Submanifolds of Einstein solvmanifolds

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VI Workshop on Differential Geometry
EGEO 2016
La Falda, Córdoba

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The Symmetric setting

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Noncompact Homogeneous Einstein Manifolds: Big Questions

- Question: Which manifolds admit a metric of constant negative Ricci curvature?
- Easier question: Which manifolds admit a homogeneous metric of constant negative Ricci curvature?

Alekseevskii conjecture [1975]: Every noncompact homogeneous Einstein manifold is a *solvmanifold*, a Riemannian manifold with a solvable group of isometries.

Solvmanifolds

All our homogeneous examples will be solvable Lie groups with a left invariant metric

Definition

A simply connected solvmanifold S with a left invariant metric g is completely determined by its *metric Lie algebra* $(\mathfrak{s}, \langle\ ,\ \rangle)$.

Definition

Write a metric solvable algebra $(\mathfrak{s}, \langle\ ,\ \rangle)$ as $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n}$ where $\mathfrak{n}$ is the nilradical of $\mathfrak{s}$. We say $\mathfrak{s}$ is of *Iwasawa type* if

- $\mathfrak{a}$ is abelian
- ad_A is symmetric relative to $\langle\ ,\ \rangle$, for all A in $\mathfrak{a}$
- for some A in $\mathfrak{a}$, $\mathrm{ad}_A|_{\mathfrak{n}}$ is positive definite.

Rank and Nilpotency

- For a solvmanifold $\mathfrak{s}$, the algebraic rank of $\mathfrak{s}$ is the dimension of $\mathfrak{a}$
- The nilpotency of the nilradical $\mathfrak{n}$ of $\mathfrak{s}$ is the step-size:
- $\mathfrak{n}$ is k-step if the $(k-1)$ derived algebra is non-zero but the k^{th} derived algebra vanishes $([\mathfrak{n},[\mathfrak{n},[\ldots[\mathfrak{n},\mathfrak{n}]..]]]=0)$.

In general, it seems difficult to find explicit examples of Einstein solvmanifolds with higher nilpotency.

Rank One Reduction

- Given any Einstein solvmanifold $(\mathfrak{s}, \langle\ ,\ \rangle)$, we can find a *rank one sub-solvmanifold*,

$$\mathfrak{s}' = \langle A_0 \rangle + \mathfrak{n}.$$

- Endowed with the induced metric, the sub-solvmanifold $\mathfrak{s}'$ is not only also Einstein, it inherits its Einstein constant from $\mathfrak{s}$.
- This suggests the structure of $\mathfrak{n}$ determines the Ricci geometry of $\mathfrak{s}$.

We will see that in some cases, a comparable reduction can be done on $\mathfrak{n}$ (and $\mathfrak{a}$), so that again, the constant Ricci curvature is inherited.

Solvmanifolds from symmetric spaces (H. Tamaru)

- Let $\mathfrak{g}$ be a semisimple Lie algebra.
- Use a Cartan involution σ to decompose $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ ($\sigma|_{\mathfrak{k}} = \mathrm{Id}$, $\sigma|_{\mathfrak{p}} = -\mathrm{Id}$).
- Define $B_\sigma(X, Y) := -B(X, \sigma(Y))$, an $\mathrm{ad}_{\mathfrak{k}}$-invariant inner product on $\mathfrak{g}$.
- Let $\mathfrak{a}$ be a maximal torus in $\mathfrak{p}$.
- $\mathfrak{g} = \mathfrak{g}_0 + \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha$ is root space decomposition, $\mathfrak{g}_0$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{g}$,
- $\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid (\mathrm{ad}\, A)X = \alpha(A)X \text{ for all } A \in \mathfrak{a}\}$.
- In Δ^+, take simple roots $\Lambda \subset \Delta^+$.

Tamaru's construction

- Define $\mathfrak{n} := \sum_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$, the nilradical of $\mathfrak{g}$.
- Define $\mathfrak{s} := \mathfrak{a} + \mathfrak{n}$, with inner product

$$\langle \, , \, \rangle = 2B_\sigma|_{\mathfrak{a}\times\mathfrak{a}} + B_\sigma|_{\mathfrak{n}\times\mathfrak{n}}.$$

- The corresponding simply connected solvmanifold (S, g) is a symmetric space.

Tamaru's construction

- Let H^i in $\mathfrak{a}$ be the dual to α_i in $\mathfrak{a}^*$, so that $\alpha_i(H^j) = \delta_{ij}$.
- Choose a subset of fundamental roots

$$\Gamma' = \{\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_k}\} \subset \Lambda.$$

- Let $Z = H^{i_1} + H^{i_2} + \cdots + H^{i_k}$, the *characteristic element*.

Definition

- Let $\mathfrak{a}' = \mathrm{span}\{H^{i_1}, H^{i_2}, \ldots, H^{i_k}\} \subset \mathfrak{a}$
- Let $\mathfrak{n}' = \sum_{\alpha(Z)>0} \mathfrak{n}_\alpha$
- Take $\mathfrak{s}' = \mathfrak{a}' + \mathfrak{n}'$, with $\langle\, , \,\rangle' = \langle\, , \,\rangle$, restricted.

Ricci curvature

Tamaru's original solvmanifold $(\mathfrak{s}, \langle\ ,\ \rangle)$ corresponds to a noncompact symmetric space with the symmetric metric: Einstein.

The subalgebra is constructed so that the constant Ricci curvature is unchanged:

- For any $A, A' \in \mathfrak{a}'$ and any $X, Y \in \mathfrak{n}'$,
- $\mathrm{Ric}^{\mathfrak{s}'}(A, A') = \mathrm{Ric}^{\mathfrak{s}}(A, A')$
- $\mathrm{Ric}^{\mathfrak{s}'}(A, X) = \mathrm{Ric}^{\mathfrak{s}}(A, X) = 0$
- $\mathrm{Ric}^{\mathfrak{s}'}(X, Y) = \mathrm{Ric}^{\mathfrak{s}}(X, Y)$.

Outline of Tamaru's Proof:

Theorem (Wolter)

Let $(\mathfrak{s} = \mathfrak{a} + \mathfrak{n}, \langle\, , \rangle)$ *be a solvable metric Lie algebra of Iwasawa type. Then the Ricci curvature satisfies*

(1) $\mathrm{Ric}^{\mathfrak{s}}(A, A') = \mathrm{tr}(\mathrm{ad}_A) \circ (\mathrm{ad}_{A'})$ *for all* $A, A' \in \mathfrak{a}$*,*

(2) $\mathrm{Ric}^{\mathfrak{s}}(X, A) = 0$ *for all* $A \in \mathfrak{a}$ *and* $X \in \mathfrak{n}$*,*

(3) $\mathrm{Ric}^{\mathfrak{s}}(X, Y) = \mathrm{Ric}^{\mathfrak{n}}(X, Y) - \langle \mathrm{ad}_{H_0} X, Y \rangle$ *for all* $X, Y \in \mathfrak{n}$*.*

Here H_0 is the mean curvature vector for $\mathfrak{s}$. Thanks to Wolter,

Corollary

$\mathrm{Ric}^{\mathfrak{s}'}(A_1, A_2) = \mathrm{Ric}^{\mathfrak{s}}(A_1, A_2)$ *for every* $A_1, A_2 \in \mathfrak{a}'$*.*

Outline of Tamaru's Proof, cont.

Theorem (Alekseevskii)

Let $\{E_i\}$ be an orthonormal basis for the nilpotent metric Lie algebra $(\mathfrak{n}, \langle\,,\,\rangle)$. The Ricci endomorphism $\mathrm{Ric}^{\mathfrak{n}}$ is given by

$$\mathrm{Ric}^{\mathfrak{n}} = \frac{1}{4}\sum(\mathrm{ad}_{E_i}) \circ (\mathrm{ad}_{E_i})^* - \frac{1}{2}\sum(\mathrm{ad}_{E_i})^* \circ (\mathrm{ad}_{E_i}).$$

Lemma

Let $X \in \mathfrak{n}'$ and let $H_0^{\perp} = H_0 - H_0'$. Then

$$\mathrm{Ric}^{\mathfrak{n}}(X) - \mathrm{Ric}^{\mathfrak{n}'}(X) = [H_0^{\perp}, X].$$

As above, H_0 is the MCV for $\mathfrak{s}$, and H_0' denotes the MCV for $\mathfrak{s}'$.

Outline of Tamaru's Proof, cont.

Using the Lemma, $\mathrm{Ric}^{\mathfrak{n}}(X) - \mathrm{Ric}^{\mathfrak{n}'}(X) = [H_0^{\perp}, X]$.
Combining this with Wolter's result,

$$\begin{aligned}
\mathrm{Ric}^{\mathfrak{s}}(X, Y) - \mathrm{Ric}^{\mathfrak{s}'}(X, Y) &= \mathrm{Ric}^{\mathfrak{n}}(X, Y) - \mathrm{Ric}^{\mathfrak{n}'}(X, Y) \\
&\quad - (\langle [H_0, X], Y\rangle - \langle [H_0', X], Y\rangle) \\
&= \langle [H_0^{\perp}, X], Y\rangle - \langle [H_0 - H_0', X], Y\rangle \\
&= \langle [H_0^{\perp}, X], Y\rangle - \langle [H_0^{\perp}, X], Y\rangle \\
&= 0.
\end{aligned}$$

Other consequences

- S' is a minimal submanifold of S.
- The Einstein condition is not needed here. The trace of the second fundamental form vanishes.
- S' in S is *not* totally geodesic, as long as Γ' and $\Lambda \setminus \Gamma'$ are not orthogonal.

Extending Tamaru's construction

Let $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n}$ be an Einstein solvable Lie algebra.

- $$\mathfrak{s} = \mathfrak{a} + \sum_{\alpha \in \Delta} \mathfrak{n}_\alpha$$

  where $\mathfrak{n}_\alpha = \{X \in \mathfrak{n} \mid [A, X] = \alpha(A)X \text{ for all } A \in \mathfrak{a}\}$.
- $$\Delta = \{\alpha \in \mathfrak{a}^* \mid \mathfrak{n}_\alpha \neq 0\}.$$
- Let $\Lambda = \{\alpha_1, \alpha_2, \ldots, \alpha_r\}$ be a set of fundamental roots.
- Let H^i in $\mathfrak{a}$ be the dual to α_i in $\mathfrak{a}^*$.
- Choose a subset of fundamental roots

  $$\Gamma' = \{\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_k}\} \subset \Lambda.$$

Sub-solvmanifold:

Let $Z = H^{i_1} + H^{i_2} + \cdots + H^{i_k} \in \mathfrak{a}$.

Definition

The subset Γ' defines the following subalgebras of $\mathfrak{s}$:

- $\mathfrak{a}' = \mathrm{span}\{H^{i_1}, H^{i_2}, \ldots, H^{i_k}\} \subset \mathfrak{a}$
- $\mathfrak{n}' = \sum_{\alpha(Z)>0} \mathfrak{n}_\alpha$
- $\mathfrak{s}' = \mathfrak{a}' + \mathfrak{n}'$

Restrict the inner product $\langle\,,\,\rangle$ on $\mathfrak{s}$ to $\mathfrak{s}'$. We say that $(\mathfrak{s}', \langle\,,\,\rangle)$ is an *attached* metric Lie subalgebra to $(\mathfrak{s}, \langle\,,\,\rangle)$.

Ricci Curvatures, outline of proof

Lemma

Let $X \in \mathfrak{n}'$ *and let* $H_0^\perp = H_0 - H_0'$. *Then*

$$\mathrm{Ric}^{\mathfrak{n}}(X) - \mathrm{Ric}^{\mathfrak{n}'}(X) = [H_0^\perp, X]$$

if and only if
(i) $\sum_i (\mathrm{ad}_{E_i'})^* E_i'$ *is in* $\mathfrak{a}'$ *(the mean curvature vector for* $\mathfrak{s}'$*), and*
(ii)

$$\sum_j (\mathrm{ad}_{E_j^\perp})^* (\mathrm{ad}_{E_j^\perp})(X) - \sum_j (\mathrm{ad}_{E_j^\perp})(\mathrm{ad}_{E_j^\perp})^*(X) = \sum_j \mathrm{ad}_{(\mathrm{ad}_{E_j^\perp})^* E_j^\perp}(X).$$

(i) Mean curvature vector

The MCV for $\mathfrak{s}$ is $H_0 = -\sum_k (\mathrm{ad}_{E_k})^* E_k$, where the sum is over an orthonormal basis $\{E_k\}$ for all of $\mathfrak{n}$.
We choose $\{E_k\} = \{E'_i\} \cup \{E_j^\perp\}$ where each basis vector is in a root space and $\{E'_i\}$ is a basis for $\mathfrak{n}'$, while $\{E_j^\perp\}$ is a basis for $\mathfrak{n}^0$.

$$H_0 = -\sum_k (\mathrm{ad}_{E_k})^* E_k = -\sum_i (\mathrm{ad}_{E'_i})^* E'_i + -\sum_j (\mathrm{ad}_{E_j^\perp})^* E_j^\perp.$$

To preserve constant Ricci curvature, we need $-\sum_i (\mathrm{ad}_{E'_i})^* E'_i \in \mathfrak{a}'$.
This is not always the case.

(ii) When is the "ad-star" condition true?

If $(\mathrm{ad}_{E_j^\perp})^* = \mathrm{ad}_X$ for some X (in some larger algebra), then write $\mathrm{ad}_{E_j^\perp} = \mathrm{ad}_Y$. Our condition (before summing)

$$(\mathrm{ad}_{E_j^\perp})^*(\mathrm{ad}_{E_j^\perp}) - (\mathrm{ad}_{E_j^\perp})(\mathrm{ad}_{E_j^\perp})^* = \mathrm{ad}_{(\mathrm{ad}_{E_j^\perp})^* E_j^\perp}$$

is exactly the Jacobi Identity: $\mathrm{ad}_X \, \mathrm{ad}_Y - \mathrm{ad}_Y \, \mathrm{ad}_X = \mathrm{ad}_{[X,Y]}$.

$$\sum_j (\mathrm{ad}_{E_j^\perp})^*(\mathrm{ad}_{E_j^\perp})(\cdot) - \sum_j (\mathrm{ad}_{E_j^\perp})(\mathrm{ad}_{E_j^\perp})^*(\cdot) = \sum_j \mathrm{ad}_{(\mathrm{ad}_{E_j^\perp})^* E_j^\perp}(\cdot).$$

For our condition, we need only sum over $\mathfrak{n}^0$ and apply to $\mathfrak{n}'$. Although this is weaker, it does not hold in all cases.