AN EXOTIC EXAMPLE OF A TENSOR PRODUCT OF A CM ELLIPTIC CURVE AND A WEIGHT 1 FORM

ARIEL PACETTI

ABSTRACT. The representation obtained as a tensor product of a rational elliptic curve with a weight 1 modular form is in general an irreducible four dimensional representation. However, there are some instances where such representation is reducible. In this short article we give an exotic way to obtain such a reducible representation.

INTRODUCTION

Let E be a rational elliptic curve, and let f be a weight 1 modular form. Given p a rational prime, let $\rho_{E,p}$ be the p-adic Galois representation attached to E and $\rho_{f,p}$ the 2-dimensional p-adic Galois representation attached to f by Deligne-Serre (see [DS74]). The Galois representation $\rho_{E,p}\otimes\rho_{f,p}$ is a rational four dimensional representation, which is generally irreducible. However there are some instances when such representation decomposes as a direct sum of two 2-dimensional ones. A first such instance occurs when E has complex multiplication by an imaginary quadratic field K and f corresponds to $\operatorname{Ind}_{G_K}^{G_\mathbb{Q}} \chi$ for χ a character of G_K (as explained in Lemma 1.1). In this article, we present a related situation, where the form f is obtained as the induction of a quadratic character of a real quadratic extension L.

The explanation behind our example is that dihedral weight 1 modular forms can have more "endomorphisms" than expected, in the sense that they can be obtained as induction of characters of different quadratic fields. Then even when our form does not seem to belong to the previous described situation (the fact that L is real quadratic implies it does not match the CM field), it does indeed.

The example appeared while studying the $\mathbb{Q}$-curve attached to a trivial solution of the Diophantine equation $x^4-5y^2=z^p$ (as described in [PT20]).

Acknowledgments. The author would like to thank David Loeffler for fruitful conversations regarding the given example.

1. THE INVOLVED OBJECTS

If E is a rational elliptic curve, and K is a number field, let E_K denote the base change of E to K.

Lemma 1.1. *Let E be a rational elliptic curve with complex multiplication by an order in an imaginary quadratic field K, and let χ be a character of K. Then $\operatorname{Ind}_{G_K}^{G_\mathbb{Q}}(\rho_{E_K,p}\otimes\chi)$ decomposes as a direct sum of two irreducible 2-dimensional representations.*

Proof. Since E is an elliptic curve with complex multiplication, there exists a Hecke character θ of infinity type $(1,0)$ such that $\rho_{E,p}=\operatorname{Ind}_{G_K}^{G_\mathbb{Q}}\theta_p$, where θ_p is the p-adic character attached to θ via class field theory. Let τ be an element of $G_\mathbb{Q}$ giving the non-trivial element of $\operatorname{Gal}(K/\mathbb{Q})$. By induction-restriction, $\rho_{E_K,p}=\theta_p\oplus\theta_p^\tau$ (where $\theta_p^\tau(\sigma)=\theta_p(\tau\sigma\tau^{-1})$) hence $\operatorname{Ind}_{G_K}^{G_\mathbb{Q}}(\rho_{E_K,p}\otimes\chi)=\operatorname{Ind}_{G_K}^{G_\mathbb{Q}}(\theta_p\otimes\chi)\oplus\operatorname{Ind}_{G_K}^{G_\mathbb{Q}}(\theta_p^\tau\otimes\chi)$ as claimed. □

If we replace the field K of complex representation by another quadratic extension L, in general the representation $\operatorname{Ind}_{G_L}^{G_\mathbb{Q}}\rho_{E_L,p}\otimes\chi$ will be an irreducible 4-dimensional representation, except for some very exceptional situations related to dihedral weight 1 modular forms. Let E be the elliptic curve 256-a1

$$E: y^2=x^3+x^2-13x-21. \tag{1}$$

2010 *Mathematics Subject Classification.* 14K22,11F80.

Key words and phrases. Quadratic twists, CM elliptic curves.

AP was partially supported by Proyecto Consolidar 2018-2021 33620180100781CB.

It has complex multiplication by $\mathbb{Z}[\sqrt{-2}]$, hence its L-series matches that of a Hecke character θ of infinity type $(1,0)$ and conductor $(\sqrt{-2})^5$ over the imaginary quadratic field $K = \mathbb{Q}(\sqrt{-2})$. Equivalently, for any prime p, the Galois representation $\rho_{E,p}$ matches $\operatorname{Ind}_{G_K}^{G_\mathbb{Q}} \theta_p$, where θ_p is the p-adic character attached to θ via class field theory.

Let $L = \mathbb{Q}(\sqrt{3})$ and let χ be the quadratic character attached to the quadratic extension $L[\sqrt{1+\sqrt{3}}]$, of conductor $\mathfrak{p}_2^5$, where $\mathfrak{p}_2$ is the unique (ramified) prime ideal of L dividing 2. Then $\operatorname{Ind}_{G_L}^{G_\mathbb{Q}} \chi$ is a weight 1 modular form f of level $3 \cdot 2^7$ and Nebentypus δ, the quadratic character of conductor 24 corresponding to the imaginary quadratic field $\mathbb{Q}(\sqrt{-6})$. For each prime p, the form f has attached a 2-dimensional p-adic Galois representation. Let E_L denote the base change of E to L. The four dimensional representation $\rho_{E,p} \otimes \rho_{f,p}$ matches $\operatorname{Ind}_{G_L}^{G_\mathbb{Q}} \rho_{E_L \otimes \chi, p}$. Although the character χ is not induced from a rational character (via the norm map) and $\rho_{E_L,p}$ is absolutely irreducible, the representation $\operatorname{Ind}_{G_L}^{G_\mathbb{Q}} \rho_{E_L \otimes \chi, p}$ is reducible due to the rare properties of the weight one form f.

Theorem 1.2. *The representation* $\operatorname{Ind}_{G_L}^{G_\mathbb{Q}} \rho_{E_L \otimes \chi, p} = V_1 \oplus V_2$*, where each* V_i *is an irreducible representation corresponding to a weight* 2 *modular form of level* $3 \cdot 2^8$ *and Nebentypus* δ_K *(of conductor* 12*).*

Proof. The extension $L[\sqrt{1+\sqrt{3}}]/\mathbb{Q}$ is not Galois, its Galois group corresponds to an extension M corresponding to the decomposition field of $x^8 - 20x^6 + 160x^4 - 600x^2 + 4356$, which contains the imaginary quadratic fields $\mathbb{Q}(\sqrt{-2})$. Consider the following field diagram:

$$\begin{tikzcd}
 & M \arrow[ddl, dash] \arrow[d, dash] \arrow[ddr, dash] & \\
 & \mathbb{Q}(\sqrt{1+\sqrt{3}}) \arrow[d, dash] & \\
\mathbb{Q}(\sqrt{-6}) \arrow[dr, dash] & \mathbb{Q}(\sqrt{3}) \arrow[d, dash] & \mathbb{Q}(\sqrt{-2}) \arrow[dl, dash] \\
 & \mathbb{Q} &
\end{tikzcd}$$

The Galois group $\operatorname{Gal}(M/\mathbb{Q})$ equals D_4 (the dihedral group of eight elements) which can be represented by $\langle \rho, s \ : \ \rho^4 = 1 = s^2, \rho s = s\rho^3 \rangle$. The representation attached to f is precisely the unique irreducible two dimensional representation of such finite group. An easy computation proves that $\mathbb{Q}(\sqrt{-6}) = M^{\langle \rho \rangle}$, while the two other quadratic fields correspond to the fixed field of $\langle s, \rho^2 \rangle$ and $\langle \rho s, \rho^2 \rangle$ (there is no canonical choice of such subgroups). If we chose the rotation so that L is the fixed field of $C_1 = \langle s, \rho^2 \rangle$, then the form f matches the induction from C_1 to G of the character $\psi(s) = 1$, $\psi(\rho^2) = -1$.

Similarly, the two dimensional representation matches the induction from $C_2 = \langle \rho s, \rho^2 \rangle$ to G of the character $\tilde{\psi}(\rho s) = 1$ and $\tilde{\psi}(\rho^2) = -1$. There is also a second character giving rise to the same two dimensional representation, sending ρs to -1 and ρ^2 to -1. Identifying D_4 with $\operatorname{Gal}(L/\mathbb{Q})$ we get that $f = \operatorname{Ind}_{G_L}^{G_\mathbb{Q}} \tilde{\psi}$, where $\tilde{\psi}$ is a quadratic character of L of conductor $\mathfrak{p}_3 \cdot (\sqrt{-2})^4$, where $\mathfrak{p}_3$ is one of the primes of L dividing 3 (the second choice has conjugate conductor). The result now follows from Lemma 1.1 and a well known formula for the conductor and Nebentypus of an induced representation.

□

References

[DS74] Pierre Deligne and Jean-Pierre Serre. Formes modulaires de poids 1. *Ann. Sci. École Norm. Sup. (4)*, 7:507–530 (1975), 1974.

[PT20] Ariel Pacetti and Lucas Villagra Torcomian. $\mathbb{Q}$-curves, hecke characters and some diophantine equations, 2020.

FAMAF-CIEM, Universidad Nacional de Córdoba. C.P:5000, Córdoba, Argentina.
Email address: apacetti@famaf.unc.edu.ar