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Positive Ricci curvature and cohomogeneity-two torus actions

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KIT, INSTITUT FÜR ALGEBRA UND GEOMETRIE

Motivation

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Problem

Construct Riemannian manifolds satisfying given geometric properties.

Possible approach:

- M a compact smooth manifold (without boundary),
- G a compact connected Lie group acting effectively on M,
- g a G-invariant Riemannian metric.

Question: When does a closed G-manifold admit an invariant Riemannian metric with positive Ricci curvature?

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Background

The **cohomogeneity** of an action is the dimension of its orbit space.

Positive Ricci curvature on homogeneous spaces and cohomogeneity one manifolds

- Berestovskii (1995): $M = G/H$ admits an invariant metric of positive Ricci curvature if and only if $|\pi_1(M)| < \infty$.
- Grove, Ziller (2002): M of cohomogeneity one admits an invariant metric of positive Ricci curvature if and only if $|\pi_1(M)| < \infty$.

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Positive Ricci curvature and cohomogeneity two

- Searle, Wilhelm (2015): M of cohomogeneity two. If the fundamental group of a principal orbit is finite and the orbit space has positive Ricci curvature, then M admits an invariant metric of positive Ricci curvature.
- Bazaikin, Matvienko (2007): Every compact, simply connected 4-manifold with an effective action of T^2 admits an invariant metric of positive Ricci curvature.

Remark: Every compact, simply connected 4-manifold with an effective action of T^2 is equivariantly diffeomorphic to a connected sum of copies of $\mathbf{S}^4$, $\pm\mathbf{C}P^2$ or $\mathbf{S}^2 \times \mathbf{S}^2$ (Orlik, Raymond 1970).

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Cohomogeneity two torus actions and positive Ricci curvature

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Theorem (–, Corro) 2016

Every compact, smooth, simply connected $(n+2)$-manifold with a smooth, effective action of a torus T^n admits an invariant Riemannian metric of positive Ricci curvature.

- There exist compact, simply connected manifolds with a cohomogeneity two torus action in every dimension $n \geq 2$.
- The topological classification is only known up to dimension $n \leq 6$.

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Torus actions and positive Ricci curvature

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Corollary

For every integer k $\geqslant 4$, every connected sum of the form

$$\#(k-3)(\mathbf{S}^2 \times \mathbf{S}^3), \tag{1}$$

$$(\mathbf{S}^2 \tilde{\times} \mathbf{S}^3)\#(k-4)(\mathbf{S}^2 \times \mathbf{S}^3), \tag{2}$$

$$\#(k-4)(\mathbf{S}^2 \times \mathbf{S}^4)\#(k-3)(\mathbf{S}^3 \times \mathbf{S}^3), \tag{3}$$

$$(\mathbf{S}^2 \tilde{\times} \mathbf{S}^4)\#(k-5)(\mathbf{S}^2 \times \mathbf{S}^4)\#(k-3)(\mathbf{S}^3 \times \mathbf{S}^3), \tag{4}$$

has a metric with positive Ricci curvature invariant under a cohomogeneity-two torus action.

- Follows from the topological classification of compact, simply connected 5- and 6-manifolds with cohomogeneity two torus actions (Oh, 1983–1982).
- The manifolds in (1) are not new examples (Sha, Yang, 1991).

Torus actions and positive Ricci curvature

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Cohomogeneity two torus actions on simply connected manifolds

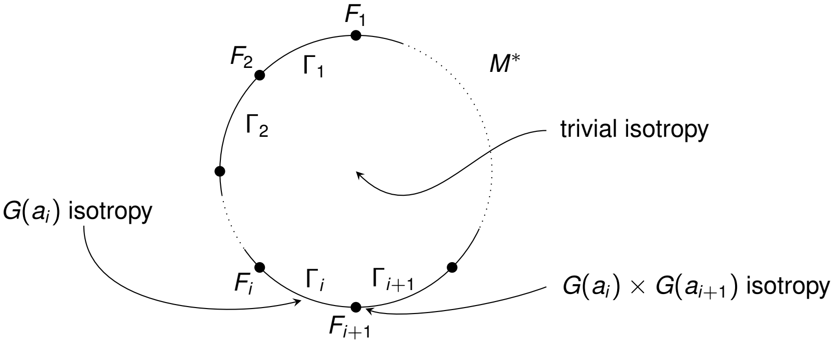
M a compact simply connected $(n+2)$-manifold, $n \geq 2$, with a cohomogeneity two action of T^n.

These manifolds were studied in the 1970s-1980s.

Kim, McGavran, Pak (1974) and Oh (1983).

- The orbit space M^* is homeomorphic to D^2.
- The only isotropy groups are T^2, T^1 and trivial.
- The boundary of M^* consists of $m \geq n$ edges Γ_i with circle isotropy $G(a_i)$ and m vertices F_i between the edges Γ_i and Γ_{i+1}, with isotropy $G(a_i) \times G(a_{i+1})$.

Cohomogeneity two torus actions on simply connected manifolds



Orbit space structure of a cohomogeneity-two torus action on a compact, simply connected manifold M.

Cohomogeneity two torus actions on simply connected manifolds

The orbit space is decorated with isotropy information, the so-called **weights**.

Definition

Let M and N be two compact, simply connected smooth $(n+2)$-manifolds with effective T^n actions. The orbit spaces M^* and N^* are ***isomorphic*** *if there exists a weight-preserving diffeomorphism between them.*

Theorem (Kim, McGavran, Pak 1974, Oh 1983)

Two closed, simply connected smooth $(n+2)$-manifolds with an effective T^n-action are equivariantly diffeomorphic if and only if their orbit spaces are isomorphic.

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Outline of the proof

Let M be a compact, simply connected $(n+2)$-manifold with a cohomogeneity two action of T^n. Assume $n \geq 2$.

- Let m be the number of vertices in the orbit space (i.e the number of orbits with isotropy T^2).
- Construct an $(m+2)$-manifold N_m with an effective T^m-action and a free action of a T^{m-n} subgroup of T^m so that N_m/T^{m-n} has an induced cohomogeneity two action of T^n with the same weights as the T^n action on M.

$$N_m = (D^2 \times T^m)/\sim$$

- By the equivariant classification theorem, M and N_m/T^{m-n} are equivariantly diffeomorphic.
- To construct the metric, one considers two cases:
  (a) the orbit space has at least 5 vertices.
  (b) the orbit space has at most 4 vertices.

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Outline of the proof

Construction of the metric

Case (a): The orbit space has at least 5 vertices.

- Construct a piecewise-smooth C^1 metric on $N_m = (D^2 \times T^m)/\sim$ that is invariant under the T^{m-n} action.
- This induces a piecewise-smooth C^1 Riemannian metric g on N_m/T^{m-n}.
- The metric g has positive Ricci curvature (O'Neill formulas).
- Smooth out the metric g while preserving positive Ricci curvature.

Case (b): The orbit space has at most 4 vertices.

- The manifold M is equivariantly diffeomorphic to $\mathbf{S}^4$, $\mathbf{S}^5$, $\mathbf{S}^3 \times \mathbf{S}^3$ or to a quotient of $\mathbf{S}^3 \times \mathbf{S}^3$ by a free linear torus action.

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Thank you