The Ricci Flow for Homogeneous Spaces from the Perspective of Evolutionary Game Theory

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Motivation

Example

The Ricci flow for left-invariant metrics on the five-dimensional Heisenberg nilpotent Lie group yields the system of differential equations

$$\dot{x}_1 = x_1(-3x_1 - x_2)$$
$$\dot{x}_2 = x_2(-x_1 - 3x_2)$$

What are exact solutions? Qualitative analysis of this system?

Observation

These are generalized Lotka-Volterra equations.

Main points of this talk

- The right conceptual setting for analyzing systems like this is evolutionary game theory.
- The most basic type of dynamic in evolutionary game theory is called the replicator dynamic. The Ricci flow for homogeneous spaces [1] is a replicator dynamic for a linear or quadratic game.
- Can relate game-theoretic notions (Nash equilibrium) to geometric notions (soliton metric).
- Can also use other evolutionary dynamics (beside the replicator dynamic) to study geometric evolution equations.
- Can study other geometric flows (beside the Ricci flow), such as the combinatorial Ricci flow.
- Can apply methods and existing results from evolutionary game theory to analyze properties of geometric flows.

[1] with appropriate basis and a natural change of variables

Game theory

Game theory is math modelling of competition

- Strategic interactions between intelligent decision makers
- Two or more players
- Each player has choices of moves.
- Different payoffs for different moves
- The payoff for a move to one player depends on what moves the other players make.

Evolutionary game theory

Use mathematics to model phenomena in biological evolution

- (1970s) Biologist John Maynard Smith applies game theory to
  - natural selection (birth and death rates)
  - repeated interactions in animal behavior

  and defines "evolutionarily stable strategy" (ESS)
- Large or infinite populations, frequency-dependent selection
- Players do not change strategies. Strategy frequencies change through variable rates of reproduction and heredity.
- (1978) Taylor and Jonker: math model for replicator dynamic and ESS
- (1979 on) Schuster, Sigmund, Hofbauer: simplify and develop math model for replicator dynamic and ESS

Evolutionary game theory

- Models repeated anonymous strategic interactions in large populations
- An optimization problem, but with multiple functions being optimized: Each agent is optimizing only its own payoff

Definition

A point $\mathbf{x}$ is a Nash equilibrium if

$$x_i > 0 \Rightarrow F_i(\mathbf{x}) \geq F_j(\mathbf{x}) \quad \text{for all } j$$

If a strategy is in use, then a player can not improve its payoff by switching from that strategy to another strategy.

Example from evolutionary game theory

Hawk-dove game

Two behaviors: Hawk (H) and dove (D)

	H	D
H	-1	2
D	0	1

Assume a large population with differing strategies. Let $\mathbf{x} = (x_H, x_D)$ be the probability density function. Payoffs for H and D depend on $\mathbf{x}$:

$$\mathbf{F}(\mathbf{x}) = \begin{bmatrix} F_H(\mathbf{x}) \\ F_D(\mathbf{x}) \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_H \\ x_D \end{bmatrix} = \begin{bmatrix} 2x_D - x_H \\ x_D \end{bmatrix}$$

Interior stable Nash equilibrium $\mathbf{x}^* = (\frac{2}{3}, \frac{1}{3})$

Formal set-up

- n strategies
- a_i = frequency that strategy i is used
- State space is the closed simplex $X \subseteq \mathbb{R}^n$ defined by

$$X = \left\{ \mathbf{a} = (a_i) \, : \, \sum_{i=1}^{n} a_i = 1; a_1, a_2, \ldots, a_n \geq 0 \right\}$$

  (discrete probability densities)
- Coordinate vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are "pure strategies"; interior points in X are "mixed strategies"
- $F_i(\mathbf{a})$ = payoff for strategy i when the overall distribution of strategies is $\mathbf{a}$

Discrete probabilistic model

- Poisson alarm clock goes off at times $t = 1, 2, 3, \ldots$
- When the alarm goes off, agents may revise strategy according to some revision protocol. Define conditional switch rates $\rho_{ij}(F, x)$ proportional to the probability of switching from Strategy i to Strategy j.
- Take limit so time becomes continuous to get a deterministic evolutionary dynamic

$$\dot{x}_i = \sum x_j \rho_{ji} - x_i \sum \rho_{ij}$$

- Different revision protocols give different ODEs
- Many revision protocols give the same ODEs

In biological settings, revision protocols describe births and deaths.

Replicator dynamic

Some imitative revision protocols

Randomly choose another agent and see their strategy.

- Imitation driven by dissatisfaction:

$$\rho_{ij} = x_j(C - F_i) \qquad (C \text{ very large})$$

- Imitation of success:

$$\rho_{ij} = x_j(F_j - C) \qquad (C \text{ very small})$$

- Pairwise proportional imitation:

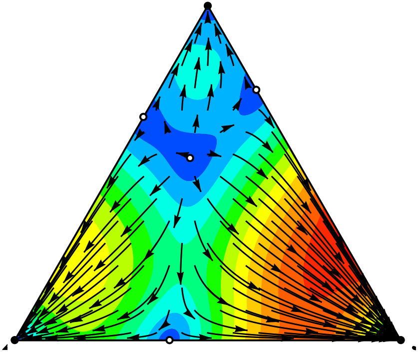
$$\rho_{ij} = x_j[F_j - F_i]_+$$

All yield the same deterministic dynamic, the *replicator dynamic*

$$\dot{x}_i = x_i(F_i - \overline{F}), \quad \text{where } \overline{F} = \sum x_i F_i.$$

Example: 1-2-3 coordination replicator dynamic

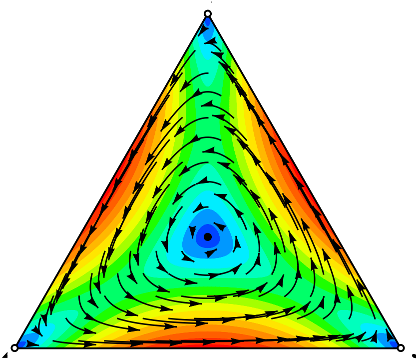
$$F(\mathbf{x}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$



(Colors show rate of convergence)

Example: Rock-paper-scissors replicator dynamic

$$F(\mathbf{x}) = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$



Best response dynamic: A target dynamic

Definition

The maximizer correspondence M maps a payoff F to the subset of X so that mass is placed on a pure strategy $\mathbf{e}_i$ if $F_i(x) \geq F_j(x)$ for all j.

Remark

Usually this is just the $\mathbf{e}_i$ so that $F_i(\mathbf{x})$ is maximal. However, if there are two or more $\mathbf{e}_i$s for which $F_i(\mathbf{x})$ is maximal, $M(\mathbf{x})$ is the convex hull of $\{\mathbf{e}_{i_1}, \mathbf{e}_{i_2}, \ldots, \mathbf{e}_{i_k}\}$

Deterministic dynamic (a differential inclusion rather than an ODE)

$$\dot{x}_i \in V(\mathbf{x}(t)) := M(F_i(\mathbf{x})) - \mathbf{x}_i.$$

Properties of the best response dynamic

Picture

Velocity vectors $\dot{\mathbf{x}}$ have their heads in $M(\mathbf{x})$ and tails at $\mathbf{x}$

Definition

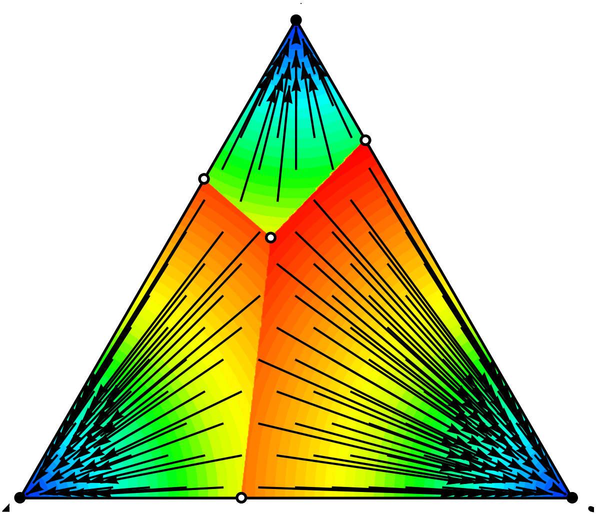
Carathéodory solutions are $\mathbf{x}(t)$ so that $\mathbf{x}$ is Lipschitz and $\dot{x} \in V(\mathbf{x}(t))$ for all t.

Solutions are nice if V is nice

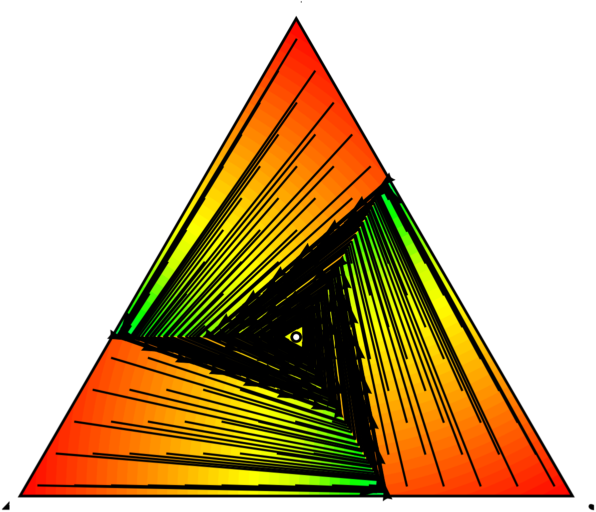
If the sets $V(\mathbf{x})$ are nonempty, convex-valued, bounded and upper-hemicontinuous, then Carathéodory solutions are well-behaved:

- existence
- initial conditions with nonunique solutions can be controlled

Example: 1-2-3 coordination best response



Example: Rock-paper-scissors best response



Geometry of Lie Groups

Let (G, g) be a three-dimensional Lie group G endowed with a left-invariant metric g.

Encode (G, g) as the metric Lie algebra $(\mathfrak{g}, Q)$ where

- $\mathfrak{g} = T_e G$ is a three-dimensional real vector space
- $Q = g|_{T_e G} = g|_{\mathfrak{g}}$ is an inner product on $\mathfrak{g}$

Let ric be the Ricci form for (G, g), evaluated at $T_e G \cong \mathfrak{g}$.

- Depends only on the structure constants for $\mathfrak{g}$ relative to a Q-orthonormal basis

The Ricci flow for 3-D Lie groups

Definition

For a simply connected Lie group with a left-invariant metric, the Ricci flow becomes the system of ODEs

$$Q_t = -2\,\mathrm{ric}(Q_t).$$

If Q and $\mathrm{ric}(Q)$ are represented with respect to a fixed basis $\mathcal{B}$, they are 3×3 matrices and

$$\begin{bmatrix} q_{11}(t) & q_{12}(t) & q_{13}(t) \\ q_{21}(t) & q_{22}(t) & q_{23}(t) \\ q_{31}(t) & q_{32}(t) & q_{33}(t) \end{bmatrix} = -2 \begin{bmatrix} r_{11}(Q_t) & r_{12}(Q_t) & r_{13}(Q_t) \\ r_{21}(Q_t) & r_{22}(Q_t) & r_{23}(Q_t) \\ r_{31}(Q_t) & r_{32}(Q_t) & r_{33}(Q_t) \end{bmatrix}$$

is a system of 9 ODEs in 9 variables.

Unimodular Lie algebras in dimension 3

Theorem (Milnor, 1976)

Let $(\mathfrak{g}, Q)$ *be a three-dimensional unimodular metric Lie algebra. Then there exists an orthonormal basis* $\{e_1, e_2, e_3\}$ *(a Milnor basis or Milnor frame) and scalars* a_1, a_2, a_3 *so that*

$$[e_1, e_2] = a_3 e_3 \qquad [e_2, e_3] = a_1 e_1 \qquad [e_3, e_1] = a_2 e_2.$$

Fact

The Ricci form is diagonal with respect to the Milnor basis.

Definition

Let Q be the standard inner product on $\mathbb{R}^3 = \operatorname{span}\{e_1, e_2, e_3\}$. For $\mathbf{a} = (a_1, a_2, a_3)$ in $\mathbb{R}^3$, let $(\mathfrak{g}_{\mathbf{a}}, Q)$ be the metric Lie algebra with structure constants a_1, a_2, a_3 relative to a Milnor basis.

Unimodular metric Lie algebras in dimension 3

$$[e_1, e_2] = a_3 e_3 \qquad [e_2, e_3] = a_1 e_1 \qquad [e_3, e_1] = a_2 e_2$$

Signs of $\{a_1, a_2, a_3\}$	Associated Lie algebra	Associated Lie group
$+, +, +$	$\mathfrak{su}(2) \cong \mathfrak{so}(3)$	$SU(2) \cong SO(3) \cong \mathrm{Isom}_+(\mathbb{S}^2)$
$+, +, -$	$\mathfrak{sl}_2(\mathbb{R})$	$SL_2(\mathbb{R}) \cong \mathrm{Isom}_+(\mathbb{H}^2)$
$+, +, 0$	$\mathfrak{e}(2)$	$E(2) \cong \mathrm{Isom}_+(\mathbb{R}^2)$
$+, -, 0$	$\mathfrak{e}(1,1)$	$E(1,1) \cong Sol$
$+, 0, 0$	$\mathfrak{h}_3$	$H_3 \cong Nil$

Stably diagonal Ricci flow

Definition

Let $\mathcal{B}$ be a basis for the metric Lie algebra $(\mathfrak{g}, Q)$. Suppose that $\mathcal{B}$ is an orthogonal Ricci eigenvector basis. Let Q_t denote the solution to the Ricci flow with $Q_0 = Q$. Say that $\mathcal{B}$ is stably diagonal if for $t > 0$, both the inner product Q_t and the Ricci endomorphism $\mathrm{Ric}(Q_t)$ remain diagonal with respect to $\mathcal{B}$.

Proposition

Let $\mathcal{B}$ be a Milnor basis for a three-dimensional unimodular metric Lie algebra. Then $\mathcal{B}$ is stably diagonal.

With respect to the Milnor basis, the Ricci flow for $(\mathfrak{g}_{\mathbf{a}}, Q)$ is a a system of ODEs in $q_{11}(t), q_{22}(t), q_{33}(t)$.

The bracket flow

Change of variables

Instead of evolving the inner product Q, evolve

$$\mathbf{a}(t) = (a_1(t), a_2(t), a_3(t))$$

(structure constants relative the Milnor basis at time t)

Since $\mathrm{ric}(Q(t))$ is a function of $\mathbf{a}(t)$ we can find $Q(t)$ from $\mathbf{a}(t)$ using

$$Q_t = -2\,\mathrm{ric}(Q_t).$$

Evolution equations for the bracket flow on 3D unimodular Lie groups

Define $F : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$F(a_1, a_2, a_3) = -2(\mathrm{ric}_{\mathbf{a}}(e_1, e_1), \mathrm{ric}_{\mathbf{a}}(e_2, e_2), \mathrm{ric}_{\mathbf{a}}(e_3, e_3)).$$

The map F sends the Lie algebra $\mathfrak{g}_{\mathbf{a}}$ to the spectrum of its Ricci form. The coordinate functions of F are

$$F_i(\mathbf{a}) = 2\mathbf{a}^T B_i \mathbf{a}, \quad i = 1, 2, 3,$$

where

$$B_1 = \begin{bmatrix} -3 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}, B_2 = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -3 & 1 \\ -1 & 1 & 1 \end{bmatrix}, B_3 = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & -3 \end{bmatrix}.$$

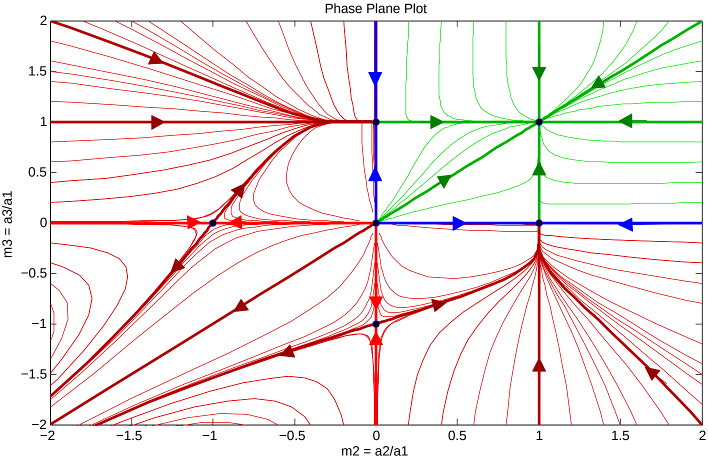
The bracket flow for 3D unimodular Lie groups

The bracket flow for 3D unimodular Lie groups normalizes to the replicator equation

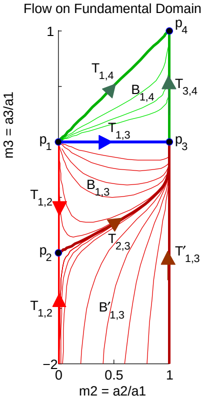
$$\dot{a}_1 = a_1(F_1(\mathbf{a}) - \overline{F})$$
$$\dot{a}_2 = a_2(F_2(\mathbf{a}) - \overline{F})$$
$$\dot{a}_3 = a_3(F_3(\mathbf{a}) - \overline{F})$$

Since the functions F_i are quadratic, this is a quadratic game.

Phase portrait for a_2/a_1, a_3/a_1 (Glickenstein-P., 2010)



Moduli space a_2/a_1, a_3/a_1 (Glickenstein-P., 2010)



Alternate renormalization gives replicator equations of quadratic type

- Consider the flow on QI and QIV separately.
- Evolve $|a_1|, |a_2|, |a_3|$ separately.
- Normalize so the simplex $X \subseteq \mathbb{R}^3$ is invariant:

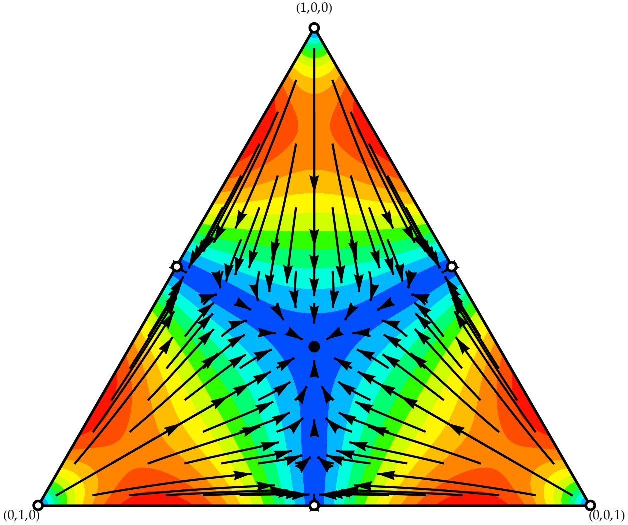
$$\dot{a}_1 = a_1(F_1(\mathbf{a}) - \overline{F})$$
$$\dot{a}_2 = a_2(F_2(\mathbf{a}) - \overline{F})$$
$$\dot{a}_3 = a_3(F_3(\mathbf{a}) - \overline{F})$$

(a replicator equation)

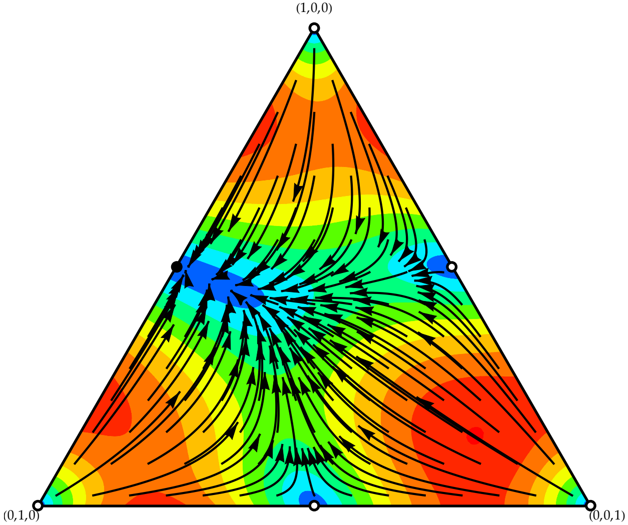
Relationship to EGT

Can not always optimize scalar curvature. Instead try to optimize eigenvalues of ric (as with payoffs for different strategies)

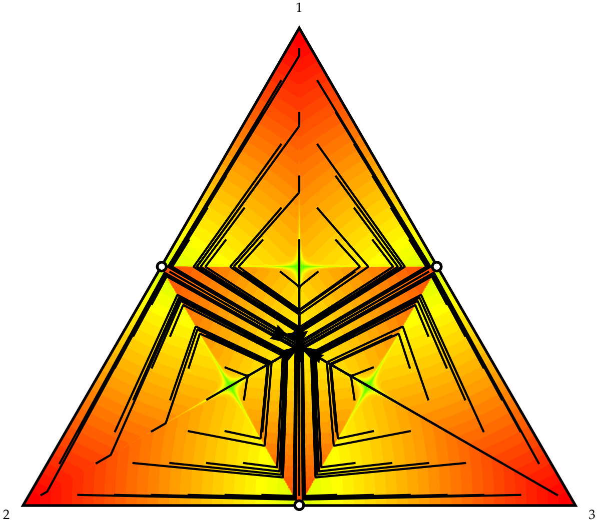
The bracket flow replicator dynamic in QI



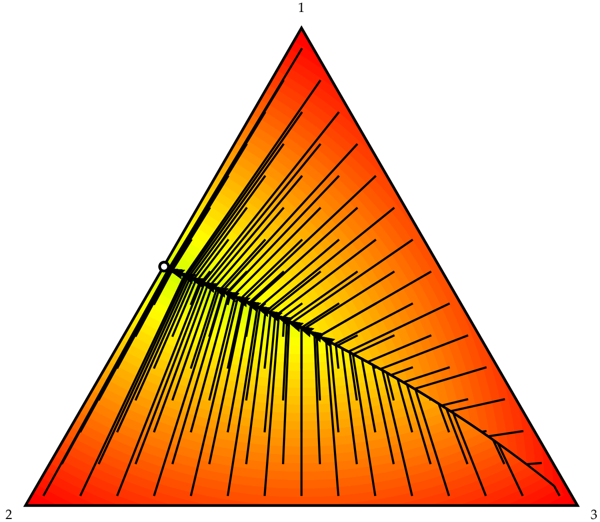
The bracket flow replicator dynamic in QIV



The bracket flow best response dynamic in QI



The bracket flow best response dynamic in QIV



The bracket flow for simply connected homogeneous spaces

Theorem (T.P.)

Let $(G/K, g)$ be a homogeneous space. Let $\{e_i\}$ be an orthonormal basis for $(T_{eK}(G/K), Q)$ which remains orthogonal under the Ricci flow.

- *With respect to this basis, after a change of variables, the bracket flow is a replicator equation with quadratic fitness functions.*
- *If G/K is a nilmanifold, then the quadratic forms are all diagonal, and after a change of variables, the bracket flow is encoded as a replicator equation with linear fitness functions.*
- $\{\textit{interior fixed points}\} = \{\textit{soliton metrics}\} = \{\textit{interior Nash equilibria}\}$

Generalized Wallach spaces

Definition

Let G/H be a compact homogeneous space, where G is a connected semisimple Lie group, and H is a closed subgroup. Assume G/H is almost effective. Let $\langle \cdot, \cdot \rangle = -B(\cdot, \cdot)$ be the bi-invariant metric on G defined by the Killing form B on $\mathfrak{g}$. Write $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{h}$. If $\mathfrak{p}$ decomposes into the direct sum of three pairwise orthogonal $\mathrm{ad}_{\mathfrak{h}}$-invariant irreducible modules

$$\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3$$

such that $[\mathfrak{p}_i, \mathfrak{p}_i] \subseteq \mathfrak{k}$ for $i = 1, 2, 3$, then G/H is a *generalized Wallach space*.

Associated algebraic parameters: a_1, a_2, a_3. Classified by Nikonorov (2015), Einstein metrics analyzed by Nikonorov and others.

Evolution equations for the bracket flow on generalized Wallach spaces

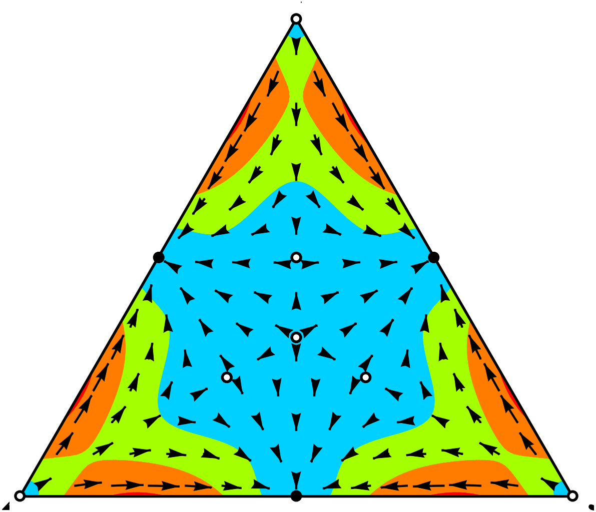
Define $F : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$F(x_1, x_2, x_3) = -2(\mathrm{ric}_{\mathbf{x}}(e_1, e_1), \mathrm{ric}_{\mathbf{x}}(e_2, e_2), \mathrm{ric}_{\mathbf{x}}(e_3, e_3))$$
$$= (\mathbf{x}^T B_1 \mathbf{x}, \mathbf{x}^T B_2 \mathbf{x}, \mathbf{x}^T B_3 \mathbf{x}),$$

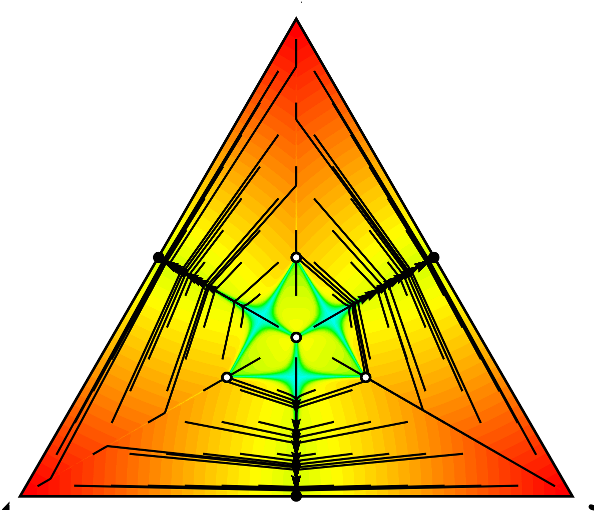
where

$$B_1 = \frac{1}{2}\begin{bmatrix} -(a_1 + a_2 + a_3) & 1/2 & 1/2 \\ 1/2 & a_1 + a_2 - a_3 & -1/2 \\ 1/2 & -1/2 & a_1 - a_2 + a_3 \end{bmatrix},$$
$$B_2 = \frac{1}{2}\begin{bmatrix} a_1 + a_2 - a_3 & 1/2 & -1/2 \\ 1/2 & -(a_1 + a_2 + a_3) & 1/2 \\ -1/2 & 1/2 & -a_1 + a_2 + a_3 \end{bmatrix}$$
$$B_3 = \frac{1}{2}\begin{bmatrix} a_1 - a_2 + a_3 & -1/2 & 1/2 \\ -1/2 & -a_1 + a_2 - a_3 & 1/2 \\ 1/2 & 1/2 & -(a_1 + a_2 + a_3) \end{bmatrix}.$$

The replicator dynamic for the bracket flow on $SU(3)/T^2$ $(a_1 = a_2 = a_3 = 1/6)$



The best response dynamic for the bracket flow on $SU(3)/T^2$ ($a_1 = a_2 = a_3 = 1/6$)



Nash equilibria

Nash equilibria

- For 4/6 kinds of dynamics, $\{\text{Nash equilibria}\} \subseteq \{\text{fixed points}\}$
- $\mathbf{x}$ in $\mathrm{int}(X)$ is a Nash equilibrium if and only if $F_i(\mathbf{x}) = F_j(\mathbf{x})$ for all i, j.
  - For nilpotent N, this yields "$U\mathbf{v} = [1]$" theorem
  - For quadratic $F_i(\mathbf{x}) = \mathbf{x}^T B_i \mathbf{x}$, $i = 1, 2, 3$, we get $\mathbf{x}^T(B_1 - B_2)\mathbf{x} = \mathbf{x}^T(B_2 - B_3)\mathbf{x}$ (projectivized).

Circle-packing metrics on a triangulated surface

Circle-packing metric

T = a triangulation of a closed connected surface S

$V = \{v_1, v_2, \ldots, v_n\}$ = the set of vertices in T

For each vertex v_i, let $r_i \in [0, \infty)$.

If there is an edge between vertices v_i and v_j, define its length to be $l_{ij} = r_i + r_j$.

The triangle is isometric to a flat triangle in Euclidean space.

We get a flat cone metric on the surface S with singularities at each vertex.

Combinatorial Ricci flow (Chow-Luo, 2003)

Combinatorial Ricci flow

Let $(r_1, r_2, \ldots, r_n)$ be in the simplex $X \subseteq \mathbb{R}^n$. Let $\mathbf{K} = (K_1, K_2, \ldots, K_n)$ with

$$K_i(r_1, \ldots, r_n) = 2\pi - \sum \cos^{-1}\left(\frac{r_i - r_j r_k}{r_i + r_j r_k}\right)$$

define "fitness." The replicator dynamic gives a renormalization of the combinatorial Ricci flow:

$$\dot{r}_i = r_i(K_i - \overline{K_i})$$

Good numerical convergence for the tetrahedron.

References

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