Maximal Ideals in Semilocal Categories

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A category $\mathcal{C}$ is *preadditive* if each of its Hom sets $\text{Hom}_{\mathcal{C}}(A, B)$ $(A, B \in \text{Ob}(\mathcal{C}))$ is endowed with an abelian group structure in such a way that composition is $\mathbb{Z}$-bilinear. That is, $f \circ (h + \ell) = f \circ h + f \circ \ell$ and $(f + g) \circ h = f \circ h + f \circ \ell$ for every $f, g \in \text{Hom}_{\mathcal{C}}(B, C)$, $h, \ell \in \text{Hom}_{\mathcal{C}}(A, B)$, $A, B, C \in \text{Ob}(\mathcal{C})$.

An *additive category* is a preadditive category with a zero object in which any two pair of objects has a coproduct (equivalenty, a product — in a preadditive category, product, coproduct and biproduct of any two objects coincide).

Let $\mathcal{C}$ be a category. An object Z is called:

- An *initial object* of $\mathcal{C}$ if for every $C \in \text{Ob}(\mathcal{C})$ there is exactly one morphism $Z \to C$.

- A *terminal object* of $\mathcal{C}$ if for every $C \in \text{Ob}(\mathcal{C})$ there is exactly one morphism $C \to Z$.

- A *null object* (or a *zero object*) if it is both initial and terminal.

In a preadditive category, an object is initial if and only if it is terminal.

Recall that a ring S is *semisimple artinian* if all right S-modules are injective; equiv., if all right S-modules are projective; equiv., if $\exists$ positive integers $t, n_1, n_2, \dots, n_t$ and division rings $k_1, k_2, \dots, k_t$ such that $S \cong \prod_{i=1}^{t} M_{n_i}(k_i)$.

A *commutative* ring is *semilocal* if it has only finitely many maximal ideals.

An arbitrary (non-necessarily commutative) ring R is *semilocal* if $R/J(R)$ is a semisimple artinian ring.

(The two notions agree on commutative rings!)

A category $\mathcal{C}$ is *semilocal* if it is a preadditive category with a non-zero object such that the endomorphism ring of every non-zero object is a semilocal ring.

Examples of semilocal categories:

(1) The full subcategory of Mod-R whose objects are all artinian R-modules.

(2) If R is a semilocal ring, the full subcategory of Mod-R whose objects are all finitely presented right R-modules.

(3) If R is a semilocal commutative ring, the full subcategory of Mod-R whose objects are all finitely generated R-modules.

(4) If R is a commut. noeth. semilocal domain of Krull dim. 1, the full subcategory of Mod-R whose objects are all torsion-free R-modules of finite rank.

(5) Recall that a module is *uniserial* if its lattice of submodules is linearly ordered under inclusion. The full subcategory of Mod-R whose objects are all finite direct sums of uniserial R-modules is a semilocal category.

If $\mathcal{C}$ is a category, *idempotents split* in $\mathcal{C}$ if for every object A of $\mathcal{C}$ and every endomorphism f of A with $f^2 = f$, there exist an object B and morphisms $g\colon A \to B$ and $h\colon B \to A$ such that $hg = f$ and $gh = 1_B$, where 1_B denotes the identity morphism of B.

The previous five examples are all examples of semilocal additive categories in which idempotents split. (Since they are all full subcategories of Mod-R, this simply means that their classes of objects are classes of modules closed under direct summands and finite direct sums.)

A *commutative additive monoid* is a set M endowed with a binary operation $+$ (addition) that is associative, commutative and has a neutral element 0.

$\mathcal{C}$ additive category.

Fix a skeleton $V(\mathcal{C})$.

Then $V(\mathcal{C})$ is a commutative additive monoid: if $A, B \in V(\mathcal{C})$,

$$A + B = A \coprod B.$$

($V(\mathcal{C})$ can possibly be a *large* monoid, i.e., it can be a proper class, not necessarily a set).

Theorem 1 *If $\mathcal{C}$ is a semilocal additive category in which idempotents split, the monoid $V(\mathcal{C})$ is a Krull monoid.*

Krull monoids are the analogues for commutative monoids of what Krull domains are in Commutative Algebra.

Valuations of abelian groups, Krull monoids

[Chouinard, 1981]

M additive, commutative, cancellative monoid

$M \subseteq G(M)$ and $G(M)$ is a torsion-free abelian group

A *discrete valuation* of an abelian group G is a surjective homomorphism $v\colon G \to \mathbb{Z}$.

[$\Rightarrow$ $G \cong \mathbb{Z} \oplus \ker v$]

$\{ x \in G \mid v(x) \geq 0 \}$ is the *valuation submonoid* of v.

[$\Rightarrow$ it is isomorphic to $\mathbb{N} \oplus \ker v$]

For a commutative monoid M, set $U(M) = \{ a \in M \mid a$ has an opposite $-a \in M \}$.

M is *reduced* if $U(M) = \{0\}$. For every monoid M, the monoid $M_{\text{red}} = M/U(M)$ is reduced.

A *discrete valuation monoid* is a monoid M with $M_{\text{red}} \cong \mathbb{N}$.

M is a *Krull monoid* if there exists a family $\{ v_i \mid i \in I \}$ of discrete valuations $v_i \colon G(M) \to \mathbb{Z}$ such that:

(1) $M = \{ x \in G(M) \mid v_i(x) \geq 0$ for every $i \in I\}$;

(2) for every $x \in G(M)$ the set $\{ i \in I \mid v_i(x) \neq 0 \}$ is finite.

Theorem 2 [Ulrich Krause, 1989]

A commutative integral domain R is a Krull domain if and only if the monoid $R^* := R \setminus \{0\}$ is a Krull monoid.

principal fractional ideals

divisorial fractional ideals

$D(M)$ the set of all divisorial fractional ideals.

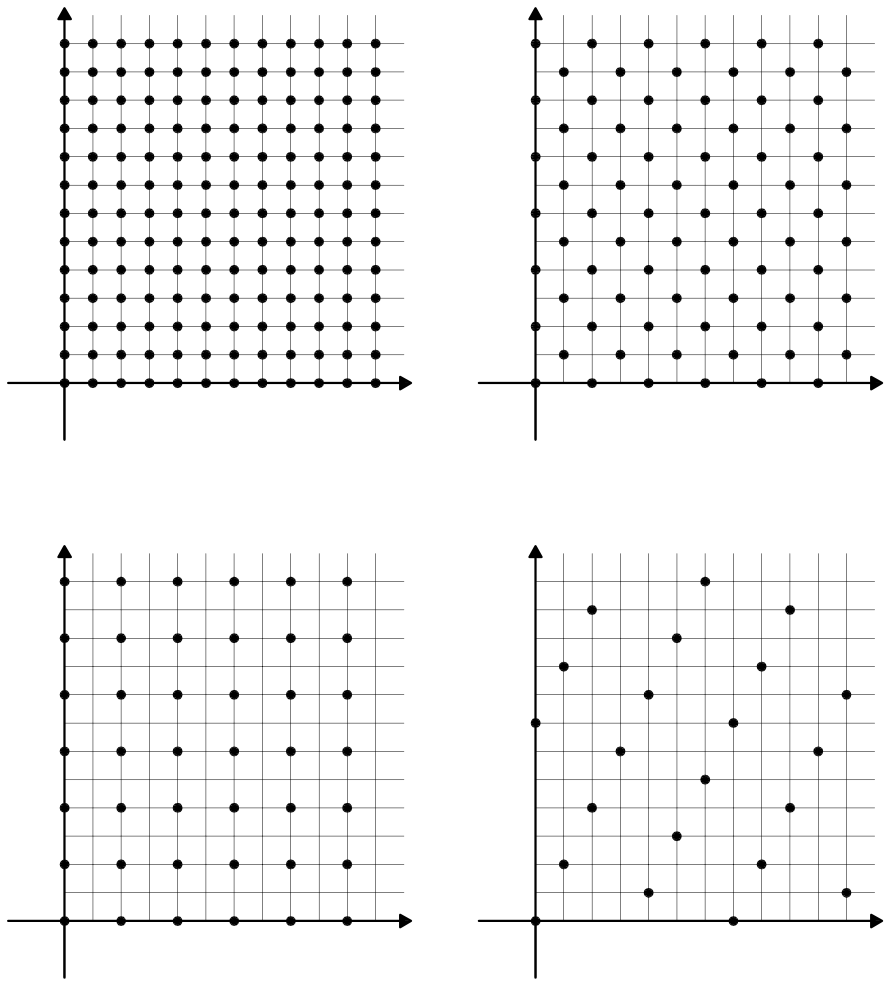
$D(M)$ is a commutative monoid with respect to the operation $*$ defined, for every $I, J \in D(M)$ by $I * J :=$ the intersection of all the principal fractional ideals containing $I + J$. Prin$(M) :=$ {non-zero principal fractional ideals}. It is a subgroup of $D(M)$.

divisor class semigroup Cl$(M) := D(M)/$Prin(M)

essential valuations (for every $x, y \in M$ with $v(x) \leq v(y)$, there exists an $s \in M$ with $x \leq y + s$ and $v(s) = 0$)

There is a natural pre-order $\leq$ on every commutative additive monoid M, called the *algebraic pre-order*, defined by $x \leq y$ if there exists $z \in M$ such that $x + z = y$.

M is a reduced Krull monoid if and only if there exist a set X and a subgroup G of $\mathbb{Z}^{(X)}$ such that $M \cong G \cap \mathbb{N}^{(X)}$, if and only if there is a monoid morphism f of M into a free commutative monoid $F = \mathbb{N}_0^{(X)}$ such that if $x, y \in M$ and $f(x) \leq f(y)$ in F implies $x \leq y$ in M. [$\Rightarrow$ reduced Krull monoid have a regular geometric structure.]



Four submonoids of $\mathbb{N}_0^2$ that are Krull monoids. Notice their regular geometric pattern.

Theorem. *If $\mathcal{C}$ is a semilocal additive category in which idempotents split, the monoid $V(\mathcal{C})$ is a Krull monoid.*

Hence, for every semilocal category $\mathcal{C}$, $V(\mathcal{C}) \hookrightarrow \mathbb{N}^{(X)}$ $\Rightarrow$ every object of a semilocal category can be described up to isomorphism by finitely many non-zero positive integers.

The technique

Let $\mathcal{C}$ be a preadditive category. An *ideal* $\mathcal{I}$ of $\mathcal{C}$: a subgroup $\mathcal{I}(A, B)$ of $\mathrm{Hom}_{\mathcal{C}}(A, B)$ for every $A, B \in \mathrm{Ob}(\mathcal{C})$, such that, for every $\varphi \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, $\psi \in \mathcal{I}(B, C)$, $\omega \in \mathrm{Hom}_{\mathcal{C}}(C, D)$, one has that $\omega\psi\varphi \in \mathcal{I}(A, D)$.

The *factor category* $\mathcal{C}/\mathcal{I}$: $\mathrm{Ob}(\mathcal{C}) = \mathrm{Ob}(\mathcal{C}/\mathcal{I})$ and, for every $A, B \in \mathrm{Ob}(\mathcal{C}) = \mathrm{Ob}(\mathcal{C}/\mathcal{I})$, $\mathrm{Hom}_{\mathcal{C}/\mathcal{I}}(A, B) := \mathrm{Hom}_{\mathcal{C}}(A, B)/\mathcal{I}(A, B)$.

Two examples:

(1) *The Jacobson radical.* It is the ideal $\mathcal{J}$ of $\mathcal{C}$ defined as follows. If A, B are objects of $\mathcal{A}$, $\mathcal{J}(A, B) := \{ f \in \mathrm{Hom}_{\mathcal{C}}(A, B) \mid 1_A - gf$ has a left inverse for all $g \in \mathrm{Hom}_{\mathcal{C}}(B, A) \}$.

(2) *The ideal of $\mathcal{C}$ associated to an ideal I of* $\mathrm{End}_{\mathcal{C}}(A)$, where A is a non-zero object of $\mathcal{C}$ [F.-Příhoda].

For a non-zero object A of $\mathcal{C}$ and a two-sided ideal I of $\mathrm{End}_{\mathcal{C}}(A)$, let $\mathcal{A}_I$ be the ideal of the category $\mathcal{C}$ defined as follows: a morphism $f\colon X \to Y$ in $\mathcal{C}$ is in $\mathcal{A}_I(X,Y)$ if and only if $\beta f \alpha \in I$ for every pair of morphisms $\alpha\colon A \to X$ and $\beta\colon Y \to A$ in $\mathcal{C}$. We call $\mathcal{A}_I$ the *ideal of $\mathcal{C}$ associated to I*. The ideal $\mathcal{A}_I$ is the greatest of the ideals $\mathcal{I}$ of $\mathcal{C}$ with $\mathcal{I}(A,A) \subseteq I$. It is easily seen that $\mathcal{A}_I(A,A) = I$. Clearly, the ideals associated to two distinct ideals of $\mathrm{End}_{\mathcal{C}}(A)$ are two distinct ideals of the category $\mathcal{C}$.

What about the maximal ideals of $\mathcal{C}$?

Lemma 3 [F.-Perone] *Let $\mathcal{C}$ be a preadditive category and $\mathcal{M}$ be a proper ideal of $\mathcal{C}$. Then $\mathcal{M}$ is a maximal ideal if and only if, for every object A of $\mathcal{C}$ with $\mathcal{M}(A,A) \neq \mathrm{End}_{\mathcal{C}}(A)$, one has that:*
(1) $\mathcal{M}(A,A)$ is a maximal ideal of $\mathrm{End}_{\mathcal{C}}(A)$, and (2) $\mathcal{M}$ is the ideal of $\mathcal{C}$ associated to $\mathcal{M}(A,A)$.

A preadditive category is *simple* if it has exactly two ideals, necessarily the trivial ones.

The factor category $\mathcal{C}/\mathcal{M}$ is simple for every preadditive category $\mathcal{C}$ and every maximal ideal $\mathcal{M}$.

Proposition 4 *The following conditions are equivalent for a preadditive category $\mathcal{C}$:*

(1) *$\mathcal{C}$ is a simple category.*

(2) *$\mathcal{C}$ has a non-zero object, and every non-zero object of $\mathcal{C}$ is a generator and a cogenerator for $\mathcal{C}$ and has a simple endomorphism ring.*

(3) *$\mathcal{C}$ has a non-zero object and there exists a simple ring R such that $\mathcal{C}$ is equivalent to a full subcategory of the category* proj-R *of all finitely generated projective right R-modules.*

Maximal ideals may not exist (even in small preadditive categories).

Example 5 Let k be a division ring and V_n a right vector space of dimension $\aleph_n$ for every $n < \omega$. Let $\mathcal{C}$ be the full subcategory of Mod-k whose objects are the vector spaces V_n, $n < \omega$, so that $\mathcal{C}$ is a small preadditive category with countably many objects. For every $V_k, W_k \in$ Ob($\mathcal{C}$) and $c \leq \aleph_\omega$, set $\mathcal{I}_c(V_k, W_k) :=$ $\{ f \in \mathrm{Hom}(V_k, W_k) \mid \mathrm{rank}(f) < c \}$. The ideals of $\mathcal{C}$ are

$0 = \mathcal{I}_1 \subset \mathcal{I}_{\aleph_0} \subset \mathcal{I}_{\aleph_1} \subset \mathcal{I}_{\aleph_2} \subset \cdots \subset \mathcal{I}_{\aleph_\omega}$.

Maximal ideals do not exist in $\mathcal{C}$.

This example also shows that, though every maximal ideal of a category $\mathcal{C}$ is the ideal associated to a maximal ideal of the endomorphism ring of a non-zero object of $\mathcal{C}$ (Lemma 2), the converse is not always true.

The converse is true when the category $\mathcal{C}$ is semilocal.

Proposition 6 *Let $\mathcal{C}$ be a semilocal category. Then:*

(1) *Every ideal of $\mathcal{C}$ associated to a maximal ideal of the endomorphism ring of a non-zero object of $\mathcal{C}$ is a maximal ideal of $\mathcal{C}$.*

(2) *In $\mathcal{C}$, every proper ideal is contained in a maximal ideal.*

(3) *Maximal ideals exist in $\mathcal{C}$.*

Let Max($\mathcal{C}$) be the *maximal spectrum* of a preadditive category $\mathcal{C}$, that is, the "class" of all maximal ideals of $\mathcal{C}$.

$\mathcal{C}_\lambda$ a preadditive category for every $\lambda \in \Lambda$, where Λ is a class. Define the *weak direct sum* $\oplus_{\lambda\in\Lambda}\mathcal{C}_\lambda$ of the categories $\mathcal{C}_\lambda$ as follows. The objects of $\oplus_{\lambda\in\Lambda}\mathcal{C}_\lambda$ are the finite sets $\{ (\lambda_1, A_1), (\lambda_2, A_2), \dots, (\lambda_n, A_n) \}$, where $n \geq 0$ is an integer, $\lambda_1, \dots, \lambda_n$ are distinct elements of Λ and A_i is a non-zero object of $\mathcal{C}_{\lambda_i}$ for every $i = 1, 2, \dots, n$. The set of all morphisms between two objects $\{ (\lambda_1, A_1), (\lambda_2, A_2), \dots, (\lambda_n, A_n) \}$ and $\{ (\mu_1, B_1), (\mu_2, B_2), \dots, (\mu_m, B_m) \}$ of the category $\oplus_{\lambda\in\Lambda}\mathcal{C}_\lambda$ is

$$\bigoplus_{\substack{i=1,\ldots,n \\ j=1,\ldots,m \\ \lambda_i=\mu_j}} \mathrm{Hom}_{\mathcal{C}_{\lambda_i}}(A_i, B_j).$$

Theorem 7 *Let $\mathcal{C}$ be a semilocal category. Then the Jacobson radical of $\mathcal{C}$ is the intersection of all maximal ideals of $\mathcal{C}$ and, for every object A in $\mathcal{C}$, there exist finitely many maximal ideals $\mathcal{M}_1, \ldots, \mathcal{M}_n$ $(n \geq 0)$ such that, for every maximal ideal $\mathcal{M}$ in $\mathcal{C}$, A is a non-zero object in $\mathcal{C}/\mathcal{M}$ if and only if $\mathcal{M} = \mathcal{M}_i$ for some $i \in \{1, \ldots, n\}$.*

$\mathcal{C}$ a semilocal category $\Rightarrow$ complete $\mathcal{C}$ to an additive category add($\mathcal{C}$) in which idempotents split (category of motives) $\Rightarrow$ add($\mathcal{C}$) also is semilocal and Max($\mathcal{C}$) = Max(add($\mathcal{C}$)) $\Rightarrow$ apply the functor

$$F\colon \mathrm{add}(\mathcal{C}) \to \oplus_{\mathcal{M}\in \mathrm{Max}(\mathcal{C})}\mathrm{add}(\mathcal{C})/\mathcal{M},$$

which is isomorphism reflecting and direct summand reflecting $\Rightarrow$ apply the functor V to get a monoid homomorphism

$V(F)\colon V(\mathrm{add}(\mathcal{C})) \to V(\oplus_{\mathcal{M}\in \mathrm{Max}(\mathcal{C})}\mathrm{add}(\mathcal{C})/\mathcal{M}) = \oplus_{\mathcal{M}\in \mathrm{Max}(\mathcal{C})} V(\mathrm{add}(\mathcal{C})/\mathcal{M})$, which is injective and a divisor homomorphism. Now each $\mathrm{add}(\mathcal{C})/\mathcal{M}$ is $\cong$ mod-R for some simple artinian ring $\Rightarrow$ each object A of the factor category $\mathrm{add}(\mathcal{C})/\mathcal{M} \cong$ mod-R is completely determined by its Goldie dimension (equal to the Goldie dimension of the semisimple artinian ring $\mathrm{End}_{\mathcal{C}/\mathcal{M}}(A)$). That is, $V(\mathrm{add}(\mathcal{C})/\mathcal{M}) \cong \mathbb{N}_0 \Rightarrow$ there is a divisor homomorphism $V(\mathrm{add}(\mathcal{C})) \to \mathbb{N}_0^{(\mathrm{Max}(\mathcal{C}))} \Rightarrow V(\mathrm{add}(\mathcal{C}))$ is a Krull monoid, every object of $\mathcal{C}$ can be described up to iso. with natural numbers, geometric regularity of finite d.-s. decompositions.

Example

Theorem 8 [F., TAMS 1996] *Let* U_R *be a uniserial module over an arbitrary ring* R*, let* $E = \mathrm{End}(U_R)$ *denote its endomorphism ring, and set*
$I := \{ f \in E \mid f$ *is not injective*$\}$ *and*
$K := \{ f \in E \mid f$ *is not surjective*$\}$*. Then* I *and* K *are two two-sided completely prime ideals of* E*, and every proper right ideal of* E *and every proper left ideal of* E *is contained either in* I *or in* K*. Moreover, exactly one of the following two conditions hold:*

(a) *Either* I *and* K *are comparable (that is,* $I \subseteq K$ *or* $K \subseteq I$*), in which case* E *is a local ring with maximal ideal* $I \cup K$*, or*

(b) I *and* K *are not comparable, and in this case* E/I *and* E/K *are division rings, and* $E/J(E) \cong E/I \times E/K$*.*

Two modules U and V are said to have

1. *the same monogeny class*, denoted $[U]_m = [V]_m$, if there exist a monomorphism $U \to V$ and a monomorphism $V \to U$;

2. *the same epigeny class*, denoted $[U]_e = [V]_e$, if there exist an epimorphism $U \to V$ and an epimorphism $V \to U$.

Theorem 9 [F., TAMS 1996] *Let* U_1, ..., U_n, V_1, ..., V_t *be* $n+t$ *non-zero uniserial right modules over a ring* R. *Then the direct sums* $U_1 \oplus \cdots \oplus U_n$ *and* $V_1 \oplus \cdots \oplus V_t$ *are isomorphic* R*-modules if and only if* $n = t$ *and there exist two permutations* σ *and* τ *of* $\{1, 2, \ldots, n\}$ *such that* $[U_i]_m = [V_{\sigma(i)}]_m$ *and* $[U_i]_e = [V_{\tau(i)}]_e$ *for every* $i = 1, 2, \ldots, n$.

If $\mathcal{C} = \{$uniserial R-modules$\}$, then $V(\mathcal{C}) \hookrightarrow F_m \times F_e$ is a subdirect product of two free commutative monoids F_m and F_e.