The Steklov Problem on Orbifolds
EGEO 2016

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August 1, 2016

Orbifolds

Roughly speaking, an **orbifold (with boundary)** is a space locally modeled on $\mathbb{R}^n/\Gamma$ (or $\mathbb{R}^n_+/\Gamma$), where Γ is a discrete group acting properly discontinuously on $\mathbb{R}^n$ or on $\mathbb{R}^n_+$.

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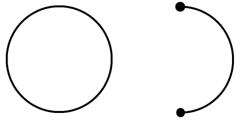
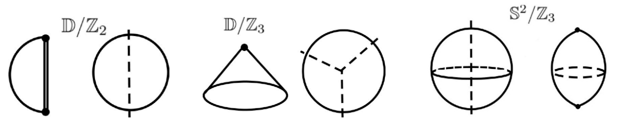
Example

- One-dimensional closed orbifolds (Type I and Type II, respectively)



- Two-dimensional orbifolds (Orbisurfaces)



The Steklov problem

Definition

The **Dirichlet-to-Neumann operator** $\mathcal{D} : C^\infty(\partial\mathcal{O}) \to C^\infty(\partial\mathcal{O})$ is defined as follows.

- Take $u \in C^\infty(\mathcal{O})$.
- Let $\tilde{u}$ be the harmonic extension of u to $\mathcal{O}$.
- $\mathcal{D}(u) = (\partial_\nu \tilde{u})|_{\partial\mathcal{O}}$.

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Definition

The **Steklov spectrum** of $\mathcal{O}$, denoted $Stek(\mathcal{O})$ is the set of all σ such that $(\partial_\nu \tilde{u}) = \sigma \tilde{u}$ on $\partial\mathcal{O}$.

$$0 = \sigma_0 \leq \sigma_1 \leq \sigma_2 \leq \cdots \nearrow \infty$$

Steklov Spectrum (Examples)

- Let $\mathbb{D}_r$ be a Euclidean disk of radius r in $\mathbb{R}^2$. The Steklov eigenvalues are

$$0, \frac{1}{r}, \frac{1}{r}, \frac{2}{r}, \frac{2}{r}, \frac{3}{r}, \frac{3}{r}, \dots$$

The eigenfunctions corresponding to $\frac{m}{r}$ are $\cos\left(\frac{m}{r}\theta\right)$ and $\sin\left(\frac{m}{r}\theta\right)$.

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- Let $\mathcal{O} = \mathbb{D}_r/\mathbb{Z}_2$. $\mathbb{Z}_2$ acts isometrically on $\mathbb{D}_r$ by a reflection across a diameter, creating a "half disc" orbifold. The Steklov eigenvlaues are

$$0, \frac{1}{r}, \frac{2}{r}, \frac{3}{r}, \ldots$$

The eigenfunctions corresponding to $\frac{m}{r}$ are $\cos\left(\frac{m}{r}\theta\right)$.

Steklov Spectrum (Examples)

- From before, we see $Stek(\mathbb{D}_{kr}) = 0, \frac{1}{kr}, \frac{1}{kr}, \frac{2}{kr}, \frac{2}{kr}, \cdots, \frac{k}{kr}, \frac{k}{kr}, \cdots$
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- Let $\mathcal{O} = \mathbb{D}_{kr}/\mathbb{Z}_k$ be a cone, where $\mathbb{Z}_k$ acts isometrically on $\mathbb{D}_{rk}$ by a rotation of angle $\frac{2\pi}{k}$. The Steklov eigenvalues are

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$$0, \cancel{\frac{1}{kr}}, \cancel{\frac{1}{kr}}, \cancel{\frac{2}{kr}}, \cancel{\frac{2}{kr}}, \cdots, \frac{k}{kr}, \frac{k}{kr}, \cdots$$

$$= 0, \frac{1}{r}, \frac{1}{r}, \frac{2}{r}, \frac{2}{r}, \frac{3}{r}, \frac{3}{r}, \cdots$$

So a disc of radius r and a cone of radius kr with angle $2\pi/k$ are Steklov isospectral.

Prior Results vs. New Setting

Theorem

For 2-dimensional Riemannian manifolds, the Laplace spectrum determines the Euler characteristic.

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Is this true for Steklov spectrum on orbifolds?

NO! Disc vs. Cone

Prior Results vs. New Setting

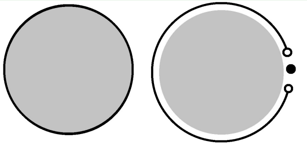
Definition (Euler characteristic of an orbifold)

Let $\{c_i\}$ be a cell division of orbisurface $\mathcal{O}$ for which the isotropy group associated to the interior points of each cell is constant. The Euler characteristic of $\mathcal{O}$ is defined by

$$\chi(\mathcal{O}) := \sum_i (-1)^{dim\, c_i} \frac{1}{|Iso(c_i)|}$$

where $|Iso(c_i)|$ is the order of the isotropy type associated to the cell c_i.

Example



$\chi(\mathbb{D}_r) = 1$

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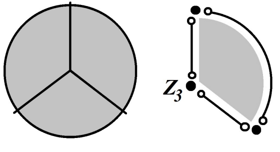

$\chi(\mathbb{D}_{3r}/\mathbb{Z}_3) = 1/3$

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For 2-dimensional Riemannian **manifolds**, *the* **Laplace** *spectrum determines the Euler characteristic.*

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Proposition (ADGH-S)

For 2-dimensional Riemannian **orbifolds**, *the* **Steklov** *spectrum does* **not** *determine the Euler characteristic.*

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For 2-dimensional Riemannian **orbifolds***, the* **Steklov** *spectrum does* **not** *detect the presence of singularities in the interior of an orbifold.*

Prior Results vs. New Setting

Theorem (Girouard, Parnovski, Polterovich and Sher, 2014)

- *Let M be a smooth compact Riemannian surface with t boundary components of lengths $\ell_1, \ldots, \ell_t$. Set $C = \{\ell_1, \ldots, \ell_t\}$. Then*

$$\sigma_j = \gamma_j^{(I)}(C) + O(j^{-\infty}),$$

where $\gamma^{(I)}(C)$ is the Steklov spectrum of a disjoint union of t disks with radii $\frac{\ell_i}{2\pi}$, $i = 1, \ldots, t$.

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Is this true for Steklov spectrum on orbifolds?

Prior Results vs. New Setting

No. . . ,

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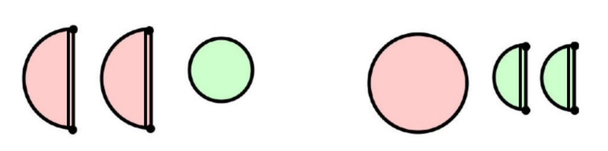
Example

$$Stek\left(\mathbb{D}_r \sqcup \mathbb{D}_R/\mathbb{Z}_2 \sqcup \mathbb{D}_R/\mathbb{Z}_2\right) = Stek\left(\mathbb{D}_R \sqcup \mathbb{D}_r/\mathbb{Z}_2 \sqcup \mathbb{D}_r/\mathbb{Z}_2\right)$$



$$\begin{aligned} Stek(\mathbb{D}_r) &= 0, \tfrac{1}{r}, \tfrac{1}{r}, \tfrac{2}{r}, \tfrac{2}{r}, \dots \\ Stek(\mathbb{D}_R/\mathbb{Z}_2) &= 0, \tfrac{1}{R}, \tfrac{2}{R}, \dots \\ Stek(\mathbb{D}_R/\mathbb{Z}_2) &= 0, \tfrac{1}{R}, \tfrac{2}{R}, \dots \end{aligned}$$

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Prior Results vs. New Setting

However,

Theorem (ADGH-S)

Let $(\mathcal{O}, g)$ be a compact Riemannian orbisurface with boundary consisting of t type I boundary components of lengths $\ell_1, \ldots, \ell_t$ and s type II boundary components of lengths $\bar{\ell_1}, \ldots, \bar{\ell_s}$. Then,

$$\sigma_j = \gamma_j(C; \bar{C}) + O(j^{-\infty}),$$

where $\gamma(C; \bar{C})$ is the Steklov spectrum of a disjoint union of t disks with boundaries of length $\ell_1, \ldots, \ell_t$, and s "half-disc" orbifolds with boundaries of length $\bar{\ell_1}, \ldots, \bar{\ell_s}$.

Prior Results vs. New Setting

Corollary (ADGH-S)

Knowing the Steklov spectrum of a compact Riemannian orbisurface up to $O(j^{-\infty})$ determines the **number** *of Type I and Type II boundary components.*

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Knowing the Steklov spectrum of a compact Riemannian orbisurface up to $O(j^{-\infty})$ *determines the number* **AND lengths** *of Type I and Type II boundary components up to an* **equivalence relation** *defined by the example switching*

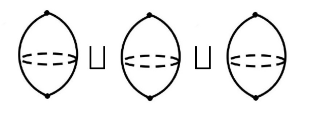
$$(\mathbb{D}_r \sqcup \mathbb{D}_R/\mathbb{Z}_2 \sqcup \mathbb{D}_R/\mathbb{Z}_2) \;\; \textit{with} \;\; (\mathbb{D}_R \sqcup \mathbb{D}_r/\mathbb{Z}_2 \sqcup \mathbb{D}_r/\mathbb{Z}_2)$$

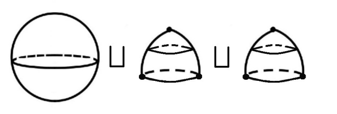
Counterexample in 3-dimensions

The results of this theorem do not hold in dimension 3.

Example

The Klein 4 group $\Gamma = \{1, \sigma, \tau, \sigma\tau\}$ acts isometrically on $M := B(0,1)$ where $\sigma =$rotation of π about the x-axis, $\tau =$rotation of π about the y-axis, and $\sigma\tau =$rotation of π about the z-axis. Take the subgroups $H_1 = \{1, \sigma\}$, $H_2 = \{1, \tau\}$, $H_3 = \{1, \sigma\tau\}$, and $K_1 = \{1\}$, $K_2 = \Gamma$, $K_3 = \Gamma$. By Parzanchevski's generalization of the Sunada theorem, $\mathcal{O}_1 := \sqcup_{i=1}^{3} (M/H_i)$ is Steklov isospectral to $\mathcal{O}_2 := \sqcup_{i=1}^{3} (M/K_i)$.





Thank you!