Level sets of the Normal Sections on Isoparametric Hypersurfaces

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Normal Sections

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- The normal sections are the curves cut out of a submanifold M of $\mathbb{R}^{n+k}$ taking as cutting tool the affine subspace generated by a unit tangent vector an the normal space, at a given point p of M.
- Let p be a point in M and consider, in the tangent space $T_p(M)$ to M at p, a unit vector X

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- γ is parametrized by arc-length and such that $\gamma\,(0) = p$, $\gamma'\,(0) = X$.
- This curve is called a normal section of M at p in the direction of X.

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- In this work, we restrict to homogeneous isoparametric hypersurface of the sphere.
- The reason for restricting to the homogeneous case is that, for these hypersurfaces, the polynomials are "independent" of the point p and this is a desirable property.

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- Let p be a point in M, since we may think that the first polynomial h_1 is the one defining the unit sphere in $\mathbb{R}^{n+2}$

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The computing of the polynomials on homogeneous isoparametric hypersurfaces in the sphere can be found in [1] and [4].

The geometric meaning

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- Now we study the other level sets of normal sections.
- the image of polynomial on unit sphere $S(T_p(M))$ is some closed interval $[-m, m] \subset \mathbb{R}$.
- where m (respectively $-m$) is the maximum (respectively minimum) of P on $S(T_p(M))$. This is so because $P(-X) = -P(X)$.

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- The curve γ is contained in the affine subspace (1) which in this case is three dimensional.
- We may consider then, at $s = 0$, the classical Frenet curvature and torsion of the curve γ. we have the Frenet frame,
- $$v_1 = \gamma'(s), \quad v_2 = \frac{1}{\|\gamma''(s)\|}\gamma''(s), \quad v_3 = v_1 \times v_2$$

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- $$\kappa(p, X) = \|\gamma''(0)\| = \|\alpha_p(X, X)\|$$
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- the possible values for our polynomial $P(X)$ are

$$P(X) = cb \quad c > 0$$

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- Namely those with $b = \dfrac{r}{c}$ and $P^{-1}(-r) = -P^{-1}(r)$.
- There are many examples are considered in the next section where the sets $P^{-1}(r)$, $(r \neq 0)$ are smooth submanifolds of the sphere $S(T_p(M))$.

Regular Values

- We present in general fashion the regular values of $P(X)$. We use Lagrange multipliers to obtain the critical points of $P(X)$ on the unit sphere $S(T_p(M))$.

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- We present in general fashion the regular values of $P(X)$. We use Lagrange multipliers to obtain the critical points of $P(X)$ on the unit sphere $S(T_p(M))$.
- Let $P(X)$ be the polynomial that define a normal section on the homogeneous isoparametric hypersurfaces in the unit sphere.
- We want to find the critical points of $P(X)$ with the restriction, $\|X\| = 1$.

Regular Values

- We obtain the system of equations,

$$\begin{cases} \dfrac{\partial P(X)}{\partial x_1} = 2\lambda x_1 \\ \quad \dots \\ \dfrac{\partial P(X)}{\partial x_i} = 2\lambda x_i \\ \quad \dots \\ \dfrac{\partial P(X)}{\partial x_n} = 2\lambda x_n \\ \|X\|^2 = 1 \end{cases}$$

Regular Values

- We know by [1] *Corollary 4.3* that the polynomial $P(X)$ is homogeneous of degree three and there are neither cubes nor squares in the polynomial then,

$$P(X) = \frac{2}{3}\lambda$$

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- We know by [1] *Corollary 4.3* that the polynomial $P(X)$ is homogeneous of degree three and there are neither cubes nor squares in the polynomial then,

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- If we write $r = \frac{2}{3}\lambda$, then the tangent vectors $X \in S(T_E(M))$ that verify:

$$P(X) = r = \frac{2}{3}\lambda$$

are singular points of $P(X)$.

Particular cases

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We label these hypersurfaces by the degree g. The simplest examples, with non-trivial polynomials $P(X)$, are the well known Cartan Hypersurfaces.

The Case $g = 3$

These are called Cartan hypersurfaces, denoted by F_R, F_C, F_H and F_O, are full flag manifolds in the projective planes RP^2, CP^2, HP^2 and OP^2(real, complex, quaternionic and Cayley), respectively.

- We work in the general case $F = \mathbb{O}$, using the following notation,

$$
\begin{aligned}
X &= (0, 0, x_1, x_2, x_3) \quad , \quad x_j \in F = \mathbb{O} \\
x_1 &= (a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7) \\
x_2 &= (b_0, b_1, b_2, b_3, b_4, b_5, b_6, b_7) \\
x_3 &= (c_0, c_1, c_2, c_3, c_4, c_5, c_6, c_7)
\end{aligned}
$$

The Case $g = 3$

- Computing our polynomial $P(X)$, one gets:

$$P(X) = 9\sqrt{3}t(x_1x_2x_3), \qquad t(x_1x_2x_3) = 2Re((x_1x_2)x_3).$$

We verified

$$\|x_1\|^2 = \|x_2\|^2 = \|x_3\|^2 = \frac{1}{3}$$

on the other hand,

$$4\lambda^2\|x_1\|^2 = \|x_2\|^2\|x_3\|^2$$

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on the other hand,

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- We obtine,

$$\lambda = \pm 9$$

and therefore the singular values are,

$$r = \frac{2}{3}\lambda = 0, \ \pm 6$$

The Case $g = 3$

Remark:

The polynomial P has only three critical values on the unit sphere in $T_p(M)$, namely 0, its maximum and its minimum. Hence the level sets $P^{-1}(r)$ for $r \in (-6, 0) \cup (0, 6)$ are smooth manifolds (hypersurface) of the sphere $S\left(T_p(M)\right)$

The Case $g = 4$

For this degree there are four spaces where the Cartan-Münzner polynomial is obtained from the *Clifford Systems* as in Ferus-Karcher-Münzner [2].

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This method is also clearly described in [3]. There are still two remaining spaces which are not obtained by this construction, for which we used different methods.

The Case $g = 4$

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To distinguish the cases we write $M_{\mathbb{R}}$, $M_{\mathbb{C}}$ and $M_{\mathbb{H}}$ associating the field $\mathbb{R}$ to the first case, $\mathbb{C}$ to the second one and $\mathbb{H}$ to the third. For each of them the corresponding ambient Euclidean space is $\mathbb{R}^{2n}$, $\mathbb{C}^{2n}$ and $\mathbb{H}^{2n}$.

The Case $g = 4$

We may denote $X = ((\alpha, B), (C, \delta)) \in T_p(M)$ by

$$
\begin{array}{ccc}
B = (u_2, \ldots, u_n), & C = (v_1, \ldots, v_{n-1}) & u_j, v_j \in \mathbb{H} \\
\alpha = a_1 i + a_2 j + a_3 k & \delta = d_1 i + d_2 j + d_3 k & \in \Im(F) \\
u_s = b_{s,0} + b_{s,1} i + b_{s,2} j + b_{s,3} k & s = 2, \ldots, n & \\
v_r = c_{r,0} + c_{r,1} i + c_{r,2} j + c_{r,3} k, & r = 1, \ldots, n-1 &
\end{array}
$$

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With this notation, in the case $F = \mathbb{R}$ the polynomial may be written as:

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With this notation, in the case $F = \mathbb{R}$ the polynomial may be written as:

$$\frac{1}{96} P(X) = \quad (t_1 c_{1,0} + t_2 b_{n,0}) \sum_{r=2}^{n-1} b_{r,0} c_{r,0} \tag{2}$$

We determine the critical values for the polynomial given by formula (2). We want to illustrate with this, the general method to follow for the cases $F = \mathbb{C}$ and $F = \mathbb{H}$.

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$$\begin{aligned}
\frac{\partial P}{\partial b_{n,0}} &= 96 t_2 \sum_{r=2}^{n-1} b_{r,0} c_{r,0} = 2\lambda b_{n,0} \\
\frac{\partial P}{\partial c_{1,0}} &= 96 t_1 \sum_{r=2}^{n-1} b_{r,0} c_{r,0} = 2\lambda c_{1,0} \\
\frac{\partial P}{\partial b_{r,0}} &= 96 \left(t_1 c_{1,0} + t_2 b_{n,0} \right) c_{r,0} = 2\lambda b_{r,0} \\
\frac{\partial P}{\partial c_{r,0}} &= 96 \left(t_1 c_{1,0} + t_2 b_{n,0} \right) b_{r,0} = 2\lambda c_{r,0} \\
\|X\|^2 &= 1
\end{aligned}$$

The Case $g = 4$

Then,

$$2\lambda\left(b_{n,0}^2 + c_{1,0}^2\right) = 2\lambda\sum_{r=2}^{n-1} b_{r,0}^2 = 2\lambda\sum_{r=2}^{n-1} c_{r,0}^2$$

We may assume that $\lambda \neq 0$ and we obtain ,

$$\left(b_{n,0}^2 + c_{1,0}^2\right) = \sum_{r=2}^{n-1} b_{r,0}^2 = \sum_{r=2}^{n-1} c_{r,0}^2 = \frac{1}{3}$$

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and,

$$(32)^2 = 4\lambda^2 \left(b_{n,0}^2 + c_{1,0}^2 \right) = \frac{4}{3}\lambda^2$$

Therefore

$$\lambda = \pm 16\sqrt{3}$$

The Case $g = 4$

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$$r = \frac{2}{3}\lambda = \pm 32\frac{\sqrt{3}}{3}$$
$$r = 0$$

Remark:

In the cases $F = \mathbb{C}$ and $F = \mathbb{H}$ we obtain the same critical values and therefore, the polynomial P has only three critical values on the unit sphere in $T_p(M)$, namely 0, its maximum and its minimum. Hence the level sets $P^{-1}(r)$ for $r \in \left(-32\frac{\sqrt{3}}{3}, 0\right) \cup \left(0, 32\frac{\sqrt{3}}{3}\right)$ are smooth manifolds (hypersurface) of the sphere $S\left(T_p(M)\right)$

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As mentioned before, there are two homogeneous isoparametric hypersurfaces on sphere which are of degree $g = 4$ but cannot be described by *Clifford Systems*.

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The case $g = 4$, $(9, 6)$ is an homogeneous submanifold as indicated in [3].
As mentioned before, there are two homogeneous isoparametric hypersurfaces on sphere which are of degree $g = 4$ but cannot be described by *Clifford Systems*. We denote these two isoparametric hypersurfaces by M_{20} and M_{10}.

The Case $g = 4$

The case $g = 4$, $(9, 6)$ is an homogeneous submanifold as indicated in [3].
As mentioned before, there are two homogeneous isoparametric hypersurfaces on sphere which are of degree $g = 4$ but cannot be described by *Clifford Systems*. We denote these two isoparametric hypersurfaces by M_{20} and M_{10}.

Remark:

In the cases $g = 4$, $(9, 6)$, M_{20} and M_{10} we also obtain the same critical values and therefore the polynomial P has only three critical values. Hence the level sets $P^{-1}(r)$ for $r \in \left(-32\frac{\sqrt{3}}{3}, 0\right) \cup \left(0, 32\frac{\sqrt{3}}{3}\right)$ are smooth hypersurfaces of the sphere $S\left(T_p(M)\right)$

Conclusions

Remark:

In this way, we have shown: Let P be a polynomial that defines a normal section on a homogeneous isoparametric hypersurfaces of the sphere, whose number of distinct curvatures is less than or equal to four. Then P has only three critical values in the unit sphere of $T_p(M)$. Namely, 0, its maximum m and its minimum $-m$. Therefore the level sets $P^{-1}(r)$ for $r \in (-m, 0) \cup (0, m)$ are smooth hypersurfaces of the sphere $S\,(T_p(M))$.

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Acknowledgement

Thank you very much

The case $g = 6$

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With this notation, the polynomial defining normal sections of M_B at p takes the form

The case $g = 6$

$$
\begin{aligned}
&\left(\frac{1}{3}\sqrt{6}\right) P_B(X) \\
= \quad & r_3 r_5 r_7 + r_3 r_6 r_8 + r_3 r_{11} r_{13} + r_3 r_{12} r_{14} + r_4 r_{12} r_{13} + r_7 r_9 r_{11} + r_8 r_9 r_{12} \\
& + \left(-r_4 r_6 r_7 - r_5 r_9 r_{13} - r_6 r_{10} r_{13} - r_6 r_9 r_{14} - r_7 r_{10} r_{12}\right) + \\
& + 3\left(r_4 r_5 r_8 + r_5 r_{10} r_{14} + r_8 r_{10} r_{11} - r_4 r_{11} r_{14}\right) + \\
& + \left(\frac{2}{\sqrt{3}}\right)\left(-r_3 r_6 r_7 - r_3 r_{12} r_{13} - r_6 r_9 r_{13} + r_7 r_9 r_{12}\right)
\end{aligned}
$$

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$$\left(\frac{1}{3}\sqrt{6}\right) P_S(X) = r_3 r_5 r_7 + r_3 r_6 r_8 + (-r_4 r_6 r_7) + \left(\frac{2}{\sqrt{3}}\right)(-r_3 r_6 r_7) + 3(r_4 r_5 r_8)$$

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For M_S the singular points are

$$\lambda \in \{0, \pm\frac{1}{2}, \pm\frac{1}{2}\sqrt{3}, \pm\frac{4}{21}\sqrt{7}\}$$