Smooth Lie supergroups

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Articles of faith

- Everything happens over $\mathbb{R}$.
- Everything is finite dimensional.
- Everything is smooth.

1. Supermanifolds

Definition

A smooth supermanifold of dimension $(m|n)$ is an object $(M|\mathcal{R}M)$ such that

- M is an n-dimensional manifold.
- $\mathcal{R}M$ is a unital superalgebra bundle over M, i.e. for every point p the fibre $\mathcal{R}_p M \cong \Lambda S_p^*$ for some vector space S of dimension n.

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Supermanifolds as algebra bundles

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Remarks

- The bundle $\mathcal{R}M$ is NOT necessarily an exterior algebra bundle.
- Nonetheless it is filtered: $\mathcal{R}_p^{\geq 1} M$ is the nilpotent ideal at p and $\mathcal{R}_p^{\geq k} M := \left(\mathcal{R}_p^{\geq 1} M\right)^k$.

Definition

$(\phi|\Phi) : (M|\mathcal{R}M) \to (N|\mathcal{R}N)$ such that

- $\phi \colon M \to N$ is smooth, and
- $\Phi \colon \Gamma(\mathcal{R}N) \to \Gamma(\mathcal{R}M)$ is a unital ($\therefore$ even) homorphism of superalgebras.

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For $f \in \mathcal{C}^\infty(N)$ and $r \in \Gamma(\mathcal{R}N)$ define

$$[\Phi|\phi|f](r) = \Phi(fr) - (f \circ \phi)\Phi(r)$$

Proposition

If $\dim(N|\mathcal{R}N) = (p|q)$ *and* $k \geq \lfloor \frac{q}{2} \rfloor$ *then*

$$[\Phi|\phi|f_0, \ldots, f_k] \equiv 0$$

and therefore Φ *is a differential operator along* ϕ*.*

A (super)rose by any other name...

Theorem

- *Let $(M, \mathcal{O})$ be a KL-supermanifold of dimension (m, n) and let S be a vector space of dimension n. There exists a bundle of superalgebras $\mathcal{R}M$ such that every fibre $\mathcal{R}_p M$ is isomorphic to ΛS^* and with the further property that $\Gamma(\mathcal{R}M) \cong \mathcal{O}$ as sheaves.*
- *Let $(M|\mathcal{R}M)$ and $(N|\mathcal{R}N)$ be supermanifolds; denote by $\mathcal{O}_M$ the sheaf of sections of $\mathcal{R}M$ and let $\Phi : \mathcal{O}_N \to \mathcal{O}_M$ be a unital superalgebra sheaf morphism along the continuous map $\phi : M \to N$. Then ϕ is smooth and Φ is a differential operator along ϕ.*

2 Tangential objects

Odd directions and codirections

Let $(M|\mathcal{R}M)$ be a supermanifold.

Definition

The bundles $\mathbf{S}^*M$ and $\mathbf{S}M$ with fibres defined respectively by

$$\mathbf{S}^*_pM := \mathcal{R}^{\geq 1}_pM/\mathcal{R}^{\geq 2}_pM$$

and

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Proposition

Let $(\phi|\Phi): (M|\mathcal{R}M) \to (N|\mathcal{R}N)$. Then for every $k \geq 1$ there exist bundle maps

$$\Phi^{(k)}: \Lambda^k\mathbf{S}^*N \to \Lambda^k\mathbf{S}^*M$$

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Proof: Φ preserves the filtration. □

Tangential maps

- The differential of $(\phi|\Phi)$ is

$$\phi_* \oplus (\Phi^{(1)})^* \colon TM \oplus \mathbf{S}M \to \phi^* (TN \oplus \mathbf{S}N)$$

- The codifferential of $(\phi|\Phi)$ is

$$\phi^* \oplus \Phi^{(1)} \colon \phi^* (T^*N \oplus \mathbf{S}^*M) \to T^*M \oplus \mathbf{S}^*M$$

- The auxiliary differential $\Phi^{(\mathrm{aux})} \colon \phi^*(T^*N) \to \Lambda^2\mathbf{S}^*M$ is given punctually by

$$df_p \mapsto \Phi(p) - f \circ \phi + \mathcal{R}_p^{\geq 3}M$$

We'll denote the dual map by $\Phi_{\mathrm{aux}} \colon \Lambda^2\mathbf{S}M \to \phi^*TN$.

The bundle of pointwise derivations has as fibres $\operatorname{der}_p \mathcal{R}M = \operatorname{der}(\mathcal{R}_p M)$.

Theorem

The sheaf of superderivations of $\Gamma(\mathcal{R}M)$ is the sheaf of sections of a vector bundle over M given by the exact sequence

$$0 \longrightarrow \operatorname{der} \mathcal{R}M \hookrightarrow \operatorname{Der} \mathcal{R}M \xrightarrow{\sigma} \mathcal{R}M \otimes TM \longrightarrow 0$$

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All of the above (and a little more) is to appear in Colloquim Mathematicum (accepted). Also: arXiv:1410:7857.

3 Lie supegroups

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Split Lie supergroups

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Theorem

A split supermanifold is a Lie supergroup if and only $M = G$ is a Lie group and the bundle $\mathbf{S}G$ is G-bihomogeneous, that is: there exists a smooth map $\beta \colon G \times \mathbf{S}G \times G \to \mathbf{S}G$ such that, if $\mathbf{b}(x, g, y) := xgy$ is the natural biaction of G on itself then the diagram

$$\begin{array}{ccc} G \times \mathbf{S}G \times G & \xrightarrow{\beta} & \mathbf{S}G \\ {\scriptstyle \mathrm{id} \times \pi \times \mathrm{id}} \downarrow & & \downarrow {\scriptstyle \pi} \\ G \times G \times G & \xrightarrow{\mathbf{b}} & G \end{array}$$

commutes.

Definition

A Lie supergroup is a group object in the category of supermanifolds.

I.e. there exist

- $(m|\mathcal{M}) : (G \times G|\mathcal{R}G\hat{\boxtimes}\mathcal{R}G) \to (G|\mathcal{R}G)$,
- $(\iota|\mathrm{I}) : (G|\mathcal{R}G) \to (G|\mathcal{R}G)$ and
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- G is a Lie group.
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You can't always get what you want...

The bundle $\mathcal{R}G$ need not be bihomogeneous; there is no naturally defined biaction.

The map $\mathcal{M}_{\rm aux} \colon \Lambda^2 m^*(\mathbf{S}G \boxplus \mathbf{S}G) \to TG$ is well-behaved: because of

Lemma

The auxiliary differential of $(\Phi \circ \Psi)$ (whenever the composition makes sense) equals

$$(\Phi \circ \Psi)^{\rm aux} = \Lambda^2 \Phi^* \circ \Psi^{\rm aux} + \psi^* \circ \Phi^{\rm aux}$$

and (co)associativity of $\mathcal{M}$ we get

$\mathcal{M}^3_{\rm aux} = \big((\mathcal{M} \otimes {\rm id}) \circ \mathcal{M}\big)_{\rm aux} = \big(({\rm id} \otimes \mathcal{M}) \circ \mathcal{M}\big)_{\rm aux}$ equals

$$\mathcal{M}_{\rm aux} \circ (\Lambda^2 \mathcal{M}_* \oplus {\rm id}) + m_* \circ (\mathcal{M}_{\rm aux} \oplus 0)$$

and

$$\mathcal{M}_{\rm aux} \circ ({\rm id} \oplus \Lambda^2 \mathcal{M}_*) + m_* \circ (0 \oplus \mathcal{M}_{\rm aux})$$

Lemma

Let $g = xy$ *then* $B_g(s,t) := \mathcal{M}_{\mathrm{aux},(x,y)}(\beta(e,s,y^{-1}),\beta(x^{-1},t,e))$ *is well-defined and bilinear; furthermore* $B^{\pm}(s,t) = \frac{1}{2}(B(s,t) \pm B(t,s))$ *are respectively symmetric* ($+$) *and skew-symmetric* ($-$).

Furthermore, since $\mathcal{M}_{\mathrm{aux}}$ is skew-symmetric we get

Theorem

$B_e^+ \colon \operatorname{Sym}^2 \mathbf{S}_e G \to T_e G$ *is biequivariant.*

Proposition

A Lie superalgebra $(\mathfrak{g}|\mathbf{S})$ *is completely determined by the following data:*

- *A Lie algebra* $\mathfrak{g}$*;*
- *a representation* $\rho\colon \mathfrak{g} \to \operatorname{End}\mathbf{S}$, *and*
- *a symmetric bilinear map* $B\colon \operatorname{Sym}^2 \mathbf{S} \to \mathfrak{g}$ *which is* ρ*-equivariant and belonging to the kernel of the composition*

$$\left(\operatorname{Sym}^2 \mathbf{S}^* \otimes \mathfrak{g}\right)^{\mathfrak{g}} \xrightarrow{\mathrm{id}\otimes\rho} \left(\operatorname{Sym}^2 \mathbf{S} \otimes \mathbf{S}^* \otimes \mathbf{S}\right)^{\mathfrak{g}} \xrightarrow{C\otimes\mathrm{id}} \left(\operatorname{Sym}^3 \mathbf{S}^* \otimes \mathbf{S}\right)^{\mathfrak{g}}$$

where $C\colon \operatorname{Sym}^2 \mathbf{S}^* \otimes \mathbf{S}^* \to \operatorname{Sym}^3 \mathbf{S}^*$ *is the tensor contraction.*

Theorem

There exist a unique unital superalgebra isomorphism $E\colon \Gamma\left(\Lambda\mathbf{S}^{*}G\right)\to\Gamma\left(\mathcal{R}G\right)$ *such that*

- E *is a differential operator (along* id*);*
- $E_{*}=\mathrm{id}$, *and*
- E *intertwines* $\mathcal{M}$ *with* M_{β} *(the multiplication on the split supergroup given by the biaction).*

(Angelical chorus)
Thank you very much for your attention!!