-To]). Assume that the level N is prime. Let l be a prime such that $l \nmid N$. Let (L, v) be a pair where L is an integral $\mathbb{Z}_l$-lattice of rank 3 with $l \nmid \det(L)$ and v satisfies $l \mid \Delta\, v$ but $v \notin lL$. The weight function $\omega_{l,v} : L \to \{0, \pm 1\}$ is defined by

$$w_{l,v}(v') = \begin{cases} 0 & \text{if } l \nmid \langle v', v' \rangle \\ \chi_l(\langle v, v' \rangle) & \text{if } l \nmid \langle v, v' \rangle \\ \chi_l(k) & \text{if } v' - kv \in lL. \end{cases}$$

Let B be the quaternion algebra ramified at N and ∞ and $R \subset B$ a maximal order. To $\mathfrak{a} \in \widetilde{\mathcal{I}}(R)$, associate the ternary lattice $L_{\mathfrak{a}} := \{x \in \mathbb{Z} + 2\mathcal{O}_r(\mathfrak{a}) : \operatorname{Tr}(x) = 0\}$. Fix $b_0 \in L_R$ such that $l \mid \mathcal{N}\, b_0$ but $b_0 \notin lL_R$. Let $a \in \mathfrak{a}$ be a generator of $\mathfrak{a}_l$, i.e. $\mathfrak{a}_l = R_l a$. For $x \in L_{\mathfrak{a}}$ define

$$w_l(\mathfrak{a}, x) := \chi_l(\mathcal{N}\, a/\mathcal{N}\, \mathfrak{a}) w_{l, a b_0 a^{-1}}(x).$$

This weight function does not depend on the generator a nor on the equivalence class of $\mathfrak{a}$. The effect of changing b_0 is to multiply all the weight functions $w_l([\mathfrak{a}], -)$ by the same factor. Abusing notation, for $[\mathfrak{a}] \in \mathcal{I}(R)$ we define the theta series

$$\Theta_{-l}([\mathfrak{a}]) = \sum_{x \in L_{\mathfrak{a}}} \omega_l([\mathfrak{a}], x) e^{2\pi i z\, \mathcal{N} x / l}$$

and extend it by linearity to $\mathcal{M}(R)$. This map satisfies the same conditions as Theorem 3.3 except that its image lies in $M_{3/2}(\Gamma_1(N))$.

The function Θ_{-l} is zero if $l \equiv 3 \pmod 4$ and a weight function at the ramified prime has to be considered (see Definition 2.2 of [Ma-RV-To]). In [Ma-RV-To] they conjecture a type of Gross formula for real discriminants, namely if d is a fundamental discriminant,

$$\sqrt{l}\, L(F, -l, 1) L(F, d, 1) = \frac{\langle F, F \rangle}{\sqrt{d}} \frac{c_{-l}(d^2)}{\langle v_F, v_F \rangle} \tag{3.1}$$

where $c_{-l}(m)$ are the Fourier coefficients of Θ_{-l}. In particular, if the L-series of F vanishes at 1, take a prime l such that $L(F, -l, 1) \neq 0$ (there exists such l by [Bu-Fr-Ho]) and use weight functions for l as before to construct a non-zero weight 3/2 modular form mapping to F by Shimura. Also, some kind of "Gross Formula" still holds.

These ideas generalize to more general levels. In [Pa-To2], the authors gave some examples of how to associate to a modular form of level N (not square free) $2^{\nu(N)+1}$ weight $3/2$ modular forms. The relation between the constructed forms and the central values of twisted L-series is as follows: let $\{q_1, \dots q_{\nu(N)}\}$ be the set of prime divisors of N (say increasingly ordered). For each vector $\varepsilon = (\varepsilon_1, \dots, \varepsilon_{\nu(N)+1})$ with $\varepsilon_i = \pm 1$, associate the set $\mathcal{P}_\varepsilon = \{d$ fundamental discriminant $: \chi_{q_i}(d) = \varepsilon_i\}$, where $\varepsilon_{\nu(N)+1}(d) = \operatorname{sign}(d)$. There is a bijection betweeen the set of weight $3/2$ constructed forms and the vectors ε, where if $f \leftrightarrow \varepsilon$ then a formula like (3.1) is conjecturally true for f and all discriminants on $\mathcal{P}_\varepsilon$ (as in Theorem 2.10). This conjecture was proved numerically on the examples considered.

In the same work, it was also shown how the weight functions can be used to work with a quaternion algebra ramified at one prime p (dividing the level) and ∞ without carying on the Atkin-Lehner involution signs. Hopefully this will give a simpler proof of a Gross type formula for the general case.

As was shown in [Bö-SP2] the case of modular forms of even higher weights is similar. One first constructs a vector on a quaternion algebra corresponding to the modular form by using spherical polynomials on the theta series. Then one defines a map Θ to half integral weight forms using (again) theta series with spherical polynomials. In the case of weights divisible by 4, the spherical polynomials used are odd functions, hence this construction is trivial! Using some weight functions on the ternary quadratic forms involved in [Ro-To] they showed how to handle this case as well.

The somehow open question is how to handle the case of weight two forms not appearing on quaternion algebras. A simple computation using the characterization of such forms (and Gauss sums) proves that if F has level p^2, then $L(F, d_1, 1)L(F, d_2, 1) = 0$ for all fundamental discriminants d_1, d_2 with d_1 positive and d_2 negative (because of the funtional equation sign). One example is the curve $E1369B$ in Cremona's notation. This case is quite interesting since there is no weight $3/2$ modular form mapping to $E1369B$ under the Shimura map. In particular although a formula like (3.1) seems to be wrong, it is not. Is this always the case? Is it true that for any level N if a modular form $F \in S_2(\Gamma_0(N))$ is not on $\mathcal{M}(R)$ for any order R on any quaternion algebra B then the central value of all the real quadratic or all the imaginary quadratic twists of F vanish? We suspect the answer should be "yes" (and some numerical experiments confirm this) although so far we do not have a proof.

References

[Bö-SP] BÖCHERER, S., SCHULZE-PILLOT, R.: On a theorem of Waldspurger and on Eisenstein series of Klingen type. *Math. Ann.* **288** (1990), 361–388.

[Bö-SP2] BÖCHERER, S., SCHULZE-PILLOT, R.: Vector valued theta series and Waldspurger's theorem. *Abh. Math. Sem. Univ. Hamburg* **64** (1994), 211–233.

[Bu-Fr-Ho] D. BUMP, S. FRIEDBERG AND J. HOFFSTEIN: Nonvanishing theorems for L-functions of modular forms and their derivatives. *Invent. Math.* **102** (1990), 543–618.

[Co-Wi] J. COATES AND A. WILES: On the conjecture of Birch and Swinnerton-Dyer. *Invent. Math.* **39** (1977), 223–251.

[Ei] EICHLER, M.: The basis problem for modular forms and the traces of the Hecke operators. *Lecture Notes in Math.* **320** (1973), 75–151.

[Fl] FLICKER, YUVAL Z.: Automorphic forms on covering groups of GL(2) *Invent. Math.* **57** (1980), 119–182.

[Gr] GROSS, BENEDICT H.: Heights and the special values of L-series *Number theory (Montreal, Que., 1985), CMS Conf. Proc.* **7** (1987), 115–187.

[Hi-Pi-Sh] HIJIKATA, HIROAKI AND PIZER, ARNOLD K. AND SHEMANSKE, THOMAS R.: The basis problem for modular forms on $\Gamma_0(N)$ *Mem. Amer. Math. Soc.* **82** Number **418** (1989), 115–187.

[J-L] JACQUET, H. AND LANGLANDS, R. P.: Automorphic forms on GL(2) *Lecture Notes in Mathematics* **114** (1970).

[Jo] STEFAN JOHANSSON: A Description of Quaternion algebras `http://www.math.chalmers.se/~sj/forskning.html`

[Ko] KOHNEN, WINFRIED: Newforms of half-integral weight *J. Reine Angew. Math.* **333** (1982), 32–72.

[Ko2] KOHNEN, WINFRIED: Fourier coefficients of modular forms of half-integral weight *Math. Ann.* **271** (1985), 237–268.

[Ma-RV-To] MAO, Z. AND RODRIGUEZ-VILLEGAS, F. AND TORNARÍA, G.: Computation of central value of quadratic twists of modular L-functions *Ranks of elliptic curves and random matrix theory, London Math. Soc. Lecture Note Ser.* **341** (2007), 273–288.

[Ni] NIWA, SHINJI: Modular forms of half integral weight and the integral of certain theta-functions *Nagoya Math. J.* **56** (1975), 147–161.

[Pa-To] PACETTI, ARIEL AND TORNARÍA, GONZALO: Shimura correspondence for level p^2 and the central values of L-series *J. Number Theory* **124** 2 (2007), 396–414.

[Pa-To2] PACETTI, ARIEL AND TORNARÍA, GONZALO: Computing central values of twisted L-series, The case of composite levels *To appear in Exp. Math.*

[Pi] PIZER, ARNOLD: An algorithm for computing modular forms on $\Gamma_0(N)$ *Journal of Algebra* **64** (1980), 340–390.

[Ro-To] ROSSON, HOLLY AND TORNARÍA, GONZALO: Central values of quadratic twists for a modular form of weight 4 *Ranks of elliptic curves and random matrix theory, London Math. Soc. Lecture Note Ser.* **341** (2007), 315–321.

[Sh] Shimura, Goro: On modular forms of half integral weight. *Ann. of Math. (2)* **97** (1973), 440–481.

[Sh2] Shimura, Goro: The critical values of certain zeta functions associated with modular forms of half-integral weight *Journal of the Mathematical Society of Japan* **33** (1981), 649–672.

[Shi] Shintani, Takuro: On construction of holomorphic cusp forms of half integral weight *Nagoya Mathematical Journal* **58** (1975), 83–126.

[Tu] Tunnell, J. B.: A classical Diophantine problem and modular forms of weight 3/2 *Invent. Math.* **72** 2 (1983), 323–334.

[RV] Rodriguez-Villegas, F.: Computing central values of L-functions *Ranks of elliptic curves and random matrix theory, London Math. Soc. Lecture Note Ser.* **341** (2007), 260–271.

[Wa] Waldspurger, J.-L.: Sur les coefficients de Fourier des formes modulaires de poids demi-entier *Journal de Mathématiques Pures et Appliquées. Neuvième Série* **60** Number **4** (1981), 375–484.

[1] Knapp, Anthony W., *Elliptic curves*. Mathematical Notes **40**. Princeton University Press, 1992.

[2] Koblitz, Neal, *Introduction to elliptic curves and modular forms*. Graduate Texts in Mathematics **97**. Springer-Verlag, New York, 1993.

[3] Shimura, Goro: *Introduction to the arithmetic theory of automorphic functions*. Publications of the Mathematical Society of Japan **11**. Princeton University Press, 1994.

Ariel Pacetti
Departamento de Matemáticas
Universidad de Buenos Aires
Pabellón I, Ciudad Universitaria. C.P:1428
Buenos Aires, Argentina
apacetti@dm.uba.ar

The author was supported by the “Red Iberoamericana de Teoría de Números” of the UAM-Grupo Santander 2007-08 (Spain)