Metric 2-step Nilpotent Lie Algebras associated with Graphs

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VI Workshop on Differential Geometry

Joint work with Rachelle DeCoste and Lisa DeMeyer

Outline

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1. Graphs and metric 2-step nilpotent Lie algebras
2. $j(Z)$ maps
3. Singularity Properties
4. Heisenberg-Like Lie Algebras
5. Density of Closed Geodesics

Construction

Let $G = (S, E)$ be a simple directed graph. Let $S = \{X_1, \ldots, X_m\}$ and $E = \{Z_1, \ldots, Z_q\}$. Define $\mathfrak{n}_G = \mathfrak{v} \oplus \mathfrak{z}$ where S is a basis for $\mathfrak{v}$ and E is a basis for $\mathfrak{z}$ over $\mathbb{R}$.

The Lie bracket on $\mathfrak{n}_G$ is defined by extending linearly the following relations.

$$[X_i, X_j] = Z_k \ \text{ if } Z_k \text{ is a directed edge from vertex } X_i \text{ to vertex } X_j.$$

All other Lie brackets are defined to be zero.

Then $\mathfrak{n}_G$ s a 2-step nilpotent Lie algebra associated with graph G. This construction was introduced by Dani and M. (2005) to study Anosov automorphisms on corresponding nilmanifolds.

Define the inner product $<,>$ on $\mathfrak{n}_G$ such that $S \cup E$ is an orthonormal basis for $\mathfrak{n}_G$.

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Example

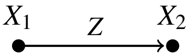

Figure: Graph K_2

The Lie algebra $\mathfrak{n}_{k_2}$ is the 3-dimensional Heisenberg Lie algebra.

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$j(Z)$ maps

Let $\mathfrak{n}$ be a 2-step nilpotent Lie algebra with inner product $\langle\ ,\ \rangle$. Let $\mathfrak{z}$ denote the center of $\mathfrak{n}$ and $\mathfrak{v} = \mathfrak{z}^{\perp}$. For each $Z \in \mathfrak{z}$, the skew-symmetric linear transformation $j(Z) : \mathfrak{v} \to \mathfrak{v}$ is defined by

$$\langle j(Z)X, Y\rangle = \langle [X, Y], Z\rangle \text{ for all } X, Y \in \mathfrak{v} \text{ and } Z \in \mathfrak{z}$$

These $j(Z)$ maps (introduced by Kaplan) capture both the bracket and metric structure on $\mathfrak{n}$ and hence they are very useful to describe the geometry of the associated simply connected Lie group N with the left invariant metric.

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Singular, Nonsingular and Almost Nonsingular

(Lee-Park) Every 2-step nilpotent Lie algebra is exactly one of the following types.

- nonsingular : $j(Z)$ is nonsingular for every nonzero $Z \in \mathfrak{z}$
- singular: $j(Z)$ is singular for all $Z \in \mathfrak{z}$
- almost nonsingular : $j(Z)$ is nonsingular for all Z in some open dense subset of $\mathfrak{z}$.

Lemma

If G is a graph with at least one edge, then the Lie algebra $\mathfrak{n}_G$ is nonsingular if and only if G is the complete graph on two vertices.

Proof.

If G has an edge Z and a vertex X such that Z is not incident to X, then $j(Z)(X) = 0 \implies j(Z)$ is singular $\implies \mathfrak{n}_G$ is not nonsingular. □

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Example: Singular

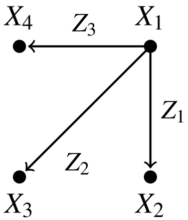

Figure: Star Graph $K_{1,3}$

- $\mathfrak{n}_{K_{1,3}}$ is singular because the matrix of $j(a_1Z_1 + a_2Z_2 + a_3Z_3)$ is

$$\begin{pmatrix} 0 & -a_1 & -a_2 & -a_3 \\ a_1 & 0 & 0 & 0 \\ a_2 & 0 & 0 & 0 \\ a_4 & 0 & 0 & 0 \end{pmatrix}.$$

Example: Almost Nonsingular

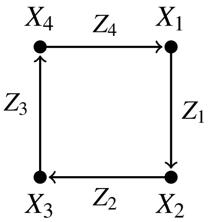

Figure: C_4

- $\mathfrak{n}_{C_4}$ is almost nonsingular. The matrix of $j(a_1Z_1 + a_3Z_3)$ is

$$\begin{pmatrix} 0 & a_1 & 0 & 0 \\ -a_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_3 \\ 0 & 0 & -a_3 & 0 \end{pmatrix}$$

and hence it is nonsingular if a_1 and a_3 are both non-zero.

Classification: Singular and Almost Nonsingular

Definition

Let $G = (S, E)$ be a graph with $|S| = 2n$, $n > 1$ and the set of vertices $S = \{X_1, \ldots, X_{2n}\}$. We say that G has a *vertex covering by n disjoint copies of* K_2 if there exists a permutation $\sigma \in S_{2n}$ such that $X_{\sigma(2i-1)}X_{\sigma(2i)} \in E$ for all $1 \leq i \leq n$, where S_{2n} is the symmetric group.

Example

- The cycle C_{2n} has a vertiex covering by n disjoint copies of K_2.
- The star graph $K_{1,n}$ does not admit a vertex covering by disjoint copies of K_2.

Proposition

Let $G = (S, E)$ be a graph with $|S| = 2n$, $n > 1$. Then $\mathfrak{n}_G$ is almost nonsingular if and only if G has a vertex covering by n disjoint copies of K_2.

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Heisenberg-Like Lie Algebras

Gornet-Mast (2009) introduced Heisenberg-like Lie algebras as a generalization of well-studied class of 2-step nilpotent Lie algebras, Heisenberg-type Lie algebras using totally geodesic condition. They also gave a characterization using eigenvalues of $j(Z)$.

Let $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ be a 2-step nilpotent Lie algebra. For $z \in \mathfrak{z}$ denote the distinct eigenvalues of $j(Z)$ by $\pm i\theta_1(Z), \ldots, \pm i\theta_m(Z)$.

Theorem (Blanchard, Gornet-Mast)

$\mathfrak{n}$ is Heisenberg-like if and only if m is a fixed integer for all nonzero $Z \in \mathfrak{z}$ and for every $i = 1, \ldots m$, there exists a constant $c_i \geq 0$ such that for every $Z \in \mathfrak{z}$, $\theta_i(Z) = c_i||Z||$.

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Heisenberg-Like and Graphs

Theorem

Let G be a connected graph. The associated metric 2-step nilpotent Lie algebra is Heisenberg-like if and only if one of the following holds:
(i) G is the star graph $K_{1,n}$ *for* $n > 1$.
(ii) G is the complete graph on 3 vertices.

Outline

Density of Closed Geodesics in Nilmanifolds

Eberlein (1994), Mast (1994), Lee-Park (1996), DeMeyer (2001), DeCoste (2008)

Definition

Let N be a simply connected nilpotent Lie group and Γ be a lattice in N. The nilmanifold $\Gamma\backslash N$ has the *density of closed geodesics property* if the vectors tangent to closed, unit speed geodesics are dense in the unit tangent bundle of $\Gamma\backslash N$.

- (Eberlein) Heisenberg type nilmanifolds have the density of closed geodesics property.
- (DeMeyer) Construction of Heisenberg-like nilmanifolds using irreducible representations of $\mathfrak{su}(2)$ having density of closed geodesics property.

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Resonance

Definition

Let $\mathfrak{n}$ be a metric 2-step nilpotent Lie algebra and let $\mathfrak{z}$ denote the center of $\mathfrak{n}$. For a nonzero $Z \in \mathfrak{z}$, we say that the map $j(Z)$ is *in resonance* if the ratio of any pair of nonzero eigenvalues of $j(Z)$ is rational.

- (Mast) Nonsingular 2-step nilmanifolds satisfying the resonance condition have the density property. Moreover, if a 2-step nilmanifold has the density of closed geodesics property, then $j(Z)$ is in resonance for a dense subset of $Z \in \mathfrak{z}$.
- (Lee-Park) In almost nonsingular case, the condition that $j(Z)$ is in resonance for Z in a dense subset of the center is a necessary and sufficient condition for the density of closed geodesics property.

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Star Graphs and Density of Closed geodesics

Let $G = K_{1,n}$ be a star graph. Note that $\mathfrak{n}_G$ is singular and the resonance condition is not known to be sufficient in order to have the density of closed geodesics property.

Theorem

Let $\mathfrak{n}_G$ denote the metric 2-step nilpotent Lie algebra associated with a star graph on 3 vertices and N denote the corresponding simply connected nilpotent Lie group. Then for any lattice Γ in N, $\Gamma \backslash N$ has the density of closed geodesics property.

Proof uses the first hit map approach (introduced by Eberlein).

Star Graphs and Density of Closed geodesics

Theorem

Let $\mathfrak{n}_G$ denote the metric 2-step nilpotent Lie algebra where either $G = K_{1,n}$ or G is a complete graph on 3 vertices. Let N denote the corresponding simply connected nilpotent Lie group. Then there exists a lattice Γ in N, $\Gamma \backslash N$ has the density of closed geodesics property.

The first hit map approach fails in these cases.

Thank You!