Algebraic Dimension of Complex Nilmanifolds

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Definition and properties

Definition

(M, J) compact complex manifold.

The algebraic dimension $a(M)$ is the transcendence degree of its field of meromorphic functions.

- $a(M) = dim_{\mathbb{C}} M$ if M is projective algebraic.
- In general, $a(M) \leq dim_{\mathbb{C}} M$.
- $a(M)$ is computed for tori and twistor spaces of ASD manifolds.

Main problem

A nilmanifold M is a compact quotient $\Gamma \backslash G$ of a real nilpotent Lie group G by a discrete subgroup Γ.

By complex nilmanifold we will mean a nilmanifold endowed with an invariant complex structure I.

Problem

Study the algebraic dimension $a(M)$ of a complex nilmanifold.

We will prove that

$$a(M) \leq dim\, \mathfrak{H}^1(M),$$

where $\mathfrak{H}^1(M)$ is the space of holomorphic differentials, i.e. of closed holomorphic 1-forms.

Algebraic Reduction

Let X be a complex manifold, and $\varphi : X \dashrightarrow \mathbb{C}^N$ a meromorphic map defined by generators of the field of meromorphic functions.

An algebraic reduction of X is a compactification of $\varphi(X)$ in $\mathbb{C}P^N \supset \mathbb{C}^N$.

It is known to be a compact algebraic variety [Ueno; Campana].

Remark

The map φ is defined for more general spaces X.

For smooth manifolds we'll use the following

Definition

M compact complex manifold $\Rightarrow \exists$ a smooth projective manifold X, a rational map $\varphi : M \dashrightarrow X$ and a diagram

$$\begin{array}{ccc} & X' & \\ {\scriptstyle a} \swarrow & & \searrow {\scriptstyle b} \\ M & \overset{\varphi}{\dashrightarrow} & X \end{array}$$

where X' is smooth and the top two arrows are proper holomorphic maps with a a proper bimeromorphic modification, such that the corresponding fields $\mathrm{Mer}(M) = \mathrm{Mer}(X)$. The map $\varphi : M \dashrightarrow X$ is the algebraic reduction of M.

Currents

Definition

A positive closed (1,1)-current T on a complex manifold is said to have analytic singularities if locally $T = \theta + dd^c\phi$ for a smooth form θ and a plurisubharmonic function $\phi = c\log(|f_1|^2 + ... + |f_n|^2)$ where $f_1, ... f_n$ are analytic functions and c a constant.

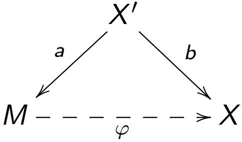
Such currents have a decomposition into absolutely continuous and singular part, where the absolutely continuous part is positive and closed.

Kähler rank

Definition

Let M be a complex manifold. The Kähler rank $k(M)$ is the maximal rank of the absolutely continuous part of a positive, closed (1,1)-current on M with analytic singularities.

Given a compact complex manifold, let $\varphi : M \dashrightarrow X$ be the algebraic reduction of M:



Definition

Let η be a positive $(1,1)$-form on X. The current T_η induced by η on M is defined as $a_* b^* \eta$.

Remark

- Since pushforward of a form is a current, T_η is a current, and not a form.
- Since a is one-to-one everywhere, except on an analytic set $E \subset X'$, the current $T_\eta = a_* b^* \eta$ is smooth outside of E.
- The positivity and closedness are preserved, as well as the rank in a general point.

If η is closed and positive, then T_η has analytic singularities.

Link between the algebraic dimension and Kähler rank

Proposition

Let M be a complex variety. Then the algebraic dimension is bounded by the Kähler rank:

$$a(M) \leq k(M).$$

Proof.

Let $\varphi : M \dashrightarrow X$ be the algebraic reduction map. Pullback of a Kähler form from X to M is a current of rank $\dim X$ at all points where it is absolutely continuous.

□

Avaraging of differential forms

Let $M = \Gamma\backslash G$ be a compact nilmanifold and ν a volume element on M induced by a the Haar measure on the Lie group G [Milnor]. After a rescaling, we can suppose that M has volume 1.

Remark

The Haar measure on G is bi-invariant, because G admits a lattice, and any Lie group admitting a lattice is unimodular.

Given any covariant k-tensor field $T : TM \times \ldots \times TM \to \mathcal{C}^\infty(M)$, one can define

$$T_{inv} : \mathfrak{g} \times \ldots \times \mathfrak{g} \to \mathbb{R}$$

$$T_{inv}(x_1, \ldots, x_k) = \int_{p \in M} T_p(x_1|_p, \ldots, x_k|_p)\nu.$$

- $T_{inv} = T$ if T comes from a left-invariant one.
- If $\alpha \in \Omega^k(M)$, then $(d\alpha)_{inv} = d(\alpha_{inv})$ and $(\alpha_{inv} \wedge \beta)_{inv} = \alpha_{inv} \wedge \beta_{inv}$ [Belgun].

Definition

We call the map $\mathrm{Av} : (T^*)^{\otimes k} \to (\mathfrak{g}^*)^{\otimes k}$, $\mathrm{Av}(T) := T_{inv}$ averaging on a nilmanifold.

- The averaging defines a linear map $\tilde{\nu} : \Omega^k(M) \to \Lambda^k \mathfrak{g}^*$, defined by $\tilde{\nu}(\alpha) = \alpha_{inv}$, which commutes with the differentials.
- By Nomizu theorem $\tilde{\nu}$ induces an isomorphism $H^k(M) \to H^k(\mathfrak{g})$. In particular, every closed k-form α on M is cohomologous to the invariant k-form α_{inv} obtained by the averaging.

Holomorphic Differentials on nilmanifolds

By using the averaging we can show the following

Proposition

Let $(M = \Gamma\backslash G, I)$ be a complex nilmanifold and h a holomorphic differential. Then h is an invariant differential form.

Proof.

A holomorphic differential h is cohomologous to the invariant form h_{inv} obtained by the averaging process. Since I is invariant, h_{inv} has to be of type $(1,0)$ and thus $h = h_{inv}$. Indeed, closed (1,0)-forms cannot be exact, because they are holomorphic, hence (if exact) equal to differentials of a global holomorphic function. □

Corollary

Let $M = \Gamma \backslash G$ be a complex nilmanifold, and $\mathfrak{H}^1(M)$ the space of holomorphic differentials. Then

$$\mathfrak{H}^1(M) = \left(\frac{\mathfrak{g} \otimes \mathbb{C}}{\mathfrak{g}^1 + I\mathfrak{g}^1} \right)^* .$$

Proof.

Let h be a holomorphic differential. Since h is invariant, it can be identified with an element of $(\mathfrak{g} \otimes \mathbb{C})^*$. Moreover, $h = \alpha + iI\alpha$, with $\alpha \in \mathfrak{g}^*$, $d\alpha = 0$ and $d(I\alpha) = 0$. By the conditions

$$d\alpha(x, y) = -\alpha([x, y]) = 0, \quad d(I\alpha)(x, y) = \alpha(I[x, y]) = 0,$$

for every $x, y \in \mathfrak{g}$, we get $\alpha(\mathfrak{g}^1) = \alpha(I\mathfrak{g}^1) = 0$. □

Average of positive currents

We can extend the previous averaging to the positive current T_η induced by the algebraic reduction $\varphi : M = \Gamma \backslash G \to X$ from some Kähler form η on X.

Proposition

Let $M = \Gamma \backslash G$ compact and I invariant. Let T_η be the positive, closed (1,1)-current induced by the algebraic reduction $\varphi : M \to X$ from some Kähler form η on X.
The average $\mathrm{Av}(T_\eta)$ is a semipositive, closed, G-invariant differential form, and $\mathrm{rank}(\mathrm{Av}(T_\eta)) \geq$ rank of the absolutely continuous part of T_η.

Proof.

If X and Y are left-invariant, then $T_\eta(X, Y)$ is a measurable function when we consider T_η as a form with distributional coefficients in local coordinates.

$\Rightarrow \mathrm{Av}(T_\eta)$ is well defined and it is a closed invariant $(1,1)$-form.

By definition $\mathrm{Av}(T_\eta)(X, IX) = 0 \iff T_p(X|_p, IX|_p) = 0$ for almost all $p \in M$.

So $X \in \mathrm{Kernel}(\mathrm{Av}(T_\eta))$ only if X is in the kernel of T_p for almost all p. □

Remark

• As a corollary we obtain that if such space admits a Kähler current, it is Kähler.

• In particular from a result by Demailly and Paun it follows that such spaces are never in Fujiki's class $\mathcal{C}$.

Note that the proof of this fact by Demailly-Paun uses also the Kähler current arising from the pull-back of a Kähler form.

Positive (1,1) forms on a nilpotent Lie algebra

Definition

A semipositive Hermitian form on $(\mathfrak{g}, I)$ is a real form $\eta \in \Lambda^2(\mathfrak{g}^*)$ which is I-invariant (that is, of Hodge type (1,1)) and satisfies $\eta(x, Ix) \geq 0$ for each $x \in \mathfrak{g}$. It is called positive definite Hermitian if this inequality is strict for all $x \neq 0$.

Definition

A subalgebra $\mathfrak{a} \subset \mathfrak{g}$ is called holomorphic if $I(\mathfrak{a}) = \mathfrak{a}$ and $[\mathfrak{g}^{0,1}, \mathfrak{a}^{1,0}]^{1,0} \subset \mathfrak{a}^{1,0}$.

Relation with foliations

Proposition

Let $\mathfrak{a} \subset \mathfrak{g}$ be a vector subspace, and $B := \mathfrak{a} \cdot G$ the corresponding left-invariant sub-bundle in TG. Then

- B is involutive (that is, Frobenius integrable) iff $\mathfrak{a}$ is a Lie subalgebra of $\mathfrak{g}$.
- B is a holomorphic sub-bundle iff $\mathfrak{a}$ is a holomorphic subalgebra.

Proof.

Let $x, y \in \mathfrak{a}$ and denote by the same letters the corresponding left-invariant vector fields. Clearly, B is involutive if and only if $\mathfrak{a}$ is a Lie subalgebra of $\mathfrak{g}$. Similarly we have that B is holomorphic if $[x + iIx, y - iIy] \in \mathfrak{a}^{1,0}$, for every $x \in \mathfrak{g}$ and $y \in \mathfrak{a}$. □

Remark

• $V = \mathfrak{g}^{(1,0)} + \mathfrak{a}^{(0,1)}$ is involutive iff $\mathfrak{a}$ is holomorphic and $V + \overline{V} = \mathfrak{g}^c$. So V is an elliptic structure in the terminology of Jacobowitz, and it defines a holomorphic foliation.

• If V_1 and V_2 are two elliptic structures complex manifold, containing the $(1, 0)$ tangent bundle, then $V_1 \cap V_2$ is also an elliptic structure.

Definition

Let η be a semipositive Hermitian form on $(\mathfrak{g}, I)$. The subspace

$$N(\eta) = \{x \in \mathfrak{g} \mid \eta(x, Ix) = 0\}$$

is called the null-space of η.

Proposition

The nullspace

$$N = \{x \in \mathfrak{g} \mid \iota_x \eta = 0\}$$

of a closed form $\eta \in \Lambda^r \mathfrak{g}^*$ is a Lie subalgebra of $\mathfrak{g}$.

Proof.

Take $x, y \in N$ and arbitrary vectors $z_1, \ldots, z_{r-1} \in \mathfrak{g}$. Then, by Cartan's formula,
$d\eta(x, y, z_1, \ldots, z_{r-1}) = \eta([x, y], z_1, \ldots z_{r-1}) = 0$, since the rest of the terms vanish, because $x, y \in N$. Therefore
$\eta([x, y], z_1, \ldots z_{r-1}) = 0$ for any $z_1, \ldots, z_{r-1} \in \mathfrak{g}$, this means that $\iota_{[x,y]}\eta = 0$, i.e $[x, y] \in N$. □

Theorem

Let η be a semipositive Hermitian form on a nilpotent Lie algebra $(\mathfrak{g}, I)$. If its nullspace $N(\eta)$ is a holomorphic subalgebra, then $N(\eta) \supseteq \mathfrak{g}^1 + I\mathfrak{g}^1$, where $\mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}]$.

Proof Since $N(\eta) = \mathfrak{a}$ is holomorphic, we have

$$[y + iIy, x - iIx]^{1,0} \in \mathfrak{a}^{1,0}, \quad \forall x \in \mathfrak{a}, \forall y \in \mathfrak{g}.$$

By using the integrability of I, the previous condition becomes

$$\eta([y, x], z) = -\eta(I[Iy, x], z), \quad \forall x \in \mathfrak{a}, \forall y, z \in \mathfrak{g}.$$

$$\Rightarrow \eta([y, x], I[y, x]) = -\eta([x, Iy], [x, y]), \quad \forall x \in \mathfrak{a}, \forall y \in \mathfrak{g}.$$

By $d\eta = 0$, one gets

$$\eta([y,x], I[y,x]) = -\eta(ad_x^2(y), Iy), \quad \forall x \in \mathfrak{a}, \forall y \in \mathfrak{g}. \quad (*)$$

By using $(*)$ it is possible to show that $\mathfrak{a}$ is an ideal of $\mathfrak{g}$.

Since η is a semipositive (1,1)-form and $\mathfrak{a}$ is its null-space, the relation $\eta([y,x], I[y,x]) = 0$ implies that $[x,y] \in \mathfrak{a}$.
Therefore, it is sufficient to show that $[x,[x,y]] \in \mathfrak{a}$ for any $x \in \mathfrak{a}$, i.e. that $[\mathfrak{a}, \mathfrak{g}^1] \subset \mathfrak{a}$.

Since $\mathfrak{g}$ is nilpotent there exists s such that $\mathfrak{g}^s = \{0\}$ and $\mathfrak{g}^{s-1} \neq \{0\}$ and we have the descending series of ideals

$$\mathfrak{g} = \mathfrak{g}^0 \supset \mathfrak{g}^1 \supset \ldots \supset \mathfrak{g}^i \supset \mathfrak{g}^{i+1} \supset \ldots \supset \mathfrak{g}^{s-1} \supset \mathfrak{g}^s = \{0\}.$$

One can prove that $[\mathfrak{a}, \mathfrak{g}^1] \subset \mathfrak{a}$ by induction on i.

Consequently, $\mathfrak{a} = N(\eta)$ is an ideal of $\mathfrak{g}$ and η induces a Kähler form on the nilpotent Lie algebra $\mathfrak{g}/\mathfrak{a}$.
By Benson-Gordon result, the Kähler nilpotent Lie algebra $\mathfrak{g}/\mathfrak{a}$ has to be abelian. Therefore $\mathfrak{g}^1 \subset \mathfrak{a}$.

Since η is (1,1)-form, its null-space $\mathfrak{a}$ is I-invariant, hence $\mathfrak{a}$ contains $\mathfrak{g}^1 + I\mathfrak{g}^1$. $\square$

Null-space foliation

Proposition

Let T be a positive closed (1,1)-current on a nilmanifold $M = G/\Gamma$, and $\mathcal{F}$ the null-space foliation of its absolutely continuous part. Then the sub-bundle associated with $\mathcal{F}$ contains a homogeneous sub-bundle Σ obtained by left translates of $\mathfrak{g}^1 + I\mathfrak{g}^1$.

Proof.

Let Av be the averaging map. The nullspace of the form $\mathrm{Av}(T)$ is contained in the intersection of all left translates of $\mathcal{F}$, hence the nullspace of $\mathrm{Av}(T)$ is also holomorphic. Then, since by previous theorem the null space of a semipositive Hermitian form on $(\mathfrak{g}, I)$ contains $\mathfrak{g}^1 + I\mathfrak{g}^1$, $N(\mathrm{Av}(T))$ contains $\mathfrak{g}^1 + I\mathfrak{g}^1$. □

Main results and consequences

Theorem

Let $(M = G/\Gamma, I)$ be a complex nilmanifold, and Σ be a a foliation obtained by left translates of $\mathfrak{g}^1 + I\mathfrak{g}^1$. Then all meromorphic functions on M are constant on the leaves of Σ.

Proof.

Let $M \to X$ be the algebraic reduction map and η the pullback of the Kähler form on X. Averaging transforms η into an invariant, closed, semipositive form. Then η vanishes on Σ by previous Proposition. □

Relation with holomorphic differentials

Theorem

Let M be a complex nilmanifold, $\mathfrak{H}^1(M)$ the space of holomorphic differentials on M, and $a(M)$ its algebraic dimension. Then

$$a(M) \leq \dim \mathfrak{H}^1(M).$$

Proof.

Let $M = \Gamma \backslash G$ be a nilmanifold, and $\varphi : M \dashrightarrow X$ the algebraic reduction map. The pullback $\varphi^* \omega_X$ of a Kähler form ω_X is a current T on M. By previous results the rank of the absolutely continuous part of T is no greater that

$$\dim \frac{\mathfrak{g}}{\mathfrak{g}^1 + I\mathfrak{g}^1} = \dim \mathfrak{H}^1(M).$$

Relation with Kähler rank

Theorem

Let (M, I) be a complex nilmanifold. Then $k(M) = \dim \mathfrak{H}^1(M)$.

Proof.

Consider the projection $\mathfrak{g} \to \mathfrak{a}$, where $\mathfrak{a} = \frac{\mathfrak{g}}{\mathfrak{g}^1 + I\mathfrak{g}^1}$. Since $\mathfrak{g}^1 + I\mathfrak{g}^1$ is I-invariant, $\mathfrak{a}$ has a complex structure and this map is compatible with it. Since $\mathfrak{a}$ is abelian, any 2-form on $\mathfrak{a}$ is closed (and gives a closed 2-form on the corresponding Lie group). Taking a positive definite Hermitian form, we obtain a positive current of rank $\dim \mathfrak{a} = \mathfrak{h}^1(M)$ on M. There are no currents with greater rank by the previous proposition. □

Corollary

Let $(M = \Gamma\backslash G, I)$ be a complex nilmanifold. Denote by $\mathfrak{h}_1$ a smallest I-invariant rational subspace of $\mathfrak{g}$ containing $\mathfrak{h} := \mathfrak{g}^1 + I\mathfrak{g}^1$. Let T be a complex torus obtained as quotient of $\mathfrak{g}/\mathfrak{h}_1$ by its integer lattice. Then any meromorphic map to a Kähler manifold is factorized through the holomorphic projection $\Psi : M \to T$.

Proof.

Let $\psi : M \dashrightarrow X$ be a meromorphic map to a Kähler (X, ω). For general $x \in X$, the zero space of the positive closed current $\psi^*\omega$ contains $\mathfrak{h}$, hence the fibers $F_x := \psi^{-1}(x)$ are tangent to $\mathfrak{h}$. The smallest compact complex subvariety of M containing a leaf of the foliation associated with $\mathfrak{h}$ is the corresponding leaf of $\mathfrak{h}_1$. Passing to the closures of the leaves of $\mathfrak{h}$, we obtain that F_x contain leaves of $\mathfrak{h}_1$. However, T is the leaf space of $\mathfrak{h}_1$. □

Relation with Albanese map

For a general compact complex manifold X, the Albanese variety $\mathrm{Alb}(X)$ is defined as the quotient of the dual space of the space of holomorphic differentials $H^0(X, d\mathcal{O})^*$ by the minimal closed complex subgroup containing the image of $H_1(X, \mathbb{Z})$ under the map

$$H_1(X, \mathbb{Z}) \to H_1(X, \mathbb{C}) \to H^0(X, d\mathcal{O})^*.$$

$\Rightarrow$ For a complex nilmanifold (M, I)

$$\mathrm{Alb}(M) = \frac{H^0(M, d\mathcal{O})^*/p(\mathfrak{h}_1)}{\mathrm{im}(H_1(M, \mathbb{Z}) \to H^0(M, d\mathcal{O}))^*/p(\mathfrak{h}_1)} = T,$$

[Rollenske]

$\Rightarrow$ $T = \mathrm{Alb}(M)$ and $a(M) = a(\mathrm{Alb}(M))$.

Examples

2-dimensional complex nilmanifolds are either tori or primary Kodaira surfaces $\Rightarrow$ their algebraic dimension is known.

Problem

Study the algebraic dimension of 3-dimensional complex nilmanifolds.

Many complex nilmanifolds admit holomorphic fibrations.
We will use

Remark

In general if a complex manifold M is the total space of a holomorphic fibration $\pi : M \to B$ we always have the inequality

$$a(M) \geq a(B).$$

Complex 2-tori

Following Birkenhake and Lange's description:

Let $T^4 = \mathbb{R}^4/\mathbb{Z}^4$ and $J = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with B nondegenerate.

If $X = \mathbb{C}^2/(\tau, Id_2)\mathbb{Z}^4$ is a complex tori defined by a complex 2×2 matrix τ, then the complex structure J_τ on T^4 such that $X \cong (T^4, J_\tau)$ as complex manifold is given by

$$J_\tau = \begin{pmatrix} y^{-1}x & y^{-1} \\ -y - xy^{-1}x & -xy^{-1} \end{pmatrix},$$

where $x = \mathrm{Re}(\tau)$ and $y = \mathrm{Im}(\tau)$. Reversing the construction

$$J \longrightarrow \tau_J = B^{-1}A + iB^{-1}.$$

In terms of a basis of $(1,0)$-forms:
if J_0 is a fixed complex structure and $\omega_j = e_j + iJ_0 e_j$, $j = 1, 2$ is a basis of $(1,0)$-forms for J_0, we define another complex structure J

$$\alpha_1 = \omega_1 + a\overline{\omega_1} + b\overline{\omega_2}, \quad \alpha_2 = \omega_2 + c\overline{\omega_1} + d\overline{\omega_2}$$

If $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = X_1 + iX_2$, then

$$J = \begin{pmatrix} Id + X_1 & X_2 \\ X_2 & Id - X_1 \end{pmatrix}^{-1} \begin{pmatrix} 0 & Id \\ -Id & 0 \end{pmatrix} \begin{pmatrix} Id + X_1 & X_2 \\ X_2 & Id - X_1 \end{pmatrix}.$$

Let τ_{ij} be the components of τ_J and

$$E = \begin{pmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{pmatrix} \in M_4(\mathbb{Z}).$$

Then the Neron-Severi group $NS(J)$ of J is given by

$$NS(J) = \{E \in M_4(\mathbb{Z}) | a + d\tau_{11} - b\tau_{12} + f\tau_{21} - c\tau_{22} + e \det(\tau) = 0\}.$$

$\Rightarrow$ The algebraic dimension of (T^4, J) is determined by

$$a(T^4, J) = \frac{1}{2} \max\{\mathrm{rank}(J^t E) | E \in NS(J), J^t E \geq 0\}.$$

Remark

• Not all complex structures are described in this way - we have the non-degeneracy condition on B which is required for (τ, Id_2) to be a period matrix. It is well known that $a(T^4, J)$ could be any of 0,1 or 2.

• Generically $a(T^4, J) = 0$. One has $a(T^4, J) = 1$ exactly when the torus admits a period matrix (τ, Id_2) with

$$\tau = \begin{pmatrix} \tau_1 & \alpha \\ 0 & \tau_2 \end{pmatrix}, \quad \alpha \notin (\tau_1, 1) M_2(\mathbb{Q}) \begin{pmatrix} 1 \\ \tau_2 \end{pmatrix}.$$

In particular, if $X = \begin{pmatrix} 0 & \sqrt{2} - i\sqrt{3} \\ 0 & 0 \end{pmatrix}$, then $a(T^4, J) = 1$.

3-dimensional complex nilmanifolds

Let J be a complex structure on a real 6-dimensional nilpotent $\mathfrak{g}$

(a) If J is non nilpotent, then $\exists$ a basis of $(1,0)$-forms (ω^j) s.t.

$$\begin{cases} d\omega^1 = 0, \\ d\omega^2 = E\,\omega^1 \wedge \omega^3 + \omega^1 \wedge \overline{\omega}^3, \\ d\omega^3 = A\,\omega^1 \wedge \overline{\omega}^1 + ib\,\omega^1 \wedge \overline{\omega}^2 - ib\overline{E}\,\omega^2 \wedge \overline{\omega}^1, \end{cases}$$

where $A, E \in \mathbb{C}$ with $|E| = 1$ and $b \in \mathbb{R} - \{0\}$.

(b) If J is nilpotent, then $\exists$ a basis of $(1,0)$-forms (ω^j)s. t.

$$\begin{cases} d\omega^1 &= 0, \\ d\omega^2 &= \epsilon\omega^1 \wedge \overline{\omega}^1, \\ d\omega^3 &= \rho\,\omega^1 \wedge \omega^2 + (1-\epsilon)A\,\omega^1 \wedge \overline{\omega}^1 + B\,\omega^1 \wedge \overline{\omega}^2 \\ & \quad + C\,\omega^2 \wedge \overline{\omega}^1 + (1-\epsilon)D\,\omega^2 \wedge \overline{\omega}^2, \end{cases}$$

where $A, B, C, D \in \mathbb{C}$ and $\epsilon, \rho \in \{0, 1\}$.

If the real and imaginary parts of the cpx structure equations constants are rational, then G admits a lattice Γ and J is rational on $M = \Gamma \backslash G$.

If J is non nilpotent, we have that $\dim \mathfrak{H}^1(M) = 1 = a(M)$.

If J is nilpotent, we have the following cases:

(b1) $\dim \mathfrak{H}^1(M) = 1 = a(M)$ if $\epsilon = 1$ and $\rho^2 + |B|^2 + |C|^2 \neq 0$.

(b2) $G = Nil_3 \times \mathbb{R}^3$, $\dim \mathfrak{H}^1(M) = 2$ if $\epsilon = 1$ and $\rho = B = C = 0$.

(b3) $\dim \mathfrak{H}^1(M) = 2$ if $\epsilon = 0$.

In the case $(b3)$: If $\rho^2 + |B|^2 + |C|^2 + |D^2| \neq 0$, then $J\mathfrak{g}^1 = \mathfrak{g}^1$ is a rational subalgebra and $\dim_{\mathbb{C}} \mathfrak{g}^1 = 1$

$\Rightarrow$ M is the total space of a holomorphic fibre bundle over a complex torus $\mathbb{T}$ of complex dimension 2.

Therefore, if $\mathbb{T}$ is algebraic, then $a(M) = \dim \mathfrak{H}^1(M) = 2$.
An example of the case (b3) is the Iwasawa manifold $M = \Gamma \backslash G$, where

$$G = \left\{ \begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} \mid z_i \in \mathbb{C} \right\}$$

and Γ is the lattice defined by taking z_i to be Gaussian integers.

- $\omega^1 = dz_1$, $\omega^2 = dz_2$, $\omega^3 = -dz_3 + z_1 dz_2$ are left-invariant on G.
- The projection $(z_1, z_2, z_3) \mapsto (z_1, z_2)$ induces a holomorphic $p : (M, J) \to (T^4, \hat{J})$

Proposition

For the invariant complex structures J on the Iwasawa manifold M we have $a(M) = a(T^4, \hat{J})$.

Proof.

From previous theorem any meromorphic function is constant on the fibers of the projection $M \to (T^4, \hat{J})$. This implies that $a(M) = a(T^4, \hat{J})$. □

$\Rightarrow$ We can have algebraic dimension equal to 0, 1 or 2.

THANK YOU VERY MUCH FOR THE ATTENTION!!