The G-invariant spectrum and non-orbifold singularities

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All work joint with Mary Sandoval

Main Ideas

- We are interested in the inverse spectral question for an orbit space M/G where M is a compact Riemannian manifold and G a closed subgroup of its isometry group.
- The spectrum we consider on M/G is the G-invariant spectrum of the Laplacian on M.

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- We generalize the Sunada-Pesce-Sutton technique to the G-invariant setting to produce isospectral non-isometric orbit spaces.

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- We are interested in the inverse spectral question for an orbit space M/G where M is a compact Riemannian manifold and G a closed subgroup of its isometry group.
- The spectrum we consider on M/G is the G-invariant spectrum of the Laplacian on M.
- We generalize the Sunada-Pesce-Sutton technique to the G-invariant setting to produce isospectral non-isometric orbit spaces.
- We show that constant sectional curvature and the presence of non-orbifold singularities are inaudible properties of the G-invariant spectrum.

G-invariant Sunada-Pesce-Sutton technique

Theorem (G-invariant Sunada-Pesce-Sutton technique)

Let M be a compact Riemannian manifold and $G \leq Isom(M)$ a compact Lie group. Suppose $H_1, H_2 \leq G$ are closed, representation equivalent subgroups. Then the orbit spaces M/H_1 and M/H_2 are isospectral in the sense that the H_i-invariant spectra of the Laplacian on M are equivalent.

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- As per the Sutton generalization, the subgroups $H_i \leq G$ are required to be closed, as opposed to finite or discrete.
- Sutton requires the actions of the H_i on M to be free, yielding isospectral manifolds M/H_i. The requirement that the actions be free is not necessary in the proof and the G-invariant version follows directly.

Representation equivalent subgroups

Theorem (An-Yu-Yu, 2013)

Let $n \geq 3$ be an odd integer and $m = (n-1)/2$. Set $H_1 = U(n)$ and $H_2 = Sp(m) \times SO(2n-2m)$. Then H_1 and H_2 are representation equivalent as subgroups of $SU(2n)$.

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Note: An-Yu-Yu let these representation equivalent subgroups H_i act on $SU(2n)$ to produce pairs of isospectral homogeneous manifolds. They then show via the long homotopy exact sequence that these pairs have distinct second homotopy groups.

Isospectral orbit spaces

We note that $H_i \leq SU(2n) \leq Isom(S^{4n-1})$ and consider the action of the H_i on S^{4n-1}.

Theorem (A.-Sandoval)

For each odd integer $n \geq 3$ the orbit spaces $O_1 = S^{4n-1}/H_1$ and $O_2 = S^{4n-1}/H_2$ are isospectral yet non-isometric.

Reduction of the orbit spaces

Principal isotropy reduction yields the following smooth SRF isometries which preserve the G-invariant spectra:

- $O_1 = S^{4n-1}/U(n) = S^7/U(2)$
- $O_2 = S^{4n-1}/Sp(m) \times SO(2n-2m) = S^7/Sp(1) \times O(2)$

First Lemma

Lemma 1

The space $O_1 = S^7/U(2)$ is isometric (and hence isospectral) to an orbifold with constant sectional curvature.

Proof sketch: In [Gorodski-Lytchak, 2015] it is shown that this space is isometric to the 3-hemisphere of constant sectional curvature 4. To show isospectrality we prove more generally that:

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Theorem (A.-Sandoval)

If M/G is isometric as a metric space to a Riemannian orbifold $\mathcal{O}$ then theses spaces are isospectral, i.e. the G-invariant spectrum on M is equivalent to the orbifold spectrum on $\mathcal{O}$.

Properties of the orbit spaces $O_i = S^7/H_i$

Table 1: $O_1 = S^7/U(2)$

Isotropy	qcodim	Points
Id	0	$v_1 \neq z \cdot v_2$
$U(1)$	1	$v_1 = z \cdot v_2$

Note: $v = (v_1, v_2) \in S^7 \subset \mathbb{C}^2 \oplus \mathbb{C}^2$ and $z \in \mathbb{C}$

Table 2: $O_2 = S^7/Sp(1) \times O(2)$

Isotropy	qcodim	Points
$Id \times Id$	0	$v_1 \neq 0,\ v_2 \neq \lambda \cdot v_3$
$Id \times O(1)$	1	$v_1 \neq 0,\ v_2 = \lambda \cdot v_3$
$Sp(1) \times Id$	1	$v_1 = 0,\ v_2 \neq \lambda \cdot v_3$
$Id \times O(2)$	3	$v_1 \neq 0,\ v_2 = v_3 = 0$
$Sp(1) \times O(1)$	2	$v_1 = 0,\ v_2 = \lambda \cdot v_3$

Note: $v = (v_1, v_2, v_3) \in S^7 \subset \mathbb{C}^2 \oplus \mathbb{C} \oplus \mathbb{C}$ and $\lambda \in \mathbb{R}$

Inverse spectral geometry

Theorem (A.-Sandoval)

The following properties are not determined by the G-invariant spectrum.

- isotropy type
- maximal isotropy dimension
- quotient codimension

Second Lemma

Lemma 2

The space $O_2 = S^7/Sp(1) \times O(2)$ admits a non-orbifold point and therefore has unbounded sectional curvature.

Proof sketch: Let $x \in S^7$ be a point with isotropy $Id \times O(2)$ and $y = \pi(x) \in O_2$. We show that the slice representation of the action at x is non-polar. It follows from [Lytchak-Thorbergsson, 2010] that y is a non-orbifold point and that sectional curvature is unbounded in any neighborhood of y.

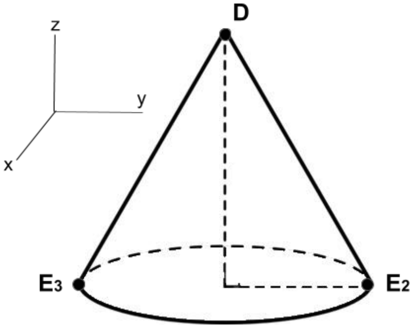
Inverse spectral geometry

Theorem (A.-Sandoval)

The following properties are not determined by the G-invariant spectrum.

- isotropy type
- maximal isotropy dimension
- quotient codimension
- constant sectional curvature
- presence of non-orbifold singularities

Non-metric parameterization of $O_2 = S^7/Sp(1) \times O(2)$



The End

Properties of the orbit space $O_2 = S^7/Sp(1) \times O(2)$

Table 2: $O_2 = S^7/Sp(1) \times O(2)$

Row	Isotropy	qcodim	Points
A	$Id \times Id$	0	$v_1 \neq 0,\ v_2 \neq \lambda \cdot v_3$
B_1	$Id \times O(1)$	1	$v_1 \neq 0,\ v_2 = \lambda \cdot v_3$
B_2	$Id \times O(1)$	2	$v_1 \neq 0,\ v_2 \neq 0,\ v_3 = 0$
B_3	$Id \times O(1)$	2	$v_1 \neq 0,\ v_2 = 0,\ v_3 \neq 0$
C	$Sp(1) \times Id$	1	$v_1 = 0,\ v_2 \neq \lambda \cdot v_3$
D	$Id \times O(2)$	3	$v_1 \neq 0,\ v_2 = v_3 = 0$
E_1	$Sp(1) \times O(1)$	2	$v_1 = 0,\ v_2 = \lambda \cdot v_3$
E_2	$Sp(1) \times O(1)$	3	$v_1 = 0,\ v_2 \neq 0,\ v_3 = 0$
E_3	$Sp(1) \times O(1)$	3	$v_1 = 0,\ v_2 = 0,\ v_3 \neq 0$

Note: $v = (v_1, v_2, v_3) \in S^7 \subset \mathbb{C}^2 \oplus \mathbb{C} \oplus \mathbb{C}$ and $\lambda \in \mathbb{R}^*$