ON GALOIS REPRESENTATIONS OF SUPERELLIPTIC CURVES

ARIEL PACETTI AND ANGEL VILLANUEVA

Abstract. A superelliptic curve over a DVR $\mathscr{O}$ of residual characteristic p is a curve given by an equation $\mathscr{C} : y^n = f(x)$. The purpose of the present article is to describe the Galois representation attached to such a curve under the hypothesis that $f(x)$ has all its roots in the fraction field of $\mathscr{O}$ and that $p \nmid n$. Our results are inspired on the algorithm given in [BW17] but our description is given in terms of a cluster picture as defined in [DDMM18].

Introduction

Galois representations play a crucial role in different aspects of modern number theory. The main source of Galois representations are the geometric ones, namely the ones obtained by looking at the étale cohomology of varieties. Among varieties, the case of curves is the easiest one, where one can understand the local L-function of a curve at a prime of good reduction by counting the number of points of the curve's equation over different finite fields. If the curve has bad reduction at the prime p, understanding the image of inertia in the ℓ-adic representation (for $\ell \neq p$) is much harder. However the situation is a little better when the curve is "semistable". There have been very important results in this direction during the last years (see [Dok18] and [DDM18]). Specific results for hyperelliptic curves are given in [DDMM18] and for the so called *superelliptic curves* in [BW17].

Let $\mathscr{O}$ be a complete DVR, π be a local uniformizer, K its field of fractions and k its residue field of characteristic p.

Definition. An *n-cyclic* or *superelliptic* curve is a non-singular curve obtained as a cyclic cover of $\mathbb{P}^1$, namely it is given by an equation of the form

$$\mathscr{C} : y^n = f(x),$$

where $f(x) \in \mathscr{O}[x]$, $\operatorname{Disc}(f(x)) \neq 0$.

Computing the local L-factor of $\mathscr{C}$ for primes of good reduction can be done quite efficiently using for example the method described in [Sut20], which gives a full description of the attached Galois representation in this case (as inertia acts trivially). For this reason, the main purpose of the present article is to describe the Galois image of inertia at p of the ℓ-adic representation of a curve $\mathscr{C}$ for $\ell \neq p$ (assuming $p \nmid n$). As a first step of this goal, we restrict to the case when $f(x)$ has all its roots in K (which is not a semistable situation, but closed to it). Over an extension K_2 of K the stable model contains many components $\mathcal{Y}_t^{(i)}$ (the notation will be explained during the article) hence the Galois representation splits as

$$V_\ell(\operatorname{Jac}(\mathscr{C})) \simeq \bigoplus V_\ell(\mathcal{Y}_t^{(i)}) \oplus \bigoplus_i \operatorname{St}(2) \otimes \chi_i, \tag{1}$$

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where $\mathrm{St}(2)$ is the Steinberg 2-dimensional representation, and the χ_i are characters. To get such a representation over K, we first study the stable model over K_2, following the results of [BW17] (where a stable model $\mathscr{Y}$ of $\mathscr{C}$ is given). Our first contribution is to relate their description to the more combinatorial one given in [DDMM18] for hyperelliptic curves in terms of *clusters*.

The way to get a stable model is to start with a stable model of a marked projective line, and compute its normalization under the natural projection of $\mathscr{C}$ to $\mathbb{P}^1$ (giving the stable model $\mathscr{Y}$). The use of clusters provides a stable model of $\mathbb{P}^1$ as explained in [DDMM18] and does not depend on the degree of the cover n. During the exposition, we make an explicit passage from clusters to triples as considered in [BW17].

The new phenomena appearing when $n > 2$ is that the normalization of projective lines might not be irreducible. This is a very interesting phenomena that we explain in Section 3. Unlike the hyperelliptic case, components not only might have positive genus (hence each of them contribute to the first part of (1)), but the different components intersect between themselves in a very combinatorial way. Understanding the intersection points is crucial to describe the component graph of the special fiber of $\mathscr{Y}$. Furthermore, the Galois action on the intersection points give rise to the characters appearing in the second part of (1) (i.e. are the characters twisting the Steinberg representation).

Let us explain the organization of the article. Let $\mathscr{R}$ denote the set of roots of $f(x)$. In this article we assume that $\mathscr{R} \subset K$ and $p \nmid n$. Since we are mostly concerned with the image of inertia (and $p \nmid n$), we also assume that $\zeta_n \in K$. Let $K_2 = K[\sqrt[n]{\tilde{\pi}}]$, where $\tilde{\pi}$ is a local uniformizer of K. In the first part of the article, we assume furthermore that $f(x)$ is a monic polynomial, in particular $f(x) = \prod_{r \in \mathscr{R}}(x - r)$.

The first section recalls the description of the stable model $(\mathscr{X}, \mathscr{D})$ of the marked $\mathbb{P}^1$ as given in [BW17]. The second section explains its relation to the cluster picture of [DDMM18] via a map from $(\mathscr{X}, \mathscr{D})$ to proper clusters. In particular, we explain how points relate to clusters, and how the different components intersect (Theorem 2.7).

In Section 3 we describe the semistable model $\mathscr{Y}$ over K_2. In particular, we give a description of the number of components in terms of the clusters and their cardinality (Proposition 3.1). The advantage of working over K_2 is that the cluster picture itself is enough to describe the representation. The last section explains how to compute the Galois representation over K. For doing that, we explain the effect of twisting the Galois representation attached to a superelliptic curve. To describe the twist, we explain the decomposition of $V_\ell(\mathrm{Jac}(\mathscr{C}))$ as a module not over $\mathbb{Z}_\ell$ but over $\mathbb{Z}_\ell[\xi_n]$. There is a contribution of the Galois representation coming from the curves

$$\mathscr{C}_{n/d} : y^{n/d} = f(x),$$

for each divisor d of n. Then our Galois representation splits as a sum of what might be called the d-new contributions. In Section 5 we give a precise description of how to compute the twist on each d-new subrepresentation.

The advantage of working with general twists is that it allows to consider a general polynomial $f(x)$ (not necessarily monic) over K (with the assumption $\mathscr{R} \subset K$). To understand the restriction of the Galois representation to the inertia subgroup, it is very convenient to work with the notion of weighted clusters as recalled in the same section. The complete weighted cluster picture contains information on whether the characters χ_i are ramified or not, as explained in the last part of the article.

Notation. Let us give a small list of the principal symbols that will be used in the article:

- By π we denote a local uniformizer of $\mathscr{O}$, k its residue field and $v(x)$ denote the π-valuation of x.
- The symbol $\mathscr{R}$ denotes the roots of the polynomial $f(x)$.
- $(\mathscr{X}, \mathscr{D})$ denotes the semistable model of the marked $(\mathbb{P}^1, D)$ line and $\mathscr{Y}$ denotes its normalization in the function field of $\mathscr{C}$.
- $\overline{X}$ denotes the special fiber of $\mathscr{X}$ and $\overline{Y}_t^{(\ell)}$ denote the components of the special fibers of $\mathscr{Y}$.

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1. The minimal stable model $(\mathscr{X}, \mathscr{D})$

The method to compute the stable minimal model is as follows: consider the cover $p : \mathscr{C} \to \mathbb{P}^1$ obtained by sending $(x, y) \to x$. This is a cyclic Galois cover of degree n ramified precisely at the points

$$D = \sum_{r \in \mathscr{R}} [r] + \begin{cases} [\infty] & \text{if } n \nmid \deg(f(x)), \\ 0 & \text{otherwise.} \end{cases}$$

Consider the marked curve $(\mathbb{P}^1, D)$ and compute its minimal semistable model $(\mathscr{X}, \mathscr{D})$. A semistable model of $\mathscr{C}$ is then obtained as the normalization $\mathscr{Y}$ of $\mathscr{X}$ in the function field of $\mathscr{C}$, i.e. $\mathscr{Y}$ fits in the following diagram

$$\begin{tikzcd} \mathscr{Y} \arrow[r, dotted] \arrow[d, dotted] & \mathscr{C} \arrow[d, "p"] \\ \mathscr{X} \arrow[r] & \mathbb{P}^1 \end{tikzcd}$$

The stable model $(\mathscr{X}, \mathscr{D})$ is obtained by gluing open affine lines blown up at a point in the special fiber of $\mathbb{P}^1$ (as explained in [DDMM18, Section 3],[BW17] and also [Liu02]). Recall the algorithm given in [BW17] to compute $(\mathscr{X}, \mathscr{D})$. Let

$$S = \mathrm{Supp}(D) = \mathscr{R} \cup \begin{cases} \infty & \text{if } n \nmid \deg(f(x)), \\ \emptyset & \text{otherwise.} \end{cases} \tag{2}$$

Remark 1.1. The assumption that $\mathscr{C}$ has positive genus implies in particular that S has at least 3 elements, as this is always the case except when $f(x)$ has degree 1 or when $f(x)$ has degree 2 and $n = 2$. In both cases the curve $\mathscr{C}$ has genus 0.

Let T denote the set of triples of distinct elements of S. The coordinate function of $t = (a, b, c) \in T$, is defined as

$$\varphi_t(x) = \frac{(b-c)}{(b-a)} \frac{(x-a)}{(x-c)}. \tag{3}$$

It corresponds to the Möbius transformation sending (a, b, c) to $(0, 1, \infty)$. Define the following equivalence relation in T: two elements $t_1, t_2 \in T$ are equivalent (which we denote $t_1 \sim t_2$) if the map $\phi_{t_2} \circ \phi_{t_1}^{-1}$ extends to an automorphism of $\mathbb{P}^1_{\mathscr{O}}$ (i.e. it corresponds to a matrix in $\mathrm{PGL}_2(\mathscr{O})$).

The semistable model $\mathscr{X}$ consists of one component (a projective line) for each equivalence class. Furthermore, the special fiber $\bar{X}$ of $\mathscr{X}$ is a tree of projective lines where each component contains at least 3 points (being either elements of S or singular points were two components meet).

For each $t \in T$ the map φ_t extends to a proper $\mathscr{O}$-morphism $\varphi_t : \mathscr{X} \to \mathbb{P}^1_{\mathscr{O}}$, whose reduction (denoted $\overline{\varphi_t}$) is a contraction morphism with contracts all but one component of $\bar{X}$ to a closed point (see [BW17, Proposition 4.2]). Furthermore, if $r \in \mathscr{R}$ then $\overline{\varphi_t}(r) = \overline{\varphi_t(r)}$. Extend the valuation v on $\mathscr{O}$ by setting $v(\infty) = -\infty$.

Lemma 1.2. *The equivalence relation in T satisfies the following properties:*

1. *The permutation of a triple (a, b, c) is equivalent to (a, b, c).*
2. *Any triple is equivalent to one with $v(b - c) = v(a - c) \leq v(a - b)$.*

Proof. The first assertion follows from a straightforward matrix computation. For the second one, note that the map from triples of elements to triples of rational numbers given by $(a, b, c) \to (v(b - c), v(a - c), v(a - b))$ is S_3 invariant, hence we can assume $v(b - c) \leq v(a - c) \leq v(a - b)$. But $a - b = (a - c) + (c - b)$ hence

$$v(a - b) \geq \min\{v(a - c), v(b - c)\} = v(b - c)$$

with equality if both values are different. Then the assumption $v(b-c) \leq v(a-c) \leq v(b-c)$ implies that $v(a - c) = v(b - c)$. $\square$

Definition 1.3. An *ordered triple* is a triple (a, b, c) with $v(a - c) = v(b - c) \leq v(a - b)$.

If (a, b, c) is an ordered triple, define its *radius* to be $\mu = v(a - b)$.

Proposition 1.4. *Let (a, b, c) be an ordered triple.*

1. *If $\infty \in S$ then any ordered triple (a, b, c) is equivalent to the ordered triple (a, b, ∞).*
2. *The radius μ depends only on the equivalent class of the triple.*
3. *The ordered triple (a, b, c) is equivalent to the ordered triple (α, β, γ), if and only if the following two properties hold:*
   - *they have the same invariant, i.e. $\mu = v(a - b) = v(\alpha - \beta)$,*
   - $a \equiv b \equiv \alpha \equiv \beta \pmod{\pi^\mu}$.

Proof. By the equivalence relation definition, an ordered triple $T_1 = (a, b, c)$ (with $c \neq \infty$) is equivalent to a triple $T_2 = (a, b, \infty)$ if and only if $\lambda_2 \circ \lambda_1^{-1}$ extends to an automorphism of $\mathbb{P}^1_{\mathscr{O}}$, where λ_i is the Möbius transformation sending the triple T_i to the triple $(0, 1, \infty)$. The Möbius matrix attached to such composition equals

$$[\lambda_2 \circ \lambda_1^{-1}] = \frac{1}{(a - c)} \begin{pmatrix} \frac{(a-c)}{(b-c)} & 0 \\ \frac{(a-b)}{(b-c)} & 1 \end{pmatrix}.$$

If we multiply the matrix by $(a - c)$ we get an invertible integral matrix, since $\frac{(a-c)}{(b-c)}$ is a unit (recall the definition of an ordered triple), hence both triples are indeed equivalent.

To prove equivalence of ordered triples (a, b, c), (α, β, γ), since we can add ∞ to the set, it is enough to restrict to the case (a, b, ∞) and (α, β, ∞). It is easy to check that the transformation sending one triple to the other one is given by the matrix

$$M = \begin{pmatrix} (a-b) & (\alpha - a) \\ 0 & (\alpha - \beta) \end{pmatrix}.$$

For a multiple of M to lie in $\mathrm{GL}_2(\mathscr{O})$, it must happen that $v(a-b) = v(\alpha - \beta)$, hence the two triples have the same *radius*, as stated. At last, under such assumption, the two triples are equivalent if and only if $v(a-\alpha) \geq \mu$ (the radius of the triples). Recall that $\mu = v(a-b)$, so $a \equiv b \pmod{\pi^\mu}$ and the same holds for α and β, as stated.

□

Remark 1.5. Given a cyclic curve $\mathscr{C} : y^n = f(x)$, there are many transformations that preserve the model (for example translation). The combinatory behind the computation of a stable model of $\mathbb{P}^1$ attached to the roots of $p(x)$ depends on the particular equation. However, the information obtained from it (number of components, discriminant, etc) does not.

Before stating the relation between equivalence classes of triples and clusters, let us illustrate the algorithm with an example.

Example 1. Let p be an odd prime number and $\mathscr{C}/\mathbb{Q}_p$ be the superelliptic curve given by the equation

$$\mathscr{C} : y^6 = x(x-p^2)(x-p)(x-p-p^2)(x-2p)(x-2p-p^2)(x-1)(x-1-p)(x-1-2p)$$

Then $\mathscr{R} = \{0, p^2, p, p+p^2, 2p, 2p+p^2, 1, 1+p, 1+2p\}$ and $S = \mathscr{R} \cup \{\infty\}$ (since $6 \nmid \deg(f)$). By Proposition 1.4 any ordered triple (a,b,c) is equivalent to the ordered triple (a,b,∞), and there are 36 such triples. The radii are given in Table 1.

Pair	$(0,1)$	$(0,1+p)$	$(0,1+2p)$	$(p^2,1)$	$(p^2,1+p)$	$(p^2,1+2p)$
Radius	0	0	0	0	0	0
Pair	$(p,1)$	$(p,1+p)$	$(p,1+2p)$	$(p+p^2,1)$	$(p+p^2,1+p)$	$(p+p^2,1+2p)$
Radius	0	0	0	0	0	0
Pair	$(2p,1)$	$(2p,1+p)$	$(2p,1+2p)$	$(2p+p^2,1)$	$(2p+p^2,1+p)$	$(2p+p^2,1+2p)$
Radius	0	0	0	0	0	0
Pair	$(0,p)$	$(0,p+p^2)$	$(0,2p)$	$(0,2p+p^2)$	(p^2,p)	$(p^2,p+p^2)$
Radius	1	1	1	1	1	1
Pair	$(p^2,2p)$	$(p^2,2p+p^2)$	$(p,2p)$	$(p,2p+p^2)$	$(p+p^2,2p)$	$(p+p^2,2p+p^2)$
Radius	1	1	1	1	1	1
Pair	$(0,p^2)$	$(p,p+p^2)$	$(2p,2p+p^2)$	$(1,1+p)$	$(1,1+2p)$	$(1+p,1+2p)$
Radius	2	2	2	1	1	1

TABLE 1.

By Proposition 1.4 (3), all elements in the first three rows are equivalent, all elements in fourth and fifth rows are equivalent, the three first elements of the last row are not equivalent, and the last three elements in the last row are equivalent, hence there are six equivalent classes. The ordered triples and the charts can be taken to be:

- $t_0 = (0,1,\infty),\ \varphi_0(x) = x$
- $t_1 = (0,p,\infty),\ \varphi_1(x) = \frac{x}{p}$
- $t_2 = (0,p^2,\infty),\ \varphi_2(x) = \frac{x}{p^2}$
- $t_3 = (p,p+p^2,\infty),\ \varphi_3(x) = \frac{x-p}{p^2}$
- $t_4 = (2p,2p+p^2,\infty),\ \varphi_4(x) = \frac{x-2p}{p^2}$
- $t_5 = (1,1+p,\infty),\ \varphi_4(x) = \frac{x-1}{p}$

Then the special fiber of $\mathscr{X}$ looks like Figure 1.

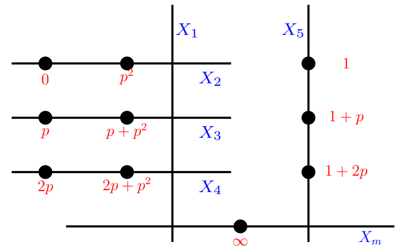


FIGURE 1. Special fiber of $\mathscr{X}$

2. Clusters and their relation with $\mathscr{X}$

Clusters were defined in [DDMM18] to study hyperelliptic curves. We strongly recommend the reader to look at such article and also the expository article [BBB$^+$20]. We follow closely the definition and notations presented in such references.

Definition 2.1. A *cluster s* is a non-empty subset of $\mathscr{R}$ of the form $\mathfrak{s} = D(z,d) \cap \mathscr{R}$, for some disc $D(z,d) = \{x \in \overline{K} \ : \ v(x-z) \geq d\}$ where $z \in \overline{K}$ and $d \in \mathbb{Q}$. A *proper* cluster is a cluster with more than one element.

We will mostly be interested in proper clusters for most of the discussions. Let $\mathrm{Cl}(\mathscr{R})$ denote the set of proper clusters of $\mathscr{R}$. For a cluster $\mathfrak{s}$, let $|\mathfrak{s}|$ denote the number of elements of $\mathscr{R}$ contained in $\mathfrak{s}$.

Lemma 2.2. *Given $\mathfrak{s}_1, \mathfrak{s}_2$ clusters, then either they are disjoint or one is contained in the other.*

Proof. The result follows from the fact that any point inside the disc defining a cluster can be taken as the ball center (see [DDMM18, Section 1.5]). □

Definition 2.3. If $\mathfrak{s}, \mathfrak{s}'$ are clusters with $\mathfrak{s}' \subsetneq \mathfrak{s}$ a maximal subcluster, we write $\mathfrak{s}' < \mathfrak{s}$ and refer to $\mathfrak{s}'$ as a *child* of $\mathfrak{s}$ and $\mathfrak{s}$ as a *parent* of $\mathfrak{s}'$.

To a proper cluster $\mathfrak{s}$ we associate its *diameter* $\mu(\mathfrak{s}) = \min\{v(z-t) \ : \ z,t \in \mathfrak{s}\}$ (in [DDMM18] the authors use term *depth* for such invariant).

Lemma 2.4. *Let $\mathfrak{s}$ be a proper cluster, and let $a, b \in \mathfrak{s}$ two elements satisfying that $v(a-b) = \mu(\mathfrak{s})$. Then $\mathfrak{s} = D(a, \mu(\mathfrak{s})) \cap \mathscr{R}$.*

Proof. Clearly, $v(a-b) \geq \mu(\mathfrak{s})$ for all $b \in \mathfrak{s}$, hence $\mathfrak{s} \subset D(a, \mu(\mathfrak{s})) \cap \mathscr{R}$. For the other inclusion, by definition $s = D(\alpha, d) \cap \mathscr{R}$, for some α, d. Since $a \in \mathfrak{s}$, $a \in D(\alpha, d)$, so we can take it as center, i.e. $s = D(a,d) \cap \mathscr{R}$. But $\mu(\mathfrak{s})$ is the minimal valuation between elements in s hence $d \geq \mu(\mathfrak{s})$ and $D(a, \mu(\mathfrak{s})) \cap \mathscr{R} \subset \mathfrak{s}$. □

Definition 2.5. Let $s_{\max}$ be the maximal cluster, i.e. the cluster containing all other clusters and all elements of S.

Let (a,b,c) be an ordered triple in T, and let $\mu = v(a-b)$ be its invariant. Define a map $\Phi : T \to \mathrm{Cl}(\mathscr{R})$ by

$$\Phi((a,b,c)) = D_\mu(a) \cap \mathscr{R}. \tag{4}$$

Theorem 2.6. *The map Φ gives a well defined map between the equivalence classes of triples in $T/\sim$ and the set of clusters of $\mathscr{R}$. Furthermore, the map Φ satisfies the following properties:*

(1) *It is injective.*
(2) *The set $\mathrm{Cl}(\mathscr{R}) \setminus \{s_{max}\} \subset \mathrm{Im}(\Phi)$.*
(3) *The cluster s_{max} lies in the image of Φ if either one of the following properties hold:*
   i. *The element $\infty \in S$,*
   ii. *there are three different elements $a, b, c \in \mathscr{R}$ satisfying*

$$\mu(s_{max}) = v(a-b) = v(b-c) = v(a-c)$$

   (equivalently, s_{max} has more than two childs).

Proof. For Φ to descend to a map on the quotient $T/\sim$, we need to prove that if $t_1 = (a, b, c)$ and $t_2 = (\alpha, \beta, \gamma)$ are equivalent ordered triples of T then $\Phi(t_1) = \Phi(t_2)$. By Proposition 1.4, the condition $t_1 \sim t_2$ implies that $\mu = v(a-b) = v(\alpha-\beta)$ and $a \equiv b \equiv \alpha \equiv \beta \mod \pi^\mu$. Then $\alpha \in D_\mu(a)$ hence $D_\mu(a) = D_\mu(\alpha)$ and $\Phi(t_1) = \Phi(t_2)$.

(1) **Injectivity:** let $t_1 = (a, b, c)$ and $t_2 = (\alpha, \beta, \gamma)$ be two ordered triples such that $\Phi(t_1) = \Phi(t_2)$. Note that

$$v(a-b) = \mu(\Phi(t_1)) = \min\{v(z-t) \ : \ z, t \in \Phi(t_1)\}. \tag{5}$$

Then we can recover the invariant μ of the triple t_1 as the diameter of $\Phi(t_1)$. Since $\Phi(t_1) = \Phi(t_2)$, the μ invariant of $\Phi(t_2)$ equals that of $\Phi(t_1)$. On the other hand, since $\Phi(t_1) = \Phi(t_2)$, $\{\alpha, \beta\} \subset \Phi(t_2)$ so $a \equiv b \equiv \alpha \equiv \beta \pmod{\pi^\mu}$ and $t_1 \sim t_2$ by Proposition 1.4.

(2) Let $\mathfrak{s} \in \mathrm{Cl}(\mathscr{R})$ be a proper cluster which is not maximal. Let $a, b \in s$ be a pair such that $v(a-b) = \mu(s)$. Clearly $\mathfrak{s} = D_\mu(a) \cap \mathscr{R}$. Let $c \notin s$, then $\mathfrak{s} = \Phi(a, b, c)$.

(3) As before, given $s_{\max}$, let $a, b \in s_{\max}$ be a pair such that $v(a-b) = \mu(s_{\max})$.
   i. If $\infty \in S$, then $s_{\max} = \Phi(a, b, \infty)$.
   ii. If there exists $a, b, c \in s_{\max}$ with $v(a-b) = v(a-c) = v(b-c)$ then $s_{\max} = \Phi(a, b, c)$.

   Finally, if $s_{\max}$ has precisely two childs, we need to prove it does not lie in the image. Let $\mathfrak{s}_1$ and $\mathfrak{s}_2$ be the maximal subclusters. Any triple (a, b, c) satisfies (without loss of generality) that two elements lie in $\mathfrak{s}_1$ and the other in $\mathfrak{s}_2$ or the three of them lie in $\mathfrak{s}_1$. In both cases it is easy to check that $\Phi(a, b, c) \subset \mathfrak{s}_1$, hence $s_{\max}$ is not in the image.

□

Example 2. The case when $s_{\max}$ is not in the image of Φ corresponds to a model $(\mathscr{X}, \mathscr{D})$ with precisely two lines intersecting in a single point. For example, if $p > 3$ is a prime number, the curve with equation:

$$\mathscr{C} : y^6 = x(x-p)(x-1)(x-1+p)(x-1+2p)(x-1+3p).$$

In Figure 2 we present its cluster picture as well as the model $(\mathscr{X}, \mathscr{D})$.

Following the description in Section 1, we know that the components of $\mathscr{X}$ correspond to elements in $T/\sim$, hence to get a complete description of $\mathscr{X}$ we need to understand how the components intersect with each other and how the points of S distribute between the

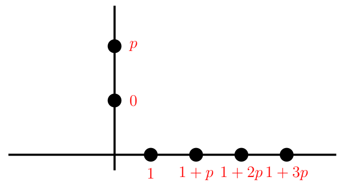


FIGURE 2. Special fiber of $\mathscr{X}$

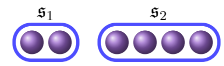


FIGURE 3. Attached clusters

components. Representing elements as clusters gives the natural answer, namely in general two components will intersect precisely when the cluster attached to one of them is a child of the other. More concretely,

Theorem 2.7. *The components attached to clusters $\mathfrak{s}_1, \mathfrak{s}_2$ in the image of Φ intersect if and only if one of the following holds:*

(1) *$\mathfrak{s}_1$ is a maximal subcluster of $\mathfrak{s}_2$ or vice-versa, or*
(2) *$\mathfrak{s}_1 \cap \mathfrak{s}_2 = \emptyset$ and both $\mathfrak{s}_1, \mathfrak{s}_2$ are maximal clusters.*

The second case corresponds precisely to the case explained in Example 2. The statement is implicit in [DDMM18] (see Section 5 and Theorem 1.10) as well as in [BW17], but we present a different proof which depends on understanding especial points on clusters and the coordinate functions evaluated at them.

Definition 2.8. Let $\mathfrak{s} \in \mathrm{Cl}(\mathscr{R})$. A *point* of $\mathfrak{s}$ is one of the following:

- a child of $\mathfrak{s}$ (i.e. maximal subclusters of $\mathfrak{s}$),
- a parent of $\mathfrak{s}$ (i.e. a minimal supercluster of $\mathfrak{s}$),
- If $\infty \in S$, one point (denoted ∞) in $s_{\max}$.
- If $s_{\max} \notin \mathrm{Im}(\Phi)$ one extra point in each maximal cluster of $\mathrm{Im}(\Phi)$ (corresponding to the intersection point of the two components, see Example 2).

Note that we did not ask the clusters to be proper while defining points. In particular, the elements of S will be points in some clusters. The component graph of the special fiber $\bar{X}$ of $\mathscr{X}$ is a stably marked tree (see section 4.2 of [BW17]). In particular, each cluster contains at least 3 points. Let $\mathcal{P}$ be the disjoint union of points in proper clusters $\mathfrak{s} \in \mathrm{Cl}(\mathscr{R})$, i.e.

$$\mathcal{P} = \bigsqcup_{\mathfrak{s} \in \mathrm{Cl}(\mathscr{R})} \{\text{points in } \mathfrak{s}\}.$$

In particular, the set $\mathscr{R} \subset \mathcal{P}$. To make computations easier, we can (and will) assume that if $\mathfrak{s}$ is a non-maximal cluster, then the ordered triple $t = (a, b, c) \in T$ mapping to it via Φ satisfies that its last coordinate does not lie in $\mathfrak{s}$ (this is always the case up to an equivalent triple). Furthermore, if $\infty \in S$, we assume that $c = \infty$.

Lemma 2.9. *Let $t \in T$, and $x_1, x_2 \in \mathscr{R}$ be roots. Let $\mathfrak{s} = \Phi(t)$ be the associate cluster. Then the coordinate function φ_t satisfies:*

(1) *if there exists a proper subcluster $\tilde{\mathfrak{s}} \subsetneq \mathfrak{s}$ such that $x_1, x_2 \in \tilde{\mathfrak{s}}$ then $\overline{\varphi_t(x_1)} = \overline{\varphi_t(x_2)}$.*
(2) *if $x_1, x_2 \in \mathfrak{s}$ but they do not lie in a common maximal subcluster then $\overline{\varphi_t(x_1)} \neq \overline{\varphi_t(x_2)}$.*
(3) *if $x_1 \notin \mathfrak{s}$ then $\overline{\varphi_t(x_1)} = \infty$.*

Proof. Let $t = (a, b, c)$, and consider first the case when $\Phi(t)$ is not the maximal cluster (hence $c \notin \mathfrak{s}$). By definition,

$$\varphi_t(x_1) - \varphi_t(x_2) = \frac{(b-c)}{(b-a)} \frac{(x_1 - x_2)(c-a)}{(x_1 - c)(x_2 - c)}. \tag{6}$$

(1) Let $x_1, x_2 \in \tilde{\mathfrak{s}}$. Recall that the valuation of the difference between one element in $\mathfrak{s}$ and one element outside $\mathfrak{s}$ is constant, then since $c \notin \mathfrak{s}$, $v(b-c) = v(c-a) = v(x_1 - c) = v(x_2 - c)$. The hypothesis that x_1, x_2 lie in a proper subcluster implies that $v(x_1 - x_2) > v(a - b) = \mu(\Phi(t))$ so the right hand side of (6) is divisible by π and $\varphi_t(x_1) \equiv \varphi_t(x_2) \pmod{\pi}$.

(2) If x_1 and x_2 do not lie in a proper subcluster, then $v(x_1 - x_2) = v(a - b)$, hence the right hand side of (6) is a unit, hence $\varphi_t(x_1) \not\equiv \varphi_t(x_2) \pmod{\pi}$.

(3) If $x_1 \notin \mathfrak{s}$, $\varphi_t(x_1) = \frac{(b-c)}{(b-a)} \frac{(x_1-a)}{(x_1-c)}$. The non-arquimedean triangle inequality implies that $v(x_1 - c) = \min\{v(x_1 - a), v(b-c)\}$ (recall that $v(a-c) = v(b-c)$). On the other hand, since (a, b, c) is an ordered triple, $v(b-a)$ is the cluster's diameter. The hypothesis $x_1, c \notin \mathfrak{s}$ imply that $v(b-a) > \max\{v(b-c), v(x_1 - a)\}$. Then $v(x_1 - c)v(b-a) > v(x_1 - a)v(b-c)$ and consequently $\overline{\varphi_t(x_1)} = \infty$.

Assume on the contrary that $\Phi(t)$ is the maximal cluster, hence $v(b-c) = v(a-b) = v(a-c)$. Distinguish two cases depending on whether x_1, x_2, c belong to a common proper subcluster or not. In the first case, $v(x_1-c) > v(x_1-a)$ for $i = 1, 2$ since x_i lies in the same subcluster as c, so $\overline{\varphi_t(x_i)} = \infty$. In particular, if x_1, x_2 lie in a common proper subcluster, $\overline{\varphi_t(x_1)} = \overline{\varphi_t(x_2)}$.

In the second case, if x_1 is not in the same proper subcluster as c, $v(x_1 - c) \leq v(x_1 - a)$ hence $\overline{\varphi_t(x_1)} \neq \infty$. If x_1, x_2 are two roots not in the same proper subcluster as c, $v(b-c) = v(c-a) = v(b-a) = v(x_1 - c) = v(x_2 - c)$ and the same proof as before applies. $\square$

2.1. **Functions on clusters.** If $s \in \mathrm{Cl}(\mathscr{R})$ lies in the image of Φ, define a function $\varphi_{\mathfrak{s}}$: $\mathcal{P} \to \mathbb{P}^1$ extending the coordinate function φ_t to $\mathcal{P}$ as follows.

Definition 2.10. Let $\mathfrak{s} \in \mathrm{Cl}(\mathscr{R})$ be in the image of Φ, say $\mathfrak{s} = \Phi(t)$ and let $p \in \mathcal{P}$ be a point, i.e. p is a point in a cluster $\tilde{\mathfrak{s}} \in \mathrm{Cl}(\mathscr{R})$. In particular, p is either a root (i.e. an element of $\mathscr{R}$) or $p = \mathfrak{s}'$ a parent/child of $\tilde{s}$. Define

$$\overline{\varphi_{\mathfrak{s}}}(p) = \begin{cases} \overline{\varphi_t}(\alpha) & \text{if } p = \alpha \in \mathscr{R} \text{ and } \alpha \in \mathfrak{s}, \\ \overline{\varphi_t}(a) & \text{if } \mathfrak{s}' = \Phi((a,b,c)) \subset \mathfrak{s}, \\ \infty & \text{otherwise.} \end{cases}$$

Remark 2.11. If $\infty \in S$, the point $\infty \in s_{\max}$ evaluates to ∞ at all functions. This is clear for the function $\varphi_{\mathfrak{s}}$ when $\mathfrak{s}$ is not the maximal cluster, and for the maximal cluster it follows from the assumption $c = \infty$ of the ordered triple attached to it.

Lemma 2.12. *Let $\mathfrak{s} \in \mathrm{Cl}(\mathscr{R})$ be an element in the image of Φ.*

- *If $\mathfrak{s}' \in \mathrm{Cl}(\mathscr{R})$ is cluster not contained in $\mathfrak{s}$, the map $\varphi_{\mathfrak{s}}$ takes the same value at all points of $\mathfrak{s}'$.*
- *If $\mathfrak{s}_1, \mathfrak{s}_2$ are two different childs of $\mathfrak{s}$, then $\varphi_{\mathfrak{s}}$ takes different values at points of $\mathfrak{s}_1$ and of $\mathfrak{s}_2$.*

Proof. The statements follow easily from Lemma 2.9. $\square$

Suppose that $p, q \in \mathcal{P}$ satisfy one of the following hypothesis:

- $\mathfrak{s} = \mathfrak{s}'$ and $p = q$,
- $\mathfrak{s}$ is a child/parent of $\mathfrak{s}'$, $p = \mathfrak{s}'$ and $q = \mathfrak{s}$,
- $s_{\max}$ is not in the image of Φ, in which case we identify the extra points of the two maximal proper clusters of $\mathrm{Im}(\Phi)$ (see Example 2).

Then Lemma 2.12 implies that $\overline{\varphi_{\mathfrak{s}}}(p) = \overline{\varphi_{\mathfrak{s}}}(q)$. In particular, all coordinate functions do not distinguish them (which explains why they are identified in the model $\mathscr{X}$). By definition, the function attached to a cluster $\mathfrak{s} = \Phi(t)$ equals the one attached to t hence they share the same properties; for example $\overline{\varphi_{\mathfrak{s}}}$ contracts all components different from t to points (see [BW17, Proposition 4.2]). We extend the function $\overline{\varphi_{\mathfrak{s}}}$ to clusters defining

$$\overline{\varphi_{\mathfrak{s}}}(\mathfrak{s}') = \begin{cases} \infty & \text{if } s \subset \mathfrak{s}', \\ \overline{\varphi_{\mathfrak{s}}}(p) & \text{otherwise} \end{cases}$$

Proposition 2.13. *Let X_1, X_2 be two components of $\mathscr{X}$. Then they do not intersect if and only if there exists a component X_t whose coordinate function $\overline{\varphi_t}$ collapses X_1 and X_2 to two different points.*

Proof. If X_1 and X_2 do not intersect, then since $\mathscr{X}$ is connected, there exists a component X_t in the path connecting them which is not equal to X_1 nor X_2. Then $\overline{\varphi_t}$ collapses X_1 to one point in $\mathbb{P}^1(\overline{\mathbb{F}_p})$ and X_2 to a different one. □

We can now give a proof of how the components intersect.

Proof of Theorem 2.7. By Proposition 2.13, the clusters $\mathfrak{s}_i = \Phi(t_i)$ do not intersect if and only if there exists t_3 such that $\overline{\varphi_{t_3}}$ takes different values at $\mathfrak{s}_1$ and $\mathfrak{s}_2$. Distinguish the following cases:

- Suppose that $\mathfrak{s}_1 \cap \mathfrak{s}_2 = \emptyset$ and they are maximal subclusters. In such case, $s_{\max}$ is not in the image of Φ. In particular, if s_3 is any other subcluster, $\mathfrak{s}_3 \subset \mathfrak{s}_1$ or $\mathfrak{s}_3 \subset \mathfrak{s}_2$ by Theorem 2.6. Then by Proposition 2.13 (4), $\overline{\varphi_{\mathfrak{s}_3}}(\mathfrak{s}_i) = \infty$ hence by Proposition 2.13 $\mathfrak{s}_1$ and $\mathfrak{s}_2$ do intersect.
- Suppose that $\mathfrak{s}_1 \cap \mathfrak{s}_2 = \emptyset$ and $\mathfrak{s}_1, \mathfrak{s}_2$ are maximal subclusters of $\tilde{s}$, $\overline{\varphi_{\tilde{\mathfrak{s}}}}$ takes different values at $\mathfrak{s}_1$ and $\mathfrak{s}_2$, hence they do not intersect. Otherwise, let $\tilde{\mathfrak{s}}$ be the minimal cluster containing both of them, and $\tilde{\mathfrak{s}_i}$, $i = 1, 2$, be a maximal subcluster of $\tilde{\mathfrak{s}}$ containing $\mathfrak{s}_i$. Then the coordinate function $\overline{\varphi_{\tilde{\mathfrak{s}}}}$ takes different values at $\mathfrak{s}_1$ and $\mathfrak{s}_2$ by Lemma 2.12.
- Suppose that $\mathfrak{s}_1$ is a subcluster of $\mathfrak{s}_2$. If it is not maximal, there exists $\mathfrak{s}_3$ subcluster such that $\mathfrak{s}_1 \subsetneq \mathfrak{s}_3 \subsetneq \mathfrak{s}_2$. Then $\overline{\varphi_{\mathfrak{s}_3}}$ takes the value ∞ at $\mathfrak{s}_2$ and sends $\mathfrak{s}_1$ to an element in $\overline{\mathbb{F}_p}$, hence they do not intersect.
- Suppose $\mathfrak{s}_1$ is a maximal subcluster of $\mathfrak{s}_2$. If they do not intersect, then there exists $\mathfrak{s}_3$ cluster such that $\overline{\varphi_{\mathfrak{s}_3}}$ takes different values at $\mathfrak{s}_1$ and $\mathfrak{s}_2$. Then Lemma 2.12 gives the following implications
  - If $\mathfrak{s}_1 \subset \mathfrak{s}_2 \subset \mathfrak{s}_3$, $\overline{\varphi_{\mathfrak{s}_3}}(\mathfrak{s}_1) = \overline{\varphi_{\mathfrak{s}_3}}(\mathfrak{s}_2)$.
  - If $\mathfrak{s}_3 \subset \mathfrak{s}_1 \subset \mathfrak{s}_2$, $\overline{\varphi_{\mathfrak{s}_3}}(\mathfrak{s}_1) = \overline{\varphi_{\mathfrak{s}_3}}(\mathfrak{s}_2) = \infty$.
  - If $\mathfrak{s}_3 \cap \mathfrak{s}_2 = \emptyset$, $\overline{\varphi_{\mathfrak{s}_3}}(\mathfrak{s}_1) = \overline{\varphi_{\mathfrak{s}_3}}(\mathfrak{s}_2)$.

  Then $\mathfrak{s}_1$ and $\mathfrak{s}_2$ must intersect.

□

Example 1 (Continued)**.** Let us go back to Example 1. The set of proper cluster equals:

$$\mathfrak{s}_1 = \{0, p^2, p, p+p^2, 2p, 2p+p^2\}, \qquad \mathfrak{s}_2 = \{0, p^2\}, \qquad \mathfrak{s}_3 = \{p, p+p^2\},$$
$$\mathfrak{s}_4 = \{2p, 2p+p^2\}, \qquad \mathfrak{s}_5 = \{1, 1+p, 1+2p\}, \qquad s_{\max} = \mathscr{R}.$$

Keeping the notation of Example 1, the map Φ maps the triples t_i to $\mathfrak{s}_i$ for $i = 1, \ldots, 5$ and t_0 to $s_{\max}$. The cluster picture is given in Figure 4, where the roots follow the same order as the one given for $\mathscr{R}$.

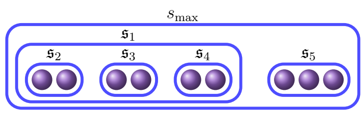
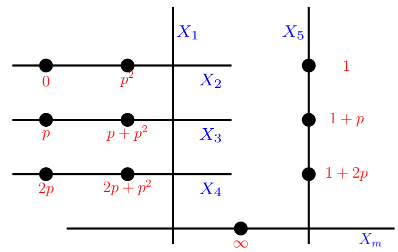


FIGURE 4. Cluster of $\mathscr{C}$.



3. The semistable model

Recall that $\mathscr{C}$ is a cyclic cover of $\mathbb{P}^1$, coming from the natural map $\phi : \mathscr{C} \to \mathbb{A}^1$ sending (x, y) to x. Let $K_2 = K(\pi^{1/d})$ and $\mathscr{Y}$ be the normalization of $\mathscr{X}$ in the function field of Y_{K_2}. The hypothesis $\mathscr{R} \subset K$ and $\pi^{1/d} \in K_2$ implies that $\mathscr{Y}$ is a semistable model of Y (by [BW17, Corollary 3.6]). The special fiber $\bar{Y}$ of $\mathscr{Y}$ is obtained as follows: let $t \in T$ with $\Phi(t) = \mathfrak{s} = D(r, d)$ (we can assume $r \in \mathscr{R}$). The cluster $\mathfrak{s}$ corresponds to a component of the special fiber of $\bar{X}$. Let $x_t := \phi_t^*(x)$ be the pullback of the standard coordinate x of X. The two variables are related via $x = \pi^d x_t + r$. Consider the polynomial $f(x_t)$ and let e_t be its content valuation (in Section 5.1 we will explain how to compute such value from a *weighted cluster*). Let $f_t(x_t) = f(x_t)\pi^{-e_t}$ and define the curve:

$$\overline{\mathcal{Y}}_t : y_t^n = \overline{f_t}(x_t). \tag{7}$$

The curve $\overline{Y_t}$ is defined as the normalization of $\overline{\mathcal{Y}}_t$. Note that the curve $\overline{\mathcal{Y}_t}$ might be reducible, and even its components might not be defined over K (but they are over an unramified extension of degree at most n). Explicitly, let $\tilde{\mathfrak{s}}_1, \ldots, \tilde{\mathfrak{s}}_N$ be the children of $\mathfrak{s}$. Let $\alpha_i \in \tilde{\mathfrak{s}}_i$ be any root and let $a_i = |\tilde{\mathfrak{s}}_i|$. Each cluster $\tilde{\mathfrak{s}}_i$ correspond to a factor of $\overline{f_t}$ and the number a_i to the multiplicity of the root α_i. Let also $c_t = \prod_{\beta \in \mathscr{R} \setminus \{\mathfrak{s}\}} \frac{(r-\beta)}{\|(r-\beta)\|_p}$.

Proposition 3.1. *Let $d := gcd(n, a_1, \ldots, a_N)$ and keep the previous notation. Then the curve $\overline{\mathcal{Y}_t}$ has d irreducible components defined over the extension $K(c_t^{1/d})$. In particular the same holds for $\overline{Y_t}$.*

Proof. If $\beta \in \mathscr{R}$ is a root not contained in $\mathfrak{s}$ then the term $x - \beta = \pi^d x_t + r - \beta$ reduces to $\frac{(r-\beta)}{\|r-\beta\|_p}$ up to a power of π (which can be removed from the equation by the assumption $\pi^{1/d} \in K_2$). Then the reduction of the polynomial $f_t(x_t)$ equals $\overline{f_t}(x_t) = c_t \prod_{i=1}^N (x_t - \alpha_i)^{a_i}$. For $\ell = 0, \ldots, d-1$ let

$$\overline{\mathcal{Y}_t^{(\ell)}} : y_t^{n/d} = \zeta_d^\ell c_t^{1/d} \prod_{i=1}^N (x_t - \alpha_i)^{a_i/d}, \tag{8}$$

where ζ_d denotes a d-th root of unity. Then clearly $\overline{\mathcal{Y}}_t = \prod_{\ell=0}^{d-1} \overline{\mathcal{Y}_t^{(\ell)}}$. Each curve $\overline{\mathcal{Y}_t^{(\ell)}}$ is irreducible, because the cover $K[x_t, y_t]/(y_t^n - f_t(x_t))$ of $K[x]$ is Galois, hence the ramification degree is the same on all its components. In particular, the number of components divide a_i for all i. $\square$

To understand the semistable model $\mathscr{Y}$ we are led to describe how different components intersect. If P is a point in $\overline{X}_t$, then number of points of $\varphi_t^{-1}(P)$ in $\overline{Y_t}$ equals $r = \gcd(n, v(f_t))$ each of them with ramification degree $\frac{n}{\gcd(n,v(f_t))}$. In particular, each component gets $\frac{\gcd(n,v(f_t))}{d}$ different points.

If $P \in \mathscr{R}$ (to easy notation suppose that $P = \alpha_1 = 0$, which can always be done after a translation) then the normalization of (7) in an open set around 0 is given by the equations

$$\begin{cases} z_t^r = c_t \displaystyle\prod_{i=2}^{N} (x_t - \alpha_i)^{a_i} \\ z_t x_t^{a_1/r} = y_t^{n/r}. \end{cases}$$

In particular, the set of r points in $\varphi_t^{-1}(0)$, with coordinates (x_t, y_t, z_t) is given by

$$\left\{ Q_i = \left(0, 0, \zeta_r^i \left(c_t \prod_{i=2}^{N} (-\alpha_i)^{a_i}\right)^{1/r}\right) \;:\; 0 \le i \le r-1 \right\}.$$

Recall that $d = \gcd(n, a_1, \ldots, a_N)$, in particular $d \mid r = \gcd(n, a_1)$. From the decomposition (8), and choosing the roots of unity in a consistent way (i.e. such that they satisfy $\zeta_{nm}^n = \zeta_m$) it follows that $Q_i \in \overline{Y_t^{(\ell)}}$ precisely when $i \equiv \ell \pmod d$. In particular, Q_0 belongs to the zeroth curve, Q_1 to the first one, and so on.

Let $\tilde{\mathfrak{s}}$ be a child of $\mathfrak{s}$ (corresponding to $\tilde{t} \in T$); say $\tilde{\mathfrak{s}} = \tilde{\mathfrak{s}}_1$ in the above notation and the center is again 0. Then the curve $\overline{\mathcal{Y}_{\tilde{t}}}$ has an equation as in (7), more concretely, let $\tilde{c} = c_t \prod_{i=2}^{N} (-\alpha_i)^{a_i}$, then

$$\overline{\mathcal{Y}_{\tilde{t}}} : y_{\tilde{t}}^n = \tilde{c} \prod_{i=1}^{\tilde{N}} (x_{\tilde{t}} - \beta_i)^{b_i}, \tag{9}$$

where the product is over childs of $\tilde{\mathfrak{s}}$, the numbers b_i equal the number of roots in each child, and β_i is a root in each of them. Note that $\sum_{i=1}^{\tilde{N}} b_i = a_1$. The gluing (as described in [DDMM18] before Remark 3.9) corresponds in our coordinates to identify the infinity point in the chart $\tilde{t}$ with the zero point in the chart t. Then equation (9) can be written as

$$\left(\frac{y_{\tilde{t}}^{n/r}}{x_{\tilde{t}}^{a_1/r}}\right)^r = \tilde{c} \prod_{i=1}^{\tilde{N}} \left(1 - \frac{\beta_i}{x_{\tilde{t}}}\right)^{b_i}. \tag{10}$$

This equation is the key to identify the points Q_i in $\mathfrak{s}$ with r-points in (10) (or its components if it happens to be reducible) and gives the intersection points of $\mathfrak{s}$ and $\tilde{\mathfrak{s}}$ (see the formulas in [DDMM18, Proposition 5.5]).

Let $\tilde{d} = \gcd(n, b_1, \ldots, b_{\tilde{N}})$, then the curve $\overline{\mathcal{Y}_{\tilde{t}}}$ consists on $\tilde{d}$ components (ordered according to powers of $\tilde{d}$-th roots of unity) and with the compatible choice of roots of unity, the point

Q_0 lies at the infinity part of the zeroth component, and so on. Let us illustrate the situation with some examples.

Example 1 (Continued II)**.** Recall that there are six components (see Figures 1 and 4):

- $\overline{\mathcal{Y}}_m : y_m^6 = x_m^6(x_m - 1)^3$ (with relations $x_m = x,$),
- $\overline{\mathcal{Y}}_1 : y_1^6 = (-1)x_1^2(x_1 - 1)^2(x_1 - 2)^2$ (with relations $x_1 = x/p$, $y_1 = y/p$),
- $\overline{\mathcal{Y}}_2 : y_2^6 = (-4)x_2(x_2 - 1)$ (with relations $x_2 = x/p^2$, $y_2 = y/p^{4/3}$),
- $\overline{\mathcal{Y}}_3 : y_3^6 = (-4)x_3(x_3 - 1)$ (with relations $x_3 = (x - p)/p^2$, $y_3 = y/p^{4/3}$),
- $\overline{\mathcal{Y}}_4 : y_4^6 = (-4)x_4(x_4 - 1)$ (with relations $x_4 = (x - 2p)/p^2$, $y_4 = y/p^{4/3}$),
- $\overline{\mathcal{Y}}_5 : y_5^6 = x_5(x_5 - 1)(x_5 - 2)$ (with relations $x_5 = (x - 1)/p$, $y_5 = y/p^{1/2}$).

The curves $\overline{\mathcal{Y}}_2, \overline{\mathcal{Y}}_3, \overline{\mathcal{Y}}_4$, are non-singular irreducible curves of genus 2, while $\overline{\mathcal{Y}}_5$ is a non-singular curve of genus 4. On the other hand, the curves $\overline{\mathcal{Y}}_m$ and $\overline{\mathcal{Y}}_1$ are reducible. The curve $\overline{\mathcal{Y}}_m$ consists of the union of three (genus 0) curves $\overline{\mathcal{Y}_m^{(\ell)}}$, $\ell = 0, 1, 2$ with equations

$$\overline{\mathcal{Y}_m^{(\ell)}} : y_m^2 = \zeta_3^\ell x_m^2(x_m - 1)\ (y_m = y),$$

where ζ_3 is a third root of unity in $\mathbb{F}_p$. The curve $\overline{\mathcal{Y}}_1$ consists of the union of two genus 1 curves $\overline{\mathcal{Y}_1^{(\ell)}}$, $\ell = 0, 1$ with equation

$$\overline{\mathcal{Y}_1^{(\ell)}} : y_1^3 = (-1)^\ell \sqrt{-1} x_1(x_1 - 1)(x_1 - 2)\ (y_1 = y/p).$$

Note that the components of $\overline{\mathcal{Y}_1}$ need not be defined over K_2, but at most over an unramified extension (since $p \nmid 6$, $K_2(\sqrt{-1})/K_2$ is unramified). The normalization explained in the previous section, in an open neighborhood of 0 (but not of 1) of the curve $\overline{\mathcal{Y}_m^{(\ell)}}$ has equation

$$\begin{cases} z_m^2 = \zeta_3^\ell (x_m - 1) \\ z_m x_m = y_m. \end{cases}$$

The preimage of 0 in the ℓ-th component corresponds to the points $P_\ell^\pm = \left(0, 0, \pm\sqrt{-1}\zeta_3^{2\ell}\right)$. In particular, it intersects $\overline{\mathcal{Y}}_1$ in 2 points. The component graph of the special fiber of $\mathscr{Y}$ is given in Figure 5.

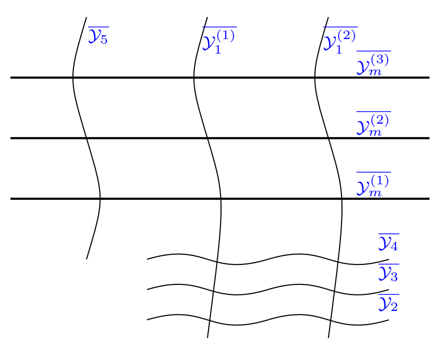


FIGURE 5. Special Fiber of $\mathscr{Y}$

Example 3. Let p be an odd prime greater than 3 and $\mathscr{C}/\mathbb{Z}_p$ be the curve with equation

$$\begin{aligned} y^6 = x(x-p^2)(x-p)(x-p-p^2)(x-2p)(x-2p-p^2)(x-1)(x-1-p^2)(x-1-2p^2) \\ (x-1-p)(x-1-p-p^2)(x-1-p-2p^2)(x-2)(x-2-p)(x-2-2p). \end{aligned}$$

It is a curve of genus 34. The set of roots of $f(x)$ equals $\mathscr{R} = \{0, p, p^2, p+p^2, 2p, 2p+p^2, 1, 1+p, 1+p^2, 1+2p^2, 1+p+p^2, 1+p+2p^2, 2, 2+p, 2+2p\}$. There are nine clusters as shown in Figure 6. They give the components:

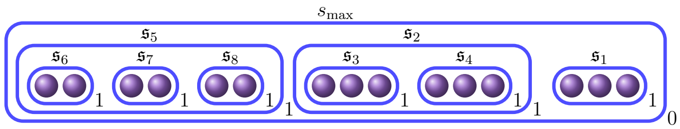


FIGURE 6. Cluster picture

- $s_{\max}$ is the disc with center $r_m = 0$ and diameter $\mu = 0$. It corresponds to a component $\overline{\mathcal{Y}_m} : y_m^6 = x_m^6(x_m-1)^6(x_m-2)^3$ consisting of 3 irreducible components $\overline{\mathcal{Y}_m^{(i)}} : y_m^2 = \zeta_3^i x_m^2(x_m-1)^2(x_m-2)$, $0 \le i \le 2$ of genus 0 (see Proposition 3.2).
- $\mathfrak{s}_1 = \{2, 2+p, 2+2p\} = D(2,1)$, with variable $x = px_1 + 2$, $y = p^{1/2}y_1$ and equation $\overline{\mathcal{Y}_1} : y_1^6 = 2^6 x_1(x_1-1)(x_1-2)$. It is an irreducible curve of genus 4.
- $\mathfrak{s}_2 = \{1, 1+p, 1+p^2, 1+2p^2, 1+p+p^2, 1+p+2p^2\} = D(1,1)$, with variable $x = px_2 + 1$, $y = py_2$ and equation $\overline{\mathcal{Y}_2} : y_2^6 = -x_2^3(x_2-1)^3$. It consists of three irreducible components $\overline{\mathcal{Y}_2^{(i)}} : y_1^2 = -\zeta_3^i x_1(x_1-1)$, $0 \le i \le 2$ of genus 0.
- $\mathfrak{s}_3 = \{1, 1+p^2, 1+2p^2\} = D(1,2)$, with variable $x = p^2x_3 + 1$, $y = p^{3/2}y_3$ and equation $\overline{\mathcal{Y}_3} : y_3^6 = x_3(x_3-1)(x_3-2)$. It is an irreducible curve of genus 4.
- $\mathfrak{s}_4 = \{1+p, 1+p+p^2, 1+p+2p^2\} = D(1+p, 2)$, with variable $x = p^2x_4 + 1 + p$, $y = p^{3/2}y_4$ and equation $\overline{\mathcal{Y}_4} : y_4^6 = -x_4(x_4-1)(x_4-2)$. It is an irreducible curve of genus 4.
- $\mathfrak{s}_5 = \{0, p, p^2, p+p^2, 2p, 2p+2p^2\} = D(0,1)$, with variable $x = px_5$, $y = py_5$ and equation $\overline{\mathcal{Y}_5} : y_5^6 = -8x_5^2(x_5-1)^2(x_5-2)^2$. It consists of two irreducible components $\overline{\mathcal{Y}_5^{(i)}} : y_5^3 = (-1)^i 2\sqrt{-2}x_5(x_5-1)(x_5-2)$, $i = 0, 1$ of genus 1.
- $\mathfrak{s}_6 = \{0, p^2\} = D(0,2)$, with variable $x = p^2x_6$, $y = p^{4/3}y_6$ and equation $\overline{\mathcal{Y}_6} : y_6^6 = -32x_6(x_6-1)$. It is an irreducible curve of genus 2.
- $\mathfrak{s}_7 = \{p, p+p^2\} = D(p,2)$, with variable $x = p^2x_7 + p$, $y = p^{4/3}y_7$ and equation $\overline{\mathcal{Y}_7} : y_7^6 = -8x_7(x_7-1)$. It is an irreducible curve of genus 2.
- $\mathfrak{s}_8 = \{2p, 2p+p^2\} = D(2p,2)$, with variable $x = p^2x_8 + 2p$, $y = p^{4/3}y_8$ and equation $\overline{\mathcal{Y}_8} : y_8^6 = -32x_8(x_8-1)$. It is an irreducible curve of genus 2.

The special fiber of $\mathscr{X}$ and $\mathscr{Y}$ are given in Figure 7 and Figure 8 respectively.

3.1. **Genus of $\overline{Y_t}$.** To get a complete understanding of the special fiber of $\mathscr{Y}$ we only need to explain how to get the genus of each component from the cluster and describe the component graph. Keeping the previous notations, let $\overline{Y_t}$ be a component of the special fiber of $\mathscr{Y}$ (we do not assume that it is irreducible), above a component X of $\mathscr{X}$, corresponding to a cluster $\mathfrak{s}$.

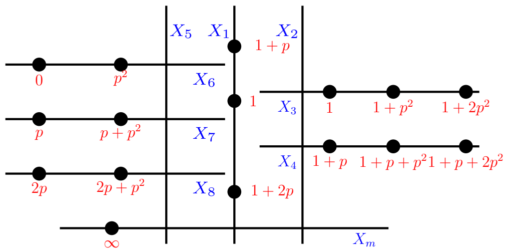


FIGURE 7. Special Fiber of $\mathscr{X}$

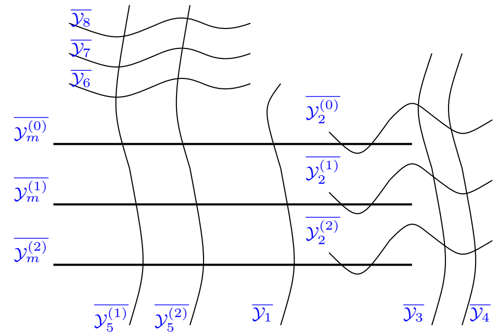


FIGURE 8. Special Fiber of $\mathscr{Y}$

Proposition 3.2. *Let $\widetilde{\mathfrak{s}}_1, \ldots, \widetilde{\mathfrak{s}}_N$ be the children of $\mathfrak{s}$, let $a_i = |\widetilde{\mathfrak{s}}_i|$ and let $d := gcd(n, a_1, \ldots, a_N)$. Then irreducible components of $\overline{Y_t}$ have genus*

$$\frac{1}{2d}\left(n(N-2) - \sum_{i=1}^{N} \gcd(n, a_i)\right) + 1 + \begin{cases} 0 & \text{if } n \mid \sum_{i=1}^{N} a_i \\ \frac{n}{2d} - \frac{\gcd(n, \sum_{i=1}^{N} a_i)}{2d} & \text{if } n \nmid \sum_{i=1}^{N} a_i \end{cases}$$

Proof. Since the genus of a curve equals that of its normalization, we can look at the components of $\overline{\mathcal{Y}_t}$. By Proposition 3.1, we know that the components are given by an equation of the form $\overline{\mathcal{Y}_t^\ell} : y_t^{n/d} = \zeta_d^\ell c^{1/d} \prod_{i=1}^{N} (x_t - \alpha_i)^{a_i/d}$.

If $\pi : X \to X'$ is a general degree D map between non-singular curves, The Riemann-Hurwitz formula (see for example [Har77, Corollary 2.4]) implies that

$$2g(X) - 2 = D(2g(X') - 2) + \sum_{P} (e_P - 1),$$

where $g(X)$ denotes the genus of X and e_P denotes the ramification degree of P. Taking $X = \overline{\mathcal{Y}_t^\ell}$ and $X' = \mathbb{P}^1$, $g(X') = 0$, $D = \frac{n}{d}$ and

- $e_P = 1$ for all points $P \neq \alpha_i$ and $P \neq \infty$,
- as mentioned before, each point α_i has ramification degree $\frac{n}{\gcd(n, a_i)}$ and there are $\frac{\gcd(n, a_i)}{d}$ points above it.
- If $n \mid \sum_{i=1}^{N} a_i = \deg(\overline{f_t(x_t)})$, ∞ is not ramified. Otherwise, it is a ramified point, with ramification degree $\frac{n}{\gcd(n, \deg(\overline{f_t(x_t)}))}$ and $\frac{\gcd(n, \deg(\overline{f_t(x_t)}))}{d}$ points.

Then Riemann-Hurwitz gives that the genus of $\overline{\mathcal{Y}_t^{(\ell)}}$ equals

$$\frac{1}{2}\left(\frac{n}{d}(N-2) - \sum_{i=1}^{N} \frac{\gcd(n, a_i)}{d}\right) + 1 + \begin{cases} 0 & \text{if } n \mid \sum_{i=1}^{N} a_i, \\ \frac{n}{2d} - \frac{\gcd(n, \sum_{i=1}^{N} a_i)}{2d} & \text{if } n \nmid \sum_{i=1}^{N} a_i. \end{cases}$$

$\square$

4. THE GALOIS REPRESENTATION OF $\mathscr{C}$

4.1. **Computing the Galois representation over K_2.** Let $\Upsilon = (V, E)$ denote the dual graph of the special fiber of $\overline{Y}$ (also referred as the graph of components in [BW17]); it is an undirected graph whose vertices V are the irreducible components of $\overline{Y}$. The set E

contains an edge joining a pair of vertices for each intersection point of the corresponding components. Under our hypothesis, the action of $\mathrm{Gal}(\overline{k}/k)$ on the set $\overline{X}$ is trivial, but its action on the set of irreducible components of $\overline{Y_t}$ (and on Υ) might not be. Let ℓ be a prime with $\ell \nmid p$. Then by [BW17, Lemma 2.7] (see also [DDM18, Corollary 1.6]) it follows that as $\mathbb{Q}_\ell[G_K]$-modules

$$\mathrm{H}^1_{\text{ét}}(\overline{Y}, \mathbb{Q}_\ell) = \sum_{\tilde{Y} \in V} \mathrm{H}^1_{\text{ét}}(\tilde{Y}, \mathbb{Q}_\ell) \oplus \mathrm{H}^1(\Upsilon, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell. \tag{11}$$

If we consider the Picard group $\mathrm{Pic}^0(Y)$, it contains an abelian part and a toric one (see for example [BLR90], Example 8, page 246). The rank of the toric part equals the rank of $\mathrm{H}^1(\Upsilon, \mathbb{Z})$, and its Galois representation consists of Jordan blocks of size 2 (see [Gro72, Proposition 3.5], page 350). The action of $\mathrm{Gal}(\overline{K}/K)$ on $Y(\overline{K})$ extends to a semilinear action on the geometric points of $\overline{Y}$ (see [DDMM18, Equation (2,18)], [DDM18, Corollary 1.6] and page 13 of [CFKS10]). In particular, we have an isomorphism of G_K-representations

$$V_\ell(\mathrm{Pic}^0(Y)) \simeq \left(\left(\mathrm{H}^1(\Upsilon, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell\right) \otimes \mathrm{Sp}_2\right) \oplus \bigoplus_{\tilde{Y} \in V} V_\ell(\mathrm{Pic}^0(\tilde{Y})).$$

Recall that the rank of $\mathrm{H}^1(\Upsilon, \mathbb{Z})$ equals $|E| - |V| + 1$ (because the graph is connected).

Remark 4.1. Since inertia acts trivially on $\mathrm{H}^1(\Upsilon, \mathbb{Z})$, the image of inertia consists of a matrix with $\mathrm{rank}_{\mathbb{Z}} \mathrm{H}^1(\Upsilon, \mathbb{Z})$ Jordan blocks of 2×2 and the identity elsewhere. The values of the Frobenius element is given by its (permutation) action on the components.

Theorem 4.2. *The rank of* $\mathrm{H}^1(\Upsilon, \mathbb{Z})$ *equals*

$$\sum_{\tilde{\mathfrak{s}}} \gcd(n, |\tilde{\mathfrak{s}}|) - \sum_{\mathfrak{s}} \gcd(n, |\tilde{\mathfrak{s}}_1|, \ldots, |\tilde{\mathfrak{s}}_N|) + 1,$$

where the first sum runs over all proper clusters except the maximal one, the second sum runs over all proper clusters, and the elements $\tilde{\mathfrak{s}}_1, \ldots, \tilde{\mathfrak{s}}_N$ *denote the children of* $\mathfrak{s}$ *(which might not be proper).*

Proof. Recall that the rank of $\mathrm{H}^1(\Upsilon, \mathbb{Z})$ equals $|E| - |V| + 1$. The value $|V|$ (the number of irreducible components) equals the second term by Proposition 3.1. The number of intersection points follows from the discussion after the same proposition, that states that $\#\varphi_t^{-1}(P) = gcd(n, v_P(f_t))$. Since $v_P(f_t) = |\tilde{\mathfrak{s}}|$ the result follows. ☐

Example 1. (Continued III). The graph of components (which can be read from Figure 5) is given in Figure 9. Its rank can be computed using the previous theorem, from the cluster description in Figure 4, implying that $\mathrm{H}^1(\Upsilon, \mathbb{Z})$ has rank 7 (which can be easily verified from the graph picture since the component graph Υ contains 9 vertices and 15 edges.). The advantage of Theorem 4.2 is that we do not need to know the graph of components! (the cluster picture is enough).

In particular, the image of inertia (of the Galois representation) equals 7 Jordan blocks of the form $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, and the identity elsewhere.

Recall by (11) that the Galois representation of $\mathscr{C}$ has two parts, one coming from the components and one coming from the graph of components. Let σ denotes the Frobenius automorphism of $\mathrm{Gal}(K_2^{\mathrm{ur}}/K_2)$. If we want to understand its action on the component graph, we need to consider two different cases: if $\sqrt{-1} \in K_2$, then all components are defined over K_2

and so are the intersection points, hence its action is trivial (and the 2×2 blocks correspond precisely to the classical Steinberg representation). However, if $\sqrt{-1}\notin K_2$, then Frobenius interchanges the two components $\overline{\mathcal{Y}_1^{(1)}}$ and $\overline{\mathcal{Y}_1^{(2)}}$. A basis for the graph cohomology are the cycles:

- $e_1 = \{,\overline{\mathcal{Y}_m^{(1)}},\overline{\mathcal{Y}_1^{(1)}},\overline{\mathcal{Y}_4},\overline{\mathcal{Y}_1^{(2)}}\}$,
- $e_2 = \{,\overline{\mathcal{Y}_m^{(1)}},\overline{\mathcal{Y}_1^{(1)}},\overline{\mathcal{Y}_3},\overline{\mathcal{Y}_1^{(2)}}\}$,
- $e_3 = \{,\overline{\mathcal{Y}_m^{(1)}},\overline{\mathcal{Y}_1^{(1)}},\overline{\mathcal{Y}_2},\overline{\mathcal{Y}_1^{(2)}}\}$,
- $e_4 = \{,\overline{\mathcal{Y}_m^{(1)}},\overline{\mathcal{Y}_1^{(1)}},\overline{\mathcal{Y}_m^{(2)}},\overline{\mathcal{Y}_5}\}$,
- $e_5 = \{,\overline{\mathcal{Y}_m^{(1)}},\overline{\mathcal{Y}_1^{(2)}},\overline{\mathcal{Y}_m^{(2)}},\overline{\mathcal{Y}_5}\}$,
- $e_6 = \{,\overline{\mathcal{Y}_m^{(1)}},\overline{\mathcal{Y}_1^{(1)}},\overline{\mathcal{Y}_m^{(3)}},\overline{\mathcal{Y}_5}\}$,
- $e_7 = \{,\overline{\mathcal{Y}_m^{(1)}},\overline{\mathcal{Y}_1^{(2)}},\overline{\mathcal{Y}_m^{(3)}},\overline{\mathcal{Y}_5}\}$.

Clearly σ fixes e_1,e_2,e_3, while it interchanges $e_4\leftrightarrow e_5$ and $e_6\leftrightarrow e_7$. Then $\{e_1,e_2,e_3,e_4+e_5,e_6+e_7,e_4-e_5,e_6-e_7\}$ is a basis of eigenvectors for σ and the Galois representation on this basis consists of 4 copies of the Steinberg representation, and 3 copies of a twist of the Steinberg representation by the unramified quadratic extension $K_2(\sqrt{-1})/K_2$.

Note that the sum of the genera of the components equals 12, and $7+12=19$ which is the genus of $\mathscr{C}$ (as it should be).

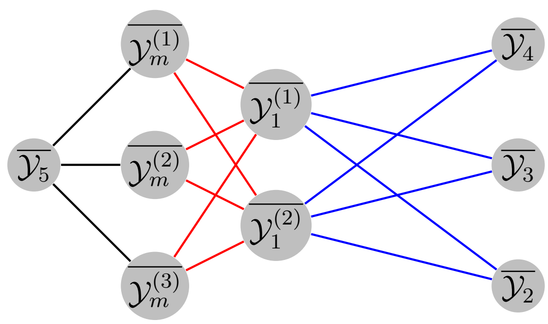


FIGURE 9. The component graph Υ.

Example 3 (Continued)**.** From the cluster picture (see Figure 6) and Theorem 4.2 we get that $\mathrm{H}^1(\Upsilon,\mathbb{Z})$ has rank 14, hence the image of inertia equals 14 Jordan blocks of size 2×2. The component graph Υ contains 14 vertices and 27 edges (which can be read from Figure 8). The sum of the genera of the components equals 20, and $20+14=34$ which is the genus of $\mathscr{C}$.

A similar analysis as the one made in the previous example can be used to determine the graph component representation. For the components $\overline{\mathcal{Y}_m^{(i)}}$ and $\overline{\mathcal{Y}_5^{(j)}}$ be defined over K we need $\sqrt{-2}$ be in K_2. Start supposing this is the case. The group G_{K_2} fixes the vertices of the graph, but still might not fix the edges (corresponding to the intersection points). The intersection of $\overline{\mathcal{Y}_m^{(i)}}$ with $\overline{\mathcal{Y}_5^{(j)}}$ correspond to two points with coordinates in $K_2(\sqrt{-2\zeta_3^i})$ which is fixed by G_{K_1} under our hypothesis, but the intersection of $\overline{\mathcal{Y}_m^{(i)}}$ with $\overline{\mathcal{Y}_2^{(i)}}$ correspond to two points with coordinates in $K_2(\sqrt{-\zeta_3^i})$.

In particular, if $\sqrt{-1}$ also belongs to K_2, the Galois representation attached to $\mathscr{C}$ decomposes as a direct sum of dimensions $8+8+8+2+2+4+4+4$ (corresponding to the curves $\overline{\mathcal{Y}_1}$, $\overline{\mathcal{Y}_3}$, $\overline{\mathcal{Y}_4}$, $\overline{\mathcal{Y}_5^{(0)}}$, $\overline{\mathcal{Y}_5^{(1)}}$, $\overline{\mathcal{Y}_6}$, $\overline{\mathcal{Y}_7}$ and $\overline{\mathcal{Y}_8}$ respectively) and 14 blocks where the action of Frobenius is trivial and a generator of inertia acts as $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ (corresponding to the Steinberg representation). However, if $\sqrt{-1} \notin K_2$, then G_{K_2} permutes the two edges joining the vertices $\overline{\mathcal{Y}_m^{(i)}}$ and $\overline{\mathcal{Y}_2^{(i)}}$. This implies that three of the fourteen blocks have Frobenius acting by -1, while the other eleven ones keep the trivial action. Note that since $p \nmid 6$ all the extensions are unramified, so the image of inertia does not change.

Suppose that $\sqrt{-2} \notin K_2$, then G_{K_2} permutes the two components $\overline{\mathcal{Y}_5^{(0)}}$ and $\overline{\mathcal{Y}_5^{(1)}}$ (with its respective intersection points), which induces another involution on the graph of components. In this case, if $\sqrt{-1} \in K_2$, five Jordan blocks have Frobenius acting by -1 while the other seven ones have trivial action while if $\sqrt{-1} \notin K_2$, eight blocks have Frobenius acting by -1 and by 1 on the other five blocks. The remaining cases can be studied similarly.

Remark 4.3. We want to emphasize that when restricting the Galois representation to K_2, the image of inertia depends only on the components graph, which can easily be read from the cluster picture. The same is true over K considering *weighted clusters* that will be introduced in the next section.

5. The Galois representation over K

5.1. **Weighted Cluster.** Recall from Proposition 3.1 that the components $\overline{Y_t}$ correspond to equations where the leading coefficient of $f(x)$ might not be 1. This corresponds to a "twist" of the representation. To determine whether the involved character is ramified or not it is important the notion of *weighted clusters*.

Definition 5.1. Let $\mathfrak{s}$ be a proper cluster (i.e. $\mathfrak{s} \neq \mathcal{R}$). Define its *relative diameter* (that will be denoted $d_{\mathfrak{s}}$) by

$$d_{\mathfrak{s}} = \mu_{\mathfrak{s}} - \mu_{P(\mathfrak{s})}$$

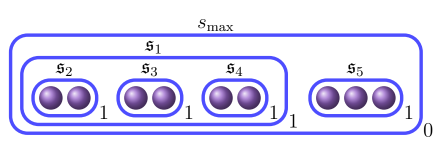
where $P(\mathfrak{s})$ denotes the parent of $\mathfrak{s}$.

Following[DDMM18], in a cluster picture we include the relative diameters as follows: in a maximal cluster include a subscript denoting its diameter; for all other clusters include a subscript given by their relative diameter (that is the difference between their diameters and that of their parent cluster).

Example 1. Recall that Table 1 gives the diameters $\mu_{s_{\max}} = 0$, $\mu_{\mathfrak{s}_1} = \mu_{\mathfrak{s}_5} = 1$, $\mu_{\mathfrak{s}_2} = \mu_{\mathfrak{s}_3} = \mu_{\mathfrak{s}_4} = 2$. Then their relative diameter equal

$$\begin{aligned} &d_{\mathfrak{s}_1} = \mu_{\mathfrak{s}_1} - \mu_{s_{\max}} = 1,\ d_{\mathfrak{s}_5} = \mu_{\mathfrak{s}_5} - \mu_{s_{\max}}, \\ &d_{\mathfrak{s}_2} = \mu_{\mathfrak{s}_2} - \mu_{\mathfrak{s}_1} = 1, d_{\mathfrak{s}_3} = \mu_{\mathfrak{s}_3} - \mu_{\mathfrak{s}_1} = 1,\ d_{\mathfrak{s}_4} = \mu_{\mathfrak{s}_4} - \mu_{\mathfrak{s}_1} = 1. \end{aligned}$$

giving the following weighted cluster.



Given $\mathfrak{s}_1, \mathfrak{s}_2$ two clusters (or roots) let $\mathfrak{s}_1 \wedge \mathfrak{s}_2$ denote the smallest cluster that contains both of them. For instance, in the previous example $0 \wedge 1 = \mathcal{R}$, $\mathfrak{s}_2 \wedge \mathfrak{s}_3 = \mathfrak{s}_1$, $\mathfrak{s}_2 \wedge p + p^2 = \mathfrak{s}_1$. Keep the notation of the previous sections, and let e_t be the valuation of the content of the polynomial $f(x_t)$ (in particular $e_t = v(c_t)$).

Proposition 5.2. *If $t \in T$ corresponds to a component of the special fiber of $\overline{X}$ associated to a cluster $\mathfrak{s}$, the content valuation of the polynomial $f(x_t)$ equals*

$$e_t = \sum_{r \in \mathscr{R}} \mu_{r \wedge \mathfrak{s}}.$$

Proof. Recall that if t corresponds to a cluster $\mathfrak{s} = D(\alpha, \mu_{\mathfrak{s}})$ then $x = \pi^{\mu_{\mathfrak{s}}} x_t + \alpha$ where $\alpha \in \mathfrak{s}$ and

$$f(x_t) = \prod_{r \in \mathscr{R}} (\pi^{\mu_{\mathfrak{s}}} x_t + \alpha - r).$$

Each factor $(\pi^{\mu_{\mathfrak{s}}} x_t + \alpha - r)$ has content valuation $\min\{\mu_{\mathfrak{s}}, v(\alpha - r)\}$ contributing to the content valuation c_t of $f(x_t)$. Consider the following two cases:

- If $r \in \mathfrak{s}$ then $\min\{\mu_{\mathfrak{s}}, v(\alpha - r)\} = \mu_{\mathfrak{s}} = \mu_{\mathfrak{s} \wedge r}$.
- Otherwise, $\min\{\mu_{\mathfrak{s}}, v(\alpha - r)\} = v(\alpha - r) = \mu_{\mathfrak{s} \wedge r}$ as well.

Then the formula follows. $\square$

5.2. **Decomposing the representation of $\mathscr{C}$.** A good reference for details on this section is [Kan85]. Let G denote the group μ_n of n-th roots of unity, whose group algebra equals

$$\mathbb{Q}[G] = \mathbb{Q}[t]/(t^n - 1) \simeq \prod_{d \mid n} \mathbb{Q}[t]/\phi_d(t), \tag{12}$$

where $\phi_d(t)$ denotes the d-th cyclotomic polynomial (with complex roots the primitive d-th roots of unity). Fix ζ_n a primitive n-th root of unity (which belongs to K). The group G acts on $\mathscr{C}$ via $t \cdot (x, y) = (x, \zeta_n y)$. This action extends to an action of $\mathbb{Q}[G]$ in $\mathrm{Aut}^0(\mathrm{Jac}(\mathscr{C})) := \mathrm{Aut}(\mathrm{Jac}(\mathscr{C})) \otimes_{\mathbb{Z}} \mathbb{Q}$. Let $V_\ell(\mathrm{Jac}(\mathscr{C}))$ denote the $\mathbb{Q}_\ell$ Tate module $T_\ell(\mathrm{Jac}(\mathscr{C})) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$. The natural injective morphism $\mathrm{End}(\mathrm{Jac}(\mathscr{C})) \otimes \mathbb{Q}_\ell \hookrightarrow \mathrm{End}(V_\ell(\mathrm{Jac}(\mathscr{C}))$ gives an action of $\mathbb{Q}_\ell[G]$ on $V_\ell(\mathrm{Jac}(\mathscr{C}))$.

If H is a subgroup of G (corresponding necessarily to the group of d-th roots of unity for some $d \mid n$) we have a natural surjective map $\pi_H : \mathscr{C} \to \mathscr{C}/H := \mathscr{C}_H$. In particular, if $H = H_{n/d}$ (corresponding to $\mu_{n/d}$), denote the quotient curve $\mathscr{C}/H$ by $\mathscr{C}_d$, with equation:

$$\mathscr{C}_d : y^d = f(x). \tag{13}$$

The quotient map is given explicitly by $\pi_d(x, y) = (x, y^{n/d})$ (an n/d to 1 map). This induces two morphisms between $\mathrm{Jac}(\mathscr{C})$ and $\mathrm{Jac}(\mathscr{C}_d)$ namely the push-forward $\pi_* : \mathrm{Jac}(\mathscr{C}) \to \mathrm{Jac}(\mathscr{C}_d)$ and the pullback $\pi_d^* : \mathrm{Jac}(\mathscr{C}_d) \to \mathrm{Jac}(\mathscr{C})$ whose kernel is contained in the n/d-torsion of $\mathrm{Jac}(\mathscr{C}_d)$. Let A_d denote the connected component of $\ker(\pi_*)$. For any prime ℓ we get an injective morphism on the $\mathbb{Q}_\ell$-Tate modules $\pi_\ell^* : V_\ell(\mathrm{Jac}(\mathscr{C}_d)) \to V_\ell(\mathrm{Jac}(\mathscr{C}))$ and

$$V_\ell(\mathrm{Jac}(\mathscr{C})) = V_\ell(A_d) \oplus \pi_\ell^*(V_\ell(\mathrm{Jac}(\mathscr{C}_d))).$$

The group μ_d acts on $\mathscr{C}_d$. For any $\alpha \in \mathbb{Q}[\mu_d]$ let $\pi^*(\alpha) = \frac{d}{n}(\pi_d^* \circ \alpha \circ \pi_*)$. Then (see the proof of Proposition 2 in [Kan85]) $\pi^*(\alpha)|_{V_\ell(A)} = 0$ and $\pi^*(\alpha)|_{\pi_\ell^*(V_\ell(\mathrm{Jac}(\mathscr{C}_d)))} = \alpha$.

In particular, the Galois representation attached to the curve $y^n = f(x)$ contains for each $d \mid n$ what might be called a *d-new* part coming from the curve $y^d = f(x)$ and $V_\ell(\mathrm{Jac}(\mathscr{C})) = \bigoplus_{d \mid n} V_\ell(\mathrm{Jac}(\mathscr{C}_d))^{d\text{-new}}$. Then in the decomposition (12) the action of the group algebra $\mathbb{Q}_\ell[t]/\phi_d(t)$ on $V_\ell(\mathrm{Jac}(\mathscr{C}))$ is non-trivial precisely in the subspace corresponding to $V_\ell(\mathrm{Jac}(\mathscr{C}_d))^{d\text{-new}}$.

Example. Suppose that $n = p \cdot q$ with p, q distinct prime numbers. Then

$$V_\ell(\mathrm{Jac}(\mathscr{C})) = V_\ell(\mathrm{Jac}(\mathscr{C}))^{pq\text{-new}} \oplus V_\ell(\mathrm{Jac}(\mathscr{C}_p)) \oplus V_\ell(\mathrm{Jac}(\mathscr{C}_q)),$$

where $V_\ell(\mathrm{Jac}(\mathscr{C}))^{pq\text{-new}} = V_\ell(A_p) \cap V_\ell(A_q)$. The group algebra $\mathbb{Q}[t]/\phi_{pq}(t)$ acts non-trivially on the first summand, $\mathbb{Q}[t]/\phi_p(t)$ on the second and $\mathbb{Q}[t]/\phi_q(t)$ on the third one.

An explicit description of $V_\ell(\mathrm{Jac}(\mathscr{C}))^{n\text{-new}}$ can be given as virtual representations using the inclusion-exclusion principle.

Remark 5.3. The contribution from $H = G$ in the above formula is trivial, as it corresponds to a genus 0 curve. This is the reason why one can remove the term with $d = 1$ in (12).

5.3. **Twisting.** Let $c \in K$ be a non-zero element, $f(x) \in K[x]$ and consider the following two curves:

$$\mathscr{C} : y^n = f(x),$$

and

$$\mathscr{C}' : y^n = c \cdot f(x).$$

It is clear that they become isomorphic over the (abelian) extension $K(c^{1/n})$, in particular, they are a twist of each other.

Problem: what is the relation between the Galois representations of $\mathscr{C}$ and that of $\mathscr{C}'$?

This problem appears in different contexts. For example, if we start with a monic polynomial $f(x)$ (which we assumed was the case) and want to consider a general polynomial (with all roots in K), we need to understand twists. Also while computing the semistable model (in Proposition 3.1) the components involve taking twists by a d-th root of c_t. The hypothesis $\zeta_n \in K$ implies that the extension associated to the twist is a Galois one, hence the extension $K(\sqrt[n]{c})/K$ corresponds to a Hecke character.

The problem is probably known to experts (as happens for example in the case of an elliptic curve twisted by a quadratic character, or an elliptic curve with CM by $\mathbb{Z}[\zeta_3]$ while twisted by a cubic or sextic character) but we did not find a good reference in the literature, so we briefly explain it.

Since both curves become isomorphic over the extension $K[\sqrt[n]{c}]$ their representations must be related by some sort of twist. More concretely, if we base extend $\mathscr{C}$ to $L = K[\sqrt[n]{c}]$ (let $\mathscr{C}_L$ denote such curve) and we do the same to $\mathscr{C}'$ then both curves become isomorphic, hence their Galois representations are the same. Recall that the representation attached to $\mathrm{Jac}(\mathscr{C})$ and $\mathrm{Res}_K(\mathrm{Jac}(\mathscr{C}_L))$ are related via

$$V_\ell(\mathrm{Res}_K(\mathrm{Jac}(\mathscr{C}_L))) = \bigoplus_{\chi} V_\ell(\mathrm{Jac}(\mathscr{C})) \otimes \chi, \tag{14}$$

where χ ranges over the characters of the (abelian) group $\mathrm{Gal}(L/K)$.

However this picture is a little misleading, as it is not true that $V_\ell(\mathrm{Jac}(\mathscr{C}'))$ equals $V_\ell(\mathrm{Jac}(\mathscr{C})) \otimes \chi$ (for some character χ) in general (note that the latter does not have the right determinant for example). What happens is that $V_\ell(\mathrm{Jac}(\mathscr{C}))$ (respectively $V_\ell(\mathrm{Jac}(\mathscr{C}')))$ has a decomposition (as explained in Section 5.1) of the form:

$$V_\ell(\mathrm{Jac}(\mathscr{C})) = \bigoplus_{d \mid n} V_\ell(\mathrm{Jac}(\mathscr{C}_d))^{d\text{-new}}.$$

Recall that $\mathbb{Q}_\ell[t]/\phi_d(t)$ acts on $V_\ell(\mathrm{Jac}(\mathscr{C}_d))^{d\text{-new}}$, hence the latter admits a decomposition in terms of the action of the d-th roots of unity. Concretely. pick a basis for the order ℓ^n points (for each n) as a $\mathbb{Z}[\zeta_d]$-modules instead of taking one as a $\mathbb{Z}$-module. Once a d-th root of unity (say ζ_d) is chosen inside the automorphism group of $\mathrm{Jac}(\mathscr{C})$, we get the decomposition

$$V_\ell(\mathrm{Jac}(\mathscr{C}_d))^{d\text{-new}} = \bigoplus_{\substack{i=1\\ \gcd(i,d)=1}}^{d} V_\ell^{(i)}(\mathrm{Jac}(\mathscr{C}_d))^{d\text{-new}}, \tag{15}$$

as $\mathbb{Q}_\ell[\mathrm{Gal}_K]$-modules where t acts on $V_\ell^{(i)}(\mathrm{Jac}(\mathscr{C}_d))^{d\text{-new}}$ as ζ_d^i. There is an explicit character χ (depending on d and c) such that

$$V_\ell^{(i)}(\mathrm{Jac}(\mathscr{C}_d))^{d\text{-new}} \simeq V_\ell^{(i)}(\mathrm{Jac}(\mathscr{C}'_d))^{d\text{-new}} \otimes \chi^i. \tag{16}$$

To describe it fix ζ_n an n-th root of unity in K. Such a choice determines an element (abusing notation) $\zeta_n \in \mathrm{End}(\mathrm{Jac}(\mathscr{C}))$ and an element ζ_n (abusing notation again) in $V_\ell(\mathrm{Jac}(\mathscr{C}))$ (its image under the map $\mathrm{End}(\mathrm{Jac}(\mathscr{C})) \otimes \mathbb{Z}_\ell \hookrightarrow \mathrm{End}(T_\ell(\mathrm{Jac}(\mathscr{C}))))$.

Lemma 5.4. *Let $L = K[\sqrt[n]{c}]$, let $r = [L : K]$, and let $V_\ell^{(i)}(\mathrm{Jac}(\mathscr{C}_d))^{d\text{-new}}$ denote the subspaces in the decomposition (15). Let $\sigma \in \mathrm{Gal}(L/K)$ be the generator sending $\sqrt[n]{c}$ to $\zeta_n^{n/r}\sqrt[n]{c}$ and let $\chi : \mathrm{Gal}(L/K) \to \overline{\mathbb{Q}_\ell}$ denote the character sending σ to $\zeta_n^{n/r}$. Then for all $1 \le i \le n$, prime to n we have*

$$V_\ell^{(i)}(\mathrm{Jac}(\mathscr{C}_d))^{d\text{-new}} \simeq V_\ell^{(i)}(\mathrm{Jac}(\mathscr{C}'_d))^{d\text{-new}} \otimes \chi^i.$$

Proof. Let $\varphi : \mathscr{C} \to \mathscr{C}'$ be the map $\varphi(x, y) = (x, \sqrt[n]{c}\, y)$ and let $\tilde{\sigma} \in \mathrm{Gal}_K$ be such that its restriction to $K[\sqrt[n]{c}]$ equals σ. We claim that

$$\tilde{\sigma} \circ \varphi = \zeta_n^{n/r} \cdot \varphi \circ \tilde{\sigma}. \tag{17}$$

If we compute both maps on a point (x, y), the left hand side equals

$$(\tilde{\sigma}(\sqrt[n]{c}) \cdot \tilde{\sigma}(x), \tilde{\sigma}(y)) = (\zeta_n^{n/r}\tilde{\sigma}(x), \tilde{\sigma}(y)),$$

which clearly equals the right hand side hence the claim. The result follows easily from (17) recalling that on $V_\ell^{(i)}(\mathrm{Jac}(\mathscr{C}_d)^{d\text{-new}})$ the element t acts by $(\zeta_n^{n/r})^i$. □

Note that there are two different types of twisting affecting the Galois representation, and the L-series p-th factor, namely unramified and ramified ones. Unramified twists already appeared while computing the Galois representation of Example 1 (in page 17). They affect the value of Frobenius, but does not change the image of inertia. To compute such twists on the components of positive genus, it is probably easier to compute the number of points of the components of the twisted curve rather than assuming the polynomial $f(x)$ is monic and then computing the twist (see [Sut20] for a fast method to count the number of points).

Ramified twists on the contrary affects the image of inertia. The use of weighted cluster is very handful to distinguish whether the twist by c_t appearing on the components of Proposition 3.1 are ramified or not.

Proposition 5.5. *Let t be a component of X corresponding to a cluster $\mathfrak{s}$. Then the components of $\mathcal{Y}_t^{(\ell)}$ are ramified twists of a non-singular superelliptic curve precisely when $d \nmid e_t$.*

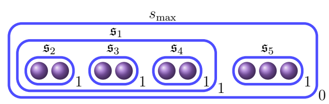
Proof. Follows from the fact that e_t is the valuation of c_t (see Proposition 3.1 for the notation). $\square$

In particular, Proposition 5.2 shows how to verify this condition from the weighted cluster picture. Note that for each $d \mid n$, if $d \nmid e_t$ the image of inertia in abelian part of the d-new part is given by t-copies of

$$\bigoplus_{\substack{i=1 \\ \gcd(i,d)=1}}^{d} \chi^i,$$

where χ is the ramified character corresponding to the extension $K(\sqrt[d]{p^{e_t}})/K$ and $t = \frac{2g(y_t^d = f_t(x_t))}{\phi(d)}$.

Example 1. Recall the weighted cluster picture:



Proposition 5.2 gives that: $e_{s_{\max}} = 0$, $e_{\mathfrak{s}_1} = 6$, $e_{\mathfrak{s}_2} = e_{\mathfrak{s}_3} = e_{\mathfrak{s}_4} = 8$ and $e_{\mathfrak{s}_5} = 3$. This implies that no ramified twist is involved on $\overline{Y_1^{(l)}}$ (its components are genus 1-curves), while the curves $\overline{Y_2}$, $\overline{Y_3}$ and $\overline{Y_4}$ (all of them of genus 2) involve a ramified twist χ corresponding to the extension $\mathbb{Q}_p(\sqrt[3]{p})/\mathbb{Q}_p$. Such curves have a 2-new part (of genus 0), a 3-new part (of genus 1) giving the representation of inertia $\chi \oplus \chi^2$ and a 6-new part (also of genus 1) giving the same representation of inertia.

Regarding the component $\overline{Y_5}$ (of genus 4), let ψ be the character attached to the representation $\mathbb{Q}_p(\sqrt{p})/\mathbb{Q}_p$. The curve has a 2-new part of genus 1, giving the representation $\psi \oplus \psi$ (since $2 \nmid e_{\mathfrak{s}_5}$); has a 3-new part (also of genus 1) which does not involve any twist (as $3 \mid 3$) hence inertia acts trivially in this 2-dimensional part; and a 6-new part (of dimension 4) where inertia acts via the quadratic character ψ.

To understand the toric part, we need to understand the action of $\mathrm{Gal}(\mathbb{Q}_p(\sqrt[3]{p})/\mathbb{Q}_p)$ on the component graph. The way to compute this action is very well explained in [DDM18] (see Examples 1.9 and 1.11). Concretely, it is given by what they call the "lift-act-reduce" procedure. Let $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ and $(\bar{x}, \bar{y})$ a point on $\overline{Y_m}$. Any lift corresponds to a point $(\tilde{x}, tildey)$ (on the curve $\mathscr{C}$), hence the reduction of its action corresponds to the point $(\sigma(\bar{x}), \sigma(\bar{y})$. In particular, it fixes the components $\overline{Y_m^{(i)}}$ and its intersection points as well. The same computation for the components of $\overline{Y_2}$ gives the action:

$$(\bar{x}, \bar{y}) \to \left(\frac{\tilde{x} - 1}{p}, \frac{\tilde{y}}{p}\right) \to \left(\frac{\sigma(\tilde{x}) - 1}{p}, \frac{\sigma(\tilde{y})}{p}\right) \to (\sigma(\bar{x}), \sigma(\bar{y})).$$

In particular it also fixes the three components as well as the intersection points. A similar computation proves the same result for the components of $\overline{Y_5}$, hence the image of inertia is the same over $\mathbb{Q}_p$ than over $\mathbb{Q}_p(\sqrt[3]{p})$ in this particular example.

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FaMAF-CIEM (CONICET), Universidad Nacional de Córdoba, Medina Allende s/n, Ciudad Universitaria, 5000 Córdoba, República Argentina.

Email address: `apacetti@famaf.unc.edu.ar`

CONICET, FCEN-Universidad Nacional de Cuyo, Padre Jorge Contreras 1300. Parque General San Martín, 5502 Mendoza, República Argentina.

Email address: `angelmza1990@gmail.com`