Einstein extensions of Riemannian manifolds

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Motivation

Suppose we are given

- a Lie group N with the Lie algebra $\mathfrak{n}$,
- an inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{n}$, and
- a symmetric derivation η of $\mathfrak{n}$.

Then $\langle \cdot, \cdot \rangle$ generates a left-invariant metric g on N.
Now consider the one-dimensional extension $\mathfrak{m}$ of $\mathfrak{n}$ by D:

- $\mathfrak{m} := \mathbb{R}A \oplus \mathfrak{n}$ as a linear space;
- Lie bracket: for $X, Y \in \mathfrak{n}$, $[X, Y]_{\mathfrak{m}} := [X, Y]_{\mathfrak{n}}$, $[A, X]_{\mathfrak{m}} := \eta(X)$;
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The inner product $\langle \cdot, \cdot \rangle_{\mathfrak{m}}$ generates a left-invariant Riemannian metric on the group $M = N \rtimes \mathbb{R}$ which can explicitly be represented by the following ansatz.
Suppose D is the left invariant operator field on M such that $D(e) = \eta$. Then the metric g^D on M is given by
$g^D = du^2 + (\exp(uD))^* g$.
Equivalently, for $t_1 A + X_1, t_2 A + X_2 \in T_{(u,x)} M$,

$$g^D(t_1 A + X_1, t_2 A + X_2) = t_1 t_2 + g(e^{uD} X_1, e^{uD} X_2).$$

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Important case:

$\mathfrak{n}$ is nilpotent and $\langle \cdot, \cdot \rangle$ is a *nilsoliton inner product.* Then if we choose η "in the correct way", the resulting metric Lie group (M, g^D) is an *Einstein solvmanifold.* (And if the conjecture of the first named author is true, any non-compact Einstein homogeneous space is isometric to a one of those).

Properties:

- $\mathrm{Ric}_{\mathfrak{n}} = (\mathrm{Tr}\,\eta)\,\eta - \mathrm{Tr}(\eta^2)\,\mathrm{id}$;
- $\mathfrak{n}$ is naturally graded by η (so that all the eigenvalues of η are integer [rational], up to scaling);
- The eigenvalues satisfy certain linear relations, and for every n, there is only a finite number of the eigenvalue types of η of dimension n, up to scaling.

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Definition and Theorem

Idea: drop the homogeneity assumption and look at what is left ...

Definition

Let (N, g) be a Riemannian manifold of dimension $n > 1$, and D be a field of symmetric operators on (N, g). For every $u \in \mathbb{R}$, define the *D-deformation* of the metric g on N by $g^u := (\exp(uD))^* g$, and the *D-extension*, by

$$(M := \mathbb{R} \times N,\ g^D := du^2 + g^u).$$

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Question:

When the extension (M, g^D) is Einstein?

Theorem 1

The extension (M, g^D) is Einstein if and only if

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Examples

1. A Ricci flat manifold (N^n, g) is id-Einstein, that is, the metric $g^{\mathrm{id}} = du^2 + e^{2u} g$ is Einstein with Einstein constant $-n$. The converse (any id-Einstein manifold is Ricci flat) follows from (2).
2. A direct product $(N_1 \times N_2, g_1 + g_2)$ of Ricci D_i-stable manifolds (N_i, g_i), $i = 1, 2$, is Ricci $D = (D_1 \oplus D_2)$-stable.
3. Suppose (N, g) is a Lie group with a left-invariant metric, D is also left invariant and is defined by a symmetric derivation of the Lie algebra of N. Then (N, g) is D-stable.
4. Let (N, g, ξ) be an η-Einstein K-contact manifold (in particular, η-Einstein Sasaki manifold) with the Ricci tensor $\mathrm{ric} = -2g + (n+1)\eta \otimes \eta$, where ξ is a unit Killing vector which defines the contact 1-form $\eta = g \circ \xi$, and D is the canonical endomorphism given by $D = \mathrm{id}_{\mathrm{Ker}\,\eta} \oplus 2\mathrm{id}_{\mathbb{R}\xi}$. Then (N, g) is D-Einstein manifold.

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- $\mathbb{R}^n$ a Euclidean space with an orthonormal basis $\{f_i\}_{i=1}^n$;
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Suppose that (M, g^D) is Einstein and that $\det D \neq 0$.

1. *The projection of $\mathbf{1}_n$ to $p^\perp$ lies in the convex cone hull of $F \cap p^\perp$.*
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1. *The projection of $\mathbf{1}_n$ to $p^\perp$ lies in the convex cone hull of $F \cap p^\perp$.*
2. *If, in addition, $\operatorname{Tr} D \neq 0$, then up to scaling,*

$$p = \mathbf{1}_n - V(V^tV)^{-1}\mathbf{1}_m.$$

In particular, up to scaling, all the p_i are integers. Moreover, for every dimension n, there is only a finite number of the eigenvalue types of the operator D, with $\det D \neq 0$ and $\operatorname{Tr} D \neq 0$.

Eigenvalue structure of D

- $\mathbb{R}^n$ a Euclidean space with an orthonormal basis $\{f_i\}_{i=1}^n$;
- $F = \{f_i + f_j - f_k \,:\, i \neq j,\, k \neq i, j\} \subset \mathbb{R}^n$;
- $p = (p_1, \dots, p_n)^t$ and $\mathbf{1}_n = (1, \dots, 1)^t$ (n ones) vectors in $\mathbb{R}^n$;
- $F_p = \{v_1, \dots, v_m\}$ maximal linearly independent subset of $F \cap p^\perp$;
- V be an $n \times m$ matrix whose vector columns are the vectors v_a (if $v_a = f_i + f_j - f_k \in F_p$, then the a-th column of V has 1 in rows i and j, -1 in row k, and zeros elsewhere).

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The conditions imposed by Theorem 2(1) on the eigenvalue structure of D are quite restrictive (although somewhat implicit). For example,

- If D has two eigenvalues, both nonzero, then the eigenvalue type is $(1, \dots, 1, 2, \dots, 2)$.
- If $n = 3$ and $\det D \neq 0$, then $(1, 1, 2)$ and $(1, 1, 1)$ (so that $D = \mathrm{id}$) are the only possible eigenvalue structures, up to scaling.
- Similarly, If $n = 4$ and $\det D \neq 0$, then all the eigenvalue types, up to scaling, are

$$(1,1,1,1),\ (2,2,3,4),\ (3,4,4,7),\ (1,2,3,4),\ (1,1,2,2),$$
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Two eigenvalues

If the operator D is non-scalar, the simplest possible case to consider is when it has an eigenvalue of multiplicity $n-1$, so that $p_1 = \cdots = p_{n-1} = \lambda,\ p_n = \nu,\ \lambda \neq \nu$. Up to scaling, we can have $(\lambda, \nu) = (0,1), (1,0)$, or $(1,2)$ (from above).

Theorem 3

Suppose that (M, g^D) is Einstein and that D has eigenvalues 0 and 1 whose multiplicities are $n-1$ and 1 respectively. Then (N, g) is locally isometric to the Riemannian product of the real line and an Einstein manifold N' of dimension $n-1$ with the Einstein constant -1. The manifold (M, g^D) is locally isometric to the Riemannian product of the hyperbolic plane of curvature -1 and N'.

If D has eigenvalues 1 and 0 with multiplicities $n-1$ and 1 respectively, then (M, g^D) is a warped product with a two-dimensional base, and it can be obtained as a "double extension", by two commuting extensions, of a Ricci flat manifold N'.

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In the $(1, 2)$ case, we prove the following Theorem.

Theorem 4

Suppose that (M, g^D) is the extension of (N, g) such that D has eigenvalues 1 and 2 whose multiplicities are $n-1$ and 1 respectively. If (M, g^D) is Einstein, then there exists an almost Kähler, Ricci flat manifold (N', ds'^2) with the fundamental form $\frac{1}{2}d\theta'$, such that the metric g on N is locally given by $\overline{ds^2} = ds'^2 + (dx^n + \theta')^2$, and the Einstein metric g^D on M is locally given by $ds^2 = du^2 + e^{2u}ds'^2 + e^{4u}(dx^n + \theta')^2$.

The manifold (N, g) is contact. The resulting metric g^D on M is a particular case of the Page metric (which proves the converse).

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Four-dimensional Einstein extensions

Theorem 5

Suppose that $\dim N = 3$ *and that the extension* (M, g^D) *is Einstein. Then both* (N, g) *and* (M, g^D) *are locally isometric to Lie groups with left-invariant metrics; N is a nilmanifold or a solvmanifold, D is a derivation, and* (M, g^D) *is an Einstein solvmanifold. All the possible cases, up to scaling, are listed in the table:*

p_i	$\mathfrak{n}$	(M, g^D)	ds^2
$0, 0, 0$	*abelian*	$\mathbb{R}^4$	$du^2 + dx_1^2 + dx_2^2 + dx_3^2$
$1, 1, 1$	*abelian*	$H^4(-1)$	$du^2 + e^{2u}(dx_1^2 + dx_2^2 + dx_3^2)$
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Extension of a Lie group

Let N be a Lie group with a left-invariant metric g and let D be left-invariant.
Is it true that if (M, g^D) is Einstein, then it is an Einstein solvmanifold?
We answer this question in positive in two cases.

Theorem 6

Suppose the extension (M, g^D) of a Lie group (N, g) by D is Einstein, and both g and D are left-invariant. Denote $\mathfrak{n}$ the Lie algebra of N.

1. *If the Killing form of the Lie algebra $\mathfrak{n}$ is nonnegative, then D is a derivation of $\mathfrak{n}$.*
2. *Suppose the Lie algebra $\mathfrak{n}$ contains a codimension one abelian ideal $\mathfrak{a}$. Then there exists a metric solvable Lie group (N', g') and an isometry $\phi : (N, g) \to (N', g')$ such that $D' = (d\phi)D$ is left-invariant relative to N' and is a derivation of the Lie algebra $\mathfrak{n}'$ of N'.*

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In the both cases the manifold (M, g^D) is an Einstein solvmanifold.

Extension of a Lie group

Let N be a Lie group with a left-invariant metric g and let D be left-invariant.
Is it true that if (M, g^D) is Einstein, then it is an Einstein solvmanifold?
We answer this question in positive in two cases.

Theorem 6

Suppose the extension (M, g^D) of a Lie group (N, g) by D is Einstein, and both g and D are left-invariant. Denote $\mathfrak{n}$ the Lie algebra of N.

1. *If the Killing form of the Lie algebra $\mathfrak{n}$ is nonnegative, then D is a derivation of $\mathfrak{n}$.*
2. *Suppose the Lie algebra $\mathfrak{n}$ contains a codimension one abelian ideal $\mathfrak{a}$. Then there exists a metric solvable Lie group (N', g') and an isometry $\phi : (N, g) \to (N', g')$ such that $D' = (d\phi)D$ is left-invariant relative to N' and is a derivation of the Lie algebra $\mathfrak{n}'$ of N'.*

In the both cases the manifold (M, g^D) is an Einstein solvmanifold.