An introduction to spectral geometry

Emilio Lauret
Universidad Nacional de Córdoba, Argentina

VI Workshop on differential geometry — EGEO
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- It is positive definite (i.e. $\langle \Delta f, f \rangle \geq 0$).

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Definition: M_1 and M_2 are called *isospectral* if $\mathrm{Spec}(M_1) = \mathrm{Spec}(M_2)$.

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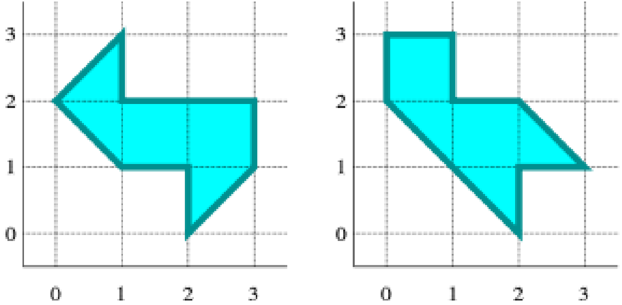

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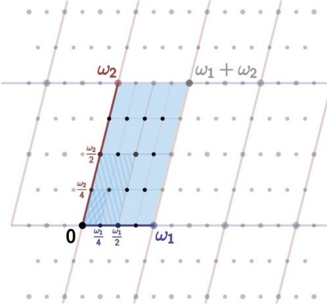

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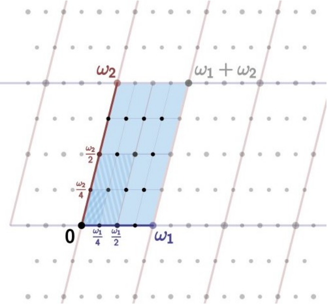

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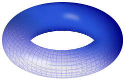
It is homeomorphic to , but it is flat.

Given a lattice Λ of $\mathbb{R}^n$, we associate the dual lattice

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$$\langle f, g\rangle = \frac{1}{\mathrm{vol}(M_\Lambda)} \int_{M_\Lambda} f(x)\overline{g(x)}dx.$$

For $v, w \in \Lambda^*$,

$$\langle f_v, f_w\rangle = \frac{1}{\mathrm{vol}(M_\Lambda)} \int_{M_\Lambda} e^{2\pi i\langle x,v-w\rangle} dx = \begin{cases} 1 & \text{if } v = w, \\ 0 & \text{if } v \neq w. \end{cases}$$

We have proven that $\{f_v\}_{v\in\Lambda^*}$ is an orthonormal set in $L^2(M_\Lambda)$.

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Hence, $\{f_v\}_{v \in \Lambda^*}$ is an orthonormal basis of $L^2(M_\Lambda)$ (since $C^\infty(M_\Lambda)$ is dense in $L^2(M_\Lambda)$).

Furthermore, $\{f_v\}_{v \in \Lambda^*}$ are eigenfunctions of Δ, Furthermore, $\{f_v\}_{v\in\Lambda^*}$ are eigenfunctions of Δ, thus

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In other words, if $\mu \in \mathbb{R}_{\geq 0}$, then

$$\mathrm{mult}(4\pi^2\mu) = \#\{v \in \Lambda^* : \|v\|^2 = \mu\}.$$

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where $d_j(\mu) = \#\{d : d \mid \mu,\ d \equiv j \pmod 4\}$.

In general $(\Lambda^*, \|\cdot\|^2)$ is a quadratic form.

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Theorem (Milnor, 1962)

The flat tori $\mathbb{R}^n/\Lambda_1$ *and* $\mathbb{R}^n/\Lambda_2$ *are isospectral if and only if the quadratic forms* $(\Lambda_1^*, \|\cdot\|^2)$ *and* $(\Lambda_2^*, \|\cdot\|^2)$ *represent the same numbers (with multiplicities)*

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It is also known that such example does not exist in dimension 3.

§3 Lens spaces

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$$\mathrm{mult}(\lambda_k) = \binom{k+n}{n} - \binom{k-2+n}{n}.$$