# The Eta Invariant on Nilmanifolds

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This is joint work with Ken Richardson, Texas Christian University

- **Dirac operators** were motivated by the need for a first-order differential operator $D$ that squared to the Laplacian $\Delta$

$$\Delta h = -\sum_{j=1}^{n} \frac{\partial^2 h}{\partial x_j^2}, \text{ for } h \in C^\infty(\mathbb{R}^n)$$

- If we let $D^2 = \Delta$, where

$$D = \sum_{j=1}^{n} c_j \frac{\partial}{\partial x_j}$$

  has constant coefficients $c_j \in \mathbb{C}$,
- we (quickly) produce a system of equations:

$$c_i^2 = -1; \qquad c_i c_j + c_j c_i = 0, \text{ if } i \neq j,$$

  that has no solutions.

- However, by allowing matrix coefficients, we obtain solutions: **Clifford Matrices**
- The vector space $\mathbb{C}^k$ on which the matrices and derivatives act is called the vector space of **spinors**.
- The minimum dimension is $k = 2^{\lfloor n/2 \rfloor}$.
- For $\mathbb{R}^3$, we may use the famous **Pauli spin matrices**

$$c_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, c_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, c_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

acting on $\mathbb{C}^2$

- The Clifford matrices can be used to form an associated **Clifford multiplication of vectors**; for $v \in \mathbb{R}^n$,

$$\mathrm{cliff}(v) = \sum_{j=1}^{n} v_j c_j \in \mathrm{End}(\mathbb{C}^k).$$

- So $\mathrm{cliff} : \mathbb{R}^n \to \mathrm{End}(\mathbb{C}^k)$ such that, for $v, w \in \mathbb{R}^n$,

$$\mathrm{cliff}(v)\mathrm{cliff}(w) + \mathrm{cliff}(w)\mathrm{cliff}(v) = -2\langle v, w \rangle .$$

- These conditions generate **Clifford Algebras**

- The **Dirac operator** acting on sections of the Clifford bundle $E$ over the Riemannian manifold $M^n$ is

$$D := \sum_{j=1}^{n} \mathrm{cliff}(e_j)\nabla_{e_j},$$

- where $\{e_j\}_{j=1}^{n}$ a local orthonormal frame of $TM$,
- $\forall s, s_1, s_2 \in \mathrm{Sect}^\infty(E)$, and $\forall V, W \in \mathrm{Sect}^\infty(TM)$.

$$\langle \mathrm{cliff}(V)s_1, s_2\rangle = -\langle s_1, \mathrm{cliff}(V)s_2\rangle\,,$$

- and $\nabla$ is a metric connection on $E$ satisfying

$$\nabla_V(\mathrm{cliff}(W)s) = \mathrm{cliff}(\nabla_V W)s + \mathrm{cliff}(W)\nabla_V s$$

- $N^n$ simply-connected $n$-dimensional nilpotent Lie group with 2-step nilpotent Lie algebra $\mathfrak{n}$
- $\Gamma$ cocompact (i.e., $\Gamma\backslash N$ compact), discrete subgroup of $N$
- $\langle \quad , \quad \rangle$ an inner product on $\mathfrak{n}$,
- which corresponds to a left-invariant metric on $N$,
- and descends to a (no longer left-invariant) Riemannian metric on $\Gamma\backslash N$

- Special Case: The $n = (2m+1)$-dim'l Heisenberg gp

$$H_m = \left\{ \begin{pmatrix} 1 & \mathbf{x} & z \\ 0 & \mathbf{I}_m & \mathbf{y} \\ 0 & 0 & 1 \end{pmatrix} : \mathbf{x}, \mathbf{y} \in \mathbb{R}^m,\ z \in \mathbb{R} \right\}$$

- $\mathfrak{h}_m = \mathrm{span}_{\mathbb{R}}\{X_1, Y_1, X_2, Y_2, \cdots, X_m, Y_m, Z\}$ with $[X_1, Y_1] = [X_2, Y_2] = \cdots = [X_m, Y_m] = Z$
- for $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ with $r | v_j w_j$ for all $j = 1, ..., m$,

$$\Gamma_{\mathbf{v},\mathbf{w},r} = \left\{ \begin{pmatrix} 1 & \boldsymbol{\gamma}_x & \gamma_z \\ 0 & \mathbf{I}_m & \boldsymbol{\gamma}_y \\ 0 & 0 & 1 \end{pmatrix} : \boldsymbol{\gamma}_x \in \mathbb{Z} \otimes \mathbf{w}, \boldsymbol{\gamma}_y \in \mathbb{Z} \otimes \mathbf{v}, \gamma_z \in \mathbb{Z} r \right\}$$

- Let $\varepsilon : \Gamma \to \{\pm 1\}$, group homomorphism.
- $k := 2^{\lfloor \frac{n}{2} \rfloor}$
- $\Sigma_\varepsilon = \Gamma \backslash (N \times \mathbb{C}^k)$ where $\gamma \in \Gamma$ acts on $N \times \mathbb{C}^k$ by

$$\gamma \cdot (x, v) = (\gamma x, \varepsilon(\gamma) v) \, .$$

- $\mathrm{Sect}^\infty(\Sigma_\varepsilon) \cong C_\varepsilon^\infty(\Gamma \backslash N, \mathbb{C}^k)$
- $= \left\{ \, f \in C^\infty \left( N, \mathbb{C}^k \right) \; : \; f(\gamma x) = \varepsilon(\gamma) f(x) \; \forall \gamma \in \Gamma, \; x \in N \, \right\}.$
- All spinor bundles over $\Gamma \backslash N$ are equivalent to one of this form.

- The Dirac operator $D$ acting on $\mathrm{Sect}^{\infty}(\Sigma_\varepsilon) \cong \left\{f \in C^{\infty}\left(N, \mathbb{C}^k\right) : f(\gamma x) = \varepsilon(\gamma) f(x) \forall \gamma \in \Gamma, x \in N\right\}$ is
- $$Df := \sum_{j=1}^{n} \mathrm{cliff}(e_j) \nabla_{e_j} f,$$
- where $\{e_j\}_{j=1}^{n}$ an orthonormal basis of $(\mathfrak{n}, \langle \quad , \quad \rangle)$,
- $\nabla$ is the Levi-Civita connection
- $\mathrm{cliff}(v)f$ is the $\mathbb{R}^n$ Clifford action on $\mathbb{C}^k$

- In great detail: (time permitting)
- Let $\mathbf{f} \in \left\{\mathbf{f} \in C^\infty\left(N, \mathbb{C}^k\right) : \mathbf{f}(\gamma x) = \varepsilon(\gamma)\mathbf{f}(x) \forall \gamma \in \Gamma, x \in N\right\}$
- So $\mathbf{f}(x) = (f_1(x), \ldots, f_k(x))^{\text{transpose}}$ where $f_q \in \{f \in C^\infty(N) : f(\gamma x) = \varepsilon(\gamma) f(x) \forall \gamma \in \Gamma, x \in N\}$
- Then $(D\mathbf{f})(x) = \sum_{j=1}^n \text{cliff}(e_j) \nabla_{e_j} \mathbf{f}(x)$
- $= \sum_{j=1}^n \text{cliff}(e_j) \left(\nabla_{e_j} f_1(x), \ldots, \nabla_{e_j} f_k(x)\right)^{\text{transpose}}$
- $= \sum_{j=1}^n \text{cliff}(e_j) \left(e_j f_1(x), \ldots, e_j f_k(x)\right)^{\text{transpose}}$
- $\text{cliff}(e_j)$ is a $k \times k$ matrix (eg, Pauli matrices if $n = 3$)

- B. Ammann and C. Bär computed the eigenvalues of the Dirac operator $(H_m, \Gamma, \langle \quad , \quad \rangle, \varepsilon)$ on rectangular Heisenberg manifolds
- Rectangular means that, after quotienting out by the center, the quotient torus has an orthogonal lattice basis
- Used Fourier analysis directly

- R. Gornet and K. Richardson have computed the eigenvalues of the Dirac operator for two-step nilmanifolds $(N, \Gamma, \langle\ ,\ \rangle, \varepsilon)$.
- Use Kirillov Theory to decompose the sections into invariant subspaces on which the Dirac operator and its eigenvalues are directly computable.
- That is, the underlying nilpotent Lie group $N$ acts on the right on the nilmanifold, and this action induces an action on the space of $L^2$ sections of the spinor bundle (the *right quasi-regular representation*).
- This $L^2$ space can be decomposed into irreducible $N$-representation spaces; usually these components are infinite-dimensional. These invariant subspaces are also invariant subspaces for the Dirac operator.
- In several cases, the irreducible components can be further decomposed into invariant subspaces of the Dirac operator that are finite dimensional.

- In these cases, the eigenvalues of the Dirac operator can be computed explicitly as the eigenvalues of (finite dim'l) matrices.
- Analogous to Pesce's computation of the eigenvalues of the Laplace operator for two-step nilmanifolds, which extended work of C.S. Gordon and E.N. Wilson on Heisenberg manifolds.
- Requires verifying that Kirillov theory results of L. Richardson, C.C. Moore, and R. Howe extend to the Dirac setting.

- Our results are most explicit for the Heisenberg manifolds, and depend on the spin structure, the lattice, and the metric. There are some examples of nonHeisenberg nilmanifolds, where we are not (yet) able to compute the eigenvalues explicitly. (infinite band matrices)

- Assume we know the eigenvalues $\{\lambda\}$ with multiplicity of a Dirac operator $D : C^\infty(E) \to C^\infty(E)$ acting on sections of a vector bundle $E \to M$.
- We define

$$\eta(s) = \sum_{\lambda \neq 0} \operatorname{sign}(\lambda) |\lambda|^{-s}$$

- This is the *eta function*, holomorphic in $s$ for large Real($s$)
- If $D$ has only nonnegative eigenvalues, this is the zeta function of $D$
- The *eta invariant* is

$$\eta(0),$$

  which means we analytically continue to $s = 0$.
- Formally, this is the number of positive eigenvalues minus the number of negative eigenvalues.

- The eta invariant was introduced and made famous by the Atiyah-Patodi-Singer Index Theorem.
- It is very difficult to calculate the eta invariant for a given operator such as a Dirac operator on a Riemannian manifold.
- Much work has been done to calculate this invariant for space forms, lens spaces and flat tori (see, for example, Donnelly 1978, Botvinnik/Gilkey 1994, Gilkey 1995).
- In 1983, M. Atiyah, H. Donnelly and I. Singer computed the eta invariant of the boundary signature operator of a framed solvmanifold in terms of the signature defect of a manifold whose boundary is that solvmanifold.
- C. Deninger and W. Singhof (1984) computed invariants of modified versions of Dirac operators on Heisenberg manifolds and were able to compute the eta invariants up to local correction terms.

- The first explicit computations of eigenvalues of Dirac operators on homogeneous spaces corresponding to noncompact Lie groups has been done by B. Ammann and C. Bär (1998), where the eigenvalues of the spin$^c$ Dirac operator on certain (rectangular) Heisenberg manifolds were computed explicitly.
- P. Loya, S. Moroianu and J. Park (2008) studied the spectrum of the Dirac operator on a certain three-dimensional circle bundle over a noncompact Riemann surface with cusps, that is, a noncompact manifold that is a cofinite quotient of $PSL(2,\mathbb{R})$. They also study the adiabatic limit of the eta invariant as the fibers are collapsed.

- R. Miatello and R. Podestá (2009) computed the eta invariant on compact flat spin manifolds with cyclic holonomy of odd prime order, which has a similar flavor to our main result
- and S. Goette (2009) has calculated formulas for the eta invariant and equivariant eta invariants on homogeneous spaces of the form $G/H$ with $G$ compact.

# Eta Invariant on 3-dim'l Heisenberg

- Metric Choice (1 continuous parameter) Recall $[X, Y] = Z$

$$\text{ONB} := \{A^{-1/2}X, A^{-1/2}Y, Z\};$$

i.e., $|X| = |Y| = A \in \mathbb{R}_{>0}$

- Lattice Choice (2 continuous and 2 discrete parameters)

$$\Gamma_{m_v;m_w,w_2;r} = \left\{ \begin{pmatrix} 1 & \frac{r(h_x m_v + h_y m_w)}{w_2} & h_z r \\ 0 & 1 & h_y w_2 \\ 0 & 0 & 1 \end{pmatrix} : h_x, h_y, h_z \in \mathbb{Z} \right\}$$

- $r, w_2 \in \mathbb{R}_{>0},\ m_v \in \mathbb{Z}_{>0},\ m_w \in \mathbb{Z}$
- Spin Structure Choice: $\varepsilon : \Gamma \to \{\pm 1\}^3$
  (either 4 or 8 choices, $m_v$ odd $\implies \varepsilon_3 = 1$)

## Theorem

***(Gt-K.Richardson)*** *The eta invariant of the spin Dirac operator on a 3-dimensional Heisenberg manifold with parameters* $A, r, w_2 \in \mathbb{R}_{>0}$, $m_v \in \mathbb{Z}_{>0}$ , $m_w \in \mathbb{Z}$ *with spin structure determined by* $\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3) \in \{\pm 1\}^3$ *satisfies*

$$\eta(0) = \frac{r^2 m_v}{96\pi^2 A^2} - N(\underset{\text{metric}}{A};\ m_v, \overset{\text{lattice}}{m_w}, w_2, r;\ \underset{\text{spin}}{\varepsilon}),$$

*where* $N(A;\ m_v, m_w, w_2, r;\ \varepsilon)$ *is the nonnegative integer given in later slides.*

Notes on proof:

- The Dirac Eigenvalues of the 3-dimensional Heisenberg manifold are symmetric about $-1/4A$ .

## Theorem

*Let $\sigma(D) - \overline{\lambda}$ be symmetric about 0 in $\mathbb{R}$. Then the eta invariant satisfies*

$$\begin{aligned}\eta(0) &= -2^{1-n}\pi^{-(n+1)/2}\left(\sum_{k=0}^{n-1}\frac{(-1)^k}{k+1}\overline{\lambda}^{k+1}a_{\frac{n-1}{2},k}\right)\\ &\quad +\operatorname{sgn}\left(\overline{\lambda}\right)2\#\left(\sigma(D)\cap I_{\overline{\lambda}}\right)+\operatorname{sgn}\left(\overline{\lambda}\right)\#\left(\sigma(D)\cap\left\{0,\overline{\lambda}\right\}\right),\end{aligned}$$

*where $I_{\overline{\lambda}}$ is the open interval between 0 and $\overline{\lambda}$, and where implicitly the last two terms include multiplicities.*

## Corollary

*For $n = 3$ , recall that every three-manifold is spin. Let $\sigma(D) - \overline{\lambda}$ be symmetric about 0 in $\mathbb{R}$. Letting $F^W$ be the twisting curvature, then*

$$\begin{aligned}\eta(0) &= -\frac{\widehat{n}\overline{\lambda}^3}{6\pi^2}\mathrm{vol}(M) + \frac{\overline{\lambda}}{4\pi^2}\left(\frac{\widehat{n}}{12}\int_M \mathrm{Scal} + \int_M Tr\left(F^W\right)\right) \\ &\quad + \mathrm{sgn}\left(\overline{\lambda}\right)\left(2\#\left(\sigma(D)\cap\left(0,\overline{\lambda}\right)\right) + \#\left(\sigma(D)\cap\left\{0,\overline{\lambda}\right\}\right)\right).\end{aligned}$$

$\square$

# RUTH, WHY R U AVOIDING SPECIFYING N?

## Theorem

*Recall that for 3-dim'l Heisenberg manifolds:*

$$\eta(0) = \frac{r^2 m_v}{96\pi^2 A^2} - N(\underset{\text{metric}}{A}; m_v; \overset{\text{lattice}}{m_w, w_2; r}; \underset{\text{spin}}{\varepsilon}),$$

$$
\begin{aligned}
N(\cdot) &= 2\#\left(\sigma(D)\cap\left(\overline{\lambda},0\right)\right)+\#\left(\sigma(D)\cap\left\{0,\overline{\lambda}\right\}\right)\\
&= 2\#\left\{\begin{array}{c}(\alpha_1,\alpha_2):\frac{\alpha_1 r m_v}{w_2}\in\mathbb{Z}+\frac{1-\varepsilon_1}{4},\\ \alpha_2 w_2+\frac{r m_w}{w_2}\alpha_1\in\mathbb{Z}+\frac{1-\varepsilon_2}{4},\ 0<\|\alpha\|<\frac{1}{8\pi\sqrt{A}}\end{array}\right\}\\
&\quad +2m_v\sum_{\substack{\mu\in\mathbb{Z}+\frac{1-\varepsilon_3}{4}\\ -\frac{r}{8\pi A}<\mu<0}}|\mu|+2m_v\sum_{p\in\mathbb{Z}_{>0}}\sum_{\substack{\mu\in\mathbb{Z}+\frac{1-\varepsilon_3}{4}\\ 0<|\mu|<\frac{r}{8\pi A\left(\sqrt{1+16p^2}+4p\right)}}}|\mu|\\
&\quad +\#\left\{\begin{array}{c}(\alpha_1,\alpha_2):\frac{\alpha_1 r m_v}{w_2}\in\mathbb{Z}+\frac{1-\varepsilon_1}{4},\\ \alpha_2 w_2+\frac{r m_w}{w_2}\alpha_1\in\mathbb{Z}+\frac{1-\varepsilon_2}{4},\ \|\alpha\|=\frac{1}{8\pi\sqrt{A}}\end{array}\right\}
\end{aligned}
$$

$$+\begin{cases} \frac{m_v r}{\pi A}\left(\sqrt{\frac{1}{16}+p^2}-p\right) & \text{if} \exists p \in \mathbb{Z}_+, \frac{r(\sqrt{\frac{1}{16}+p^2}-p)}{2\pi A} \in \mathbb{Z}+\frac{1-\varepsilon_3}{4} \\ \frac{m_v r}{8\pi A} & \text{if } \frac{r}{8\pi A} \in \mathbb{Z}+\frac{1-\varepsilon_3}{4} \\ 0 & \text{otherwise} \end{cases}$$

$$+\begin{cases} 2 & \text{if } \varepsilon_1=\varepsilon_2=\varepsilon_3=1 \\ 0 & \text{otherwise} \end{cases}$$

**Theorem**

***(Gt-K.Richardson)*** *If the lattice is rectangular* ($\iff m_w = 0$) *and* $A > r/4\pi$ *and* $rm_v/4\pi\sqrt{A} < w_2 < 4\pi\sqrt{A}$, *then if* $\varepsilon \neq$ *identity,*

$$\eta(0) = \frac{r^2 m_v}{96\pi^2 A^2}.$$

*Subtract* 2 *if* $\varepsilon =$ *identity.*

- Observe: Can vary $w_2$ continuously and $\eta(0)$ remains unchanged.

## Lemma

*(Gornet-Richardson) For any positive integer m and any spin structure on any* $4m+1$*-dimensional Heisenberg manifold, the eta invariant of the corresponding Dirac operator is zero.*

## Recent Application

- In a recent paper, Bär and A. Strohmeier derived a formula for the gravitational chiral charge creation, given by the index of the Lorentzian Dirac operator with certain APS-type boundary conditions.
- The most important contributions for Bianchi Type-II space-times come from the eta invariant of a three-dimensional Heisenberg manifold.
- The authors used our results to compute the contribution of the eta invariant, and this contribution can be shown to be arbitrarily large (positive or negative).
- This set of examples is important, because previous nonrigorous Wick rotation methods used by physicists do not have this eta invariant correction term, and because of that they incorrectly calculated the amount of quantum charge current created by the gravitational field.

## Unanswered Questions

- In general, how does each irreducible subspace of spinors for the nilpotent Lie group decompose into finite-dimensional eigenspaces for the Dirac operator?
- What manifolds have a point of spectral symmetry for the Dirac operator?

Thank You!