eLENA VII

VII Encuentro Nacional de Álgebra

UFRGS

UNIVERSIDADE FEDERAL DO RIO GRANDE DO SUL

# Liftings over the smallest Hopf algebra without Chevalley property (CP)

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CAPES FAPERGS CNPq

# Lifting method - Classification of Hopf algebras.

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Does $H$ satisfy (CP)?

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Does $H$ satisfy (CP)? $\longleftrightarrow$ Is $H_0$ a Hopf subalg. of $H$?

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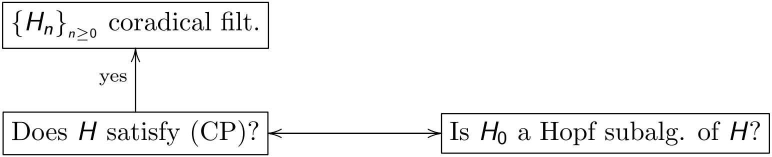
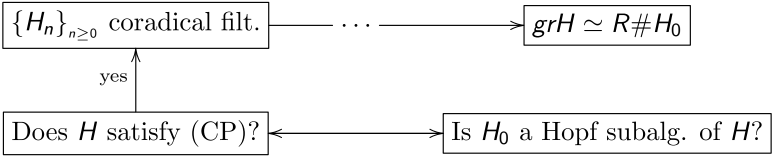
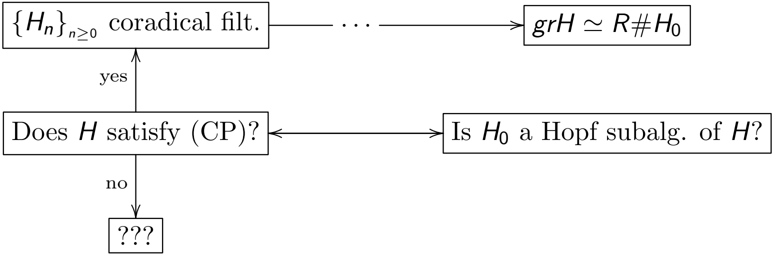


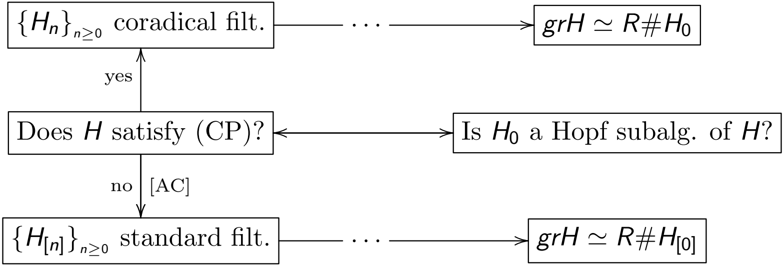
# Lifting method - Classification of Hopf algebras.



# Lifting method - Classification of Hopf algebras.



# Lifting method - Classification of Hopf algebras.



# The smallest Hopf algebra without (CP).

$$\mathcal{K} = \Bbbk\langle a, b, c, d;\ ab = \xi ba \quad ac = \xi ca \quad a^4 = 1$$
$$cb = 0 = bc \quad b^2 = 0 = c^2 \quad cd = \xi dc$$
$$a^2 c = b \quad bd = \xi db \quad ad = 1 = da\rangle$$

where $\xi$ is fourth primitive root of unity.

$$\Delta(a) = a \otimes a + b \otimes c \qquad \Delta(b) = a \otimes b + b \otimes d$$
$$\Delta(c) = c \otimes a + d \otimes c \qquad \Delta(d) = c \otimes b + d \otimes d$$
$$\varepsilon(a) = 1 = \varepsilon(d) \qquad \varepsilon(b) = 0 = \varepsilon(c)$$
$$\mathcal{S}(a) = d \quad \mathcal{S}(b) = \xi b \quad \mathcal{S}(c) = -\xi c \quad \mathcal{S}(d) = a$$

# The smallest Hopf algebra without (CP).

As coalgebra,

$$\mathcal{K} \simeq H_4 \oplus \mathcal{M}^*(2, \mathbb{k}),$$

where $H_4$ represents the Sweedler algebra. Then, again as coalgebra,

$$\mathcal{K}_0 \simeq \mathbb{k}C_2 \oplus \mathcal{M}^*(2, \mathbb{k}),$$

where $C_2$ denotes the cyclic group of order 2.

# $H_{[0]} = \mathcal{K}$, but $R = ???$

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with $D(H_{[0]})$ the Drinfel'd double of $H_{[0]}$.

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$$R \in {}^{H_{[0]}}_{H_{[0]}}\mathcal{YD} \longleftrightarrow {}_{D(H_{[0]})}\mathcal{M},$$

with $D(H_{[0]})$ the Drinfel'd double of $H_{[0]}$.

**Def. Drinfel'd double**

$D(H) = H^* \otimes H$ as vector space and:

$$(f \bowtie h)(g \bowtie k) = f(h_1 \twoheadrightarrow g_2) \bowtie (h_2 \twoheadleftarrow g_1)k \text{ and } 1_{D(\mathcal{K})} = \varepsilon \bowtie 1,$$

where $h \twoheadrightarrow f = \langle f_3 (\mathcal{S}^*)^{-1}(f_1), h \rangle f_2$ and $h \twoheadleftarrow f = \langle f, \mathcal{S}^{-1}(h_3) h_1 \rangle h_2$.

# $\mathcal{A}_4''$.

$$\mathcal{K}^* \simeq \mathcal{A}_4'' = \Bbbk\langle x, g; g^4 = 1, g^2 = 1 + x^2, gx = -xg\rangle$$

$$\begin{array}{ll} \Delta(g) = g \otimes g & \Delta(x) = x \otimes g + 1 \otimes x \\ \varepsilon(g) = 1 & \varepsilon(x) = 0 \\ \mathcal{S}(g) = g^3 & \mathcal{S}(x) = -xg^3 \end{array}$$

# Results.

## Prop.

As algebra:

$$D(\mathcal{K}) \simeq \Bbbk\langle a, b, c, d, x, g\rangle$$

satisfying the relations of $\mathcal{K}, \mathcal{A}_4''$ and more:

$$\begin{aligned} ag &= ga \quad & bg &= -gb \qquad & cg &= -gc \quad & dg &= gd \\ ax + \xi xa &= \sqrt{2}\xi(c + gb) & & & bx + \xi xb &= \sqrt{2}\xi(d - ga) \\ cx - \xi xc &= \sqrt{2}\xi(a - gd) & & & dx - \xi xd &= \sqrt{2}\xi(b + gc) \end{aligned}$$

# Results.

**Lemma 1**

There are 4 1-dimensional (simple) non-isomorphic representations over $D(\mathcal{K})$. Namely:

$$a \rightsquigarrow [\lambda_1] \qquad b \rightsquigarrow [0] \qquad c \rightsquigarrow [0]$$
$$d \rightsquigarrow [\lambda_1^{-1}] \qquad x \rightsquigarrow [0] \qquad g \rightsquigarrow [\lambda_1^2]$$

where $\lambda_1^4 = 1$.

# Results.

## Lemma 2

There are 12 2-dimensional simple non-isomorphic representations over $D(\mathcal{K})$. Namely:

$$a \rightsquigarrow \begin{bmatrix} \lambda_1 & 0 \\ 0 & -\xi\lambda_1 \end{bmatrix} \; b \rightsquigarrow \begin{bmatrix} 0 & \lambda_1^2 \\ 0 & 0 \end{bmatrix} \; c \rightsquigarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \; d \rightsquigarrow \begin{bmatrix} \lambda_1^{-1} & 0 \\ 0 & \xi\lambda_1^{-1} \end{bmatrix}$$

$$x \rightsquigarrow \begin{bmatrix} 0 & \frac{\sqrt{2}\xi}{2}(\lambda_1^3 + \lambda_1\lambda_2) \\ \sqrt{2}\xi(\lambda_1 - \lambda_1^3\lambda_2) & 0 \end{bmatrix} \qquad g \rightsquigarrow \begin{bmatrix} \lambda_2 & 0 \\ 0 & -\lambda_2 \end{bmatrix}$$

where $(\lambda_1, \lambda_2) \in \mathcal{I} = \{(\lambda_1, \lambda_2) : \lambda_1^4 = \lambda_2^4 = 1, \lambda_2 \neq \lambda_1^2\}$.

# Results.

## Cor. 1

There are 4 1-dimensional (simple) non-isomorphic modules $V_{\lambda_1}$ in ${}^{\mathcal{K}}_{\mathcal{K}}\mathcal{YD}$. Namely:

$$a \rightsquigarrow [\lambda_1] \qquad b \rightsquigarrow [0] \qquad c \rightsquigarrow [0] \qquad d \rightsquigarrow [\lambda_1^{-1}]$$

$$\lambda(v) = \begin{cases} 1 \otimes v, & \lambda_1^2 = 1 \\ a^2 \otimes v, & \lambda_1^2 = -1 \end{cases}$$

where $\lambda_1^4 = 1$.

## Results.

### Cor. 2

There are 12 2-dimensional simple non-isomorphic modules $V_{\lambda_1,\lambda_2}$ in ${}^{\mathcal{K}}_{\mathcal{K}}\mathcal{YD}$. Namely:

$$a \rightsquigarrow \begin{bmatrix} \lambda_1 & 0 \\ 0 & -\xi\lambda_1 \end{bmatrix} \quad b \rightsquigarrow \begin{bmatrix} 0 & \lambda_1^2 \\ 0 & 0 \end{bmatrix} \quad c \rightsquigarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad d \rightsquigarrow \begin{bmatrix} \lambda_1^{-1} & 0 \\ 0 & \xi\lambda_1^{-1} \end{bmatrix}$$

Case 1) $(\lambda_2 = \xi)$
$$\lambda(v) = d \otimes v + (\lambda_1 - \xi\lambda_1^{-1})c \otimes w$$
$$\lambda(w) = a \otimes w + \frac{(\lambda_1\xi + \lambda_1^{-1})}{2} b \otimes v$$

Case 2) $(\lambda_2 = -\xi)$
$$\lambda(v) = a \otimes v + (\lambda_1 + \xi\lambda_1^{-1})b \otimes w$$
$$\lambda(w) = d \otimes w + \frac{(-\lambda_1\xi + \lambda_1^{-1})}{2} c \otimes v$$

Case 3) $(\lambda_2 = 1)$ $\quad \lambda(v) = 1 \otimes v + 2\lambda_1 ac \otimes w \quad \lambda(w) = a^2 \otimes w$

Case 4) $(\lambda_2 = -1)$ $\quad \lambda(v) = a^2 \otimes v + 2\lambda_1 ab \otimes w \quad \lambda(w) = 1 \otimes w$

where $(\lambda_1, \lambda_2) \in \mathcal{I}$.

# Results.

For $V_{\lambda_1}$ :

$$c(v \otimes v) = v_{-1} \cdot v \otimes v_0 = \begin{cases} 1 \cdot \otimes v = v \otimes v, & \lambda_1^2 = 1 \\ a^2 \cdot v \otimes v = -v \otimes v, & \lambda_1^2 = -1 \end{cases}$$

Therefore,

$$\mathfrak{B}(V) = \begin{cases} \Bbbk[v], & \lambda_1^2 = 1 \\ \wedge V, & \lambda_1^2 = -1 \end{cases}$$

# Results.

For $V_{\lambda_1,\lambda_2}$ :

Case 1) $v \otimes v \mapsto \lambda_1^{-1} v \otimes v \qquad v \otimes w \mapsto \xi\lambda_1^{-1} w \otimes v + (\lambda_1 - \xi\lambda_1^{-1}) v \otimes w$

$w \otimes v \mapsto \lambda_1 v \otimes w \qquad w \otimes w \mapsto -\xi\lambda_1 w \otimes w + \dfrac{(\lambda_1 + \xi\lambda_1^{-1})}{2} v \otimes v$

Case 2) $v \otimes v \mapsto \lambda_1 v \otimes v \qquad v \otimes w \mapsto -\xi\lambda_1 w \otimes v + (\lambda_1^{-1} + \xi\lambda_1) v \otimes w$

$w \otimes v \mapsto \xi\lambda_1^{-1} v \otimes w \quad w \otimes w \mapsto \xi\lambda_1^{-1} w \otimes w + \dfrac{(\lambda_1^{-1} - \xi\lambda_1)}{2} v \otimes v$

Case 3) $v \otimes v \mapsto v \otimes v \qquad v \otimes w \mapsto w \otimes v - 2v \otimes w$

$w \otimes v \mapsto -v \otimes w \qquad w \otimes w \mapsto w \otimes w$

Case 4) $v \otimes v \mapsto v \otimes v \qquad v \otimes w \mapsto -w \otimes v + 2v \otimes w$

$w \otimes v \mapsto v \otimes w \qquad w \otimes w \mapsto w \otimes w$

where $(\lambda_1, \lambda_2) \in \mathcal{I}$.

# References.

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