Submanifolds and Holonomy

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Submanifolds

Let M and $\bar{M}$ be Riemannian manifolds, when M is a subset of $\bar{M}$ and the inclusion map is an isometric immersion, then we say that M is a **submanifold** of $\bar{M}$.

Let M be submanifold of $\bar{M}$, the Riemannian metric on $\bar{M}$ induces along M on ortogonal splitting of $T\bar{M}$. $T\bar{M}|_M = TM \oplus \nu M$.

The vector bundle νM is called the **normal bundle** of M, the fibre at $p \in M$ is the normal space at p and is denoted by $\nu_p M$.

A section of νM is called a **normal vector field**.

Let X, Y be a vector field on M and ξ a normal vector field of M, and $\bar{\nabla}, \nabla$ the Levi-Civita connections of $\bar{M}$ and M respectively. Then we have the next equations without care the extensions of the fields in the ambient space.

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Submanifolds

$\bar{\nabla}_X Y = \nabla_X Y + \alpha(X, Y)$ **Gauss formula**.
$\bar{\nabla}_X \xi = -A_\xi X + \nabla^{\perp}_X \xi$ **Weingarten formula**.
Observation: 1. α is called the second fundamental form which one C^∞-bilinear symmetric tensor field with values in the normal bundle.
2.$\nabla^{\perp}$ define a metric connection over the normal bundle and is called **normal connection**.
3. A_ξ is called the shape operator of M in direction of ξ and is related to the second fundamental form by the equation: $\langle \alpha(x, y), \xi \rangle = \langle A_\xi X, Y \rangle$.
4. A_ξ is a self-adjoint tensor field on M, $A_\xi(p)$ does not depend on the extension of ξ_p as a normal vector field.

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Submanifolds

From now $\bar{M} = \mathbb{R}^N, S^N$

Definition

A submanifold M of $\mathbb{R}^N$ is called **full** if it is not contained in any proper affine subspace of $\mathbb{R}^N$.

Definition

If $M_1 \subset \mathbb{R}^{N_1}$ and $M_2 \subset \mathbb{R}^{N_2}$ are (Riemannian) submanifolds then $M_1 \times M_2$ is a submanifold of $\mathbb{R}^{N_1+N_2}$ which is called the product of M_1 by M_2. A submanifold of euclidian space is called irreducible if it is not a product of manifolds.

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The rank of a Euclidean submanifold is the maximal number of linearly independent, locally defined, parallel normal fields.

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Holonomy

Let M be a submanifold of $\bar{M}$. Let $p \in M$ and c be a piecewise differentiable curve in M with $c(0) = c(1) = p$ then the $\nabla^{\perp}$-parallel transport along c induces a linear isometry

$$\tau_c^{\perp} : \nu_p M \to \nu_p M$$

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Definition

The normal holonomy group to p is the set $\{\tau_c^{\perp} : c \text{ is a piecewise differentiable curves in } M \text{ with } c(0) = c(1) = p$, and we denote by $\Phi(p)$, the restricted normal holonomy group $\Phi^*(p)$, it is the identity component of the holonomy.

Observations: $\Phi(p), \Phi*(p) \subset O(\nu_p M)$, they are Lie subgroups of the ortogonal group. If M is connected, the normal holonomy groups from two points are conjugated by the parallel transport, for this reason we write just Φ and $\Phi*$ instead $\Phi(p)$ or $\Phi*(p)$ respectively.

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Holonomy

Normal Holonomy Theorem (1990-Olmos)

Let M be a connected submanifold of a standard space form $\bar{M}^n(k)$. Let $p \in M$ and let Φ^* be the restricted normal holonomy group at p. Then Φ^* is compact, there exists a unique (up to order) orthogonal decomposition $\nu_p M = V_0 \oplus ... \oplus V_m$ of the normal space $\nu_p M$ into Φ^*-invariant subspaces and there exists normal subgroups $\Phi_0, ..., \Phi_m$ of Φ^* such that:

i $\Phi^* = \Phi_0 \times ... \times \Phi_m$ *(direct product)*

ii Φ_i *acts trivially on* V_j *if* $j \neq i$.

iii $\Phi_0 = \{1\}$ *and, if* $i \geq 1$, Φ_i *acts irreducibly on* V_i *as the isotropy representation of an irreducible Symmetric Riemannian space.*

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An S-representation is by definition the isotropy representation of the simply connected, semisimple and symmetric space $M = G/K$. And the orbits, orbits of S-representations.

upper bound Theorem (2015-Olmos-Riaño)

Let M^n a homogeneous Euclidean submanifold and let r the number of no-trivial irreducible factor subspaces of the normal space in the normal holonomy theorem, the $r \leq \frac{n}{2}$.

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Rank rigidity theorem for homogeneous euclidian submanifolds (1994-Olmos)

An irreducible full homogeneous submanifold of euclidean space of rank at least 2, which is not a curve, is an orbit of an S-representation, moreover a homogeneous euclidean submanifold with rank at least 1 must always be contained in a sphere.

He formulated the following conjecture, like a possible genelalization of the last theorem.

conjecture

An irreducible full homogeneous submanifold of the sphere, different from a curve, such that the normal holonomy group does not act transitively on the unit sphere of the normal space, must be an orbit of an S-representation.

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Motivation

The conjecture is true when $n = 2$ (the conjecture is empty in this case).

the conjecture is true when $n = 3$ (2015 Olmos-Riaño)

The conjecture is actually equivalent to the following two conjectures taken together.

a) Let M be a homogeneous irreducible and full submanifolds of the sphere, different from a curve, which is not an orbit of an S-representation. Then the normal holonomy group acts irreducibly.

b) Let M be a homogeneous and full submanifolds of the sphere such that the normal holonomy acts irreducibly and is non-transitive. The M is an orbit of an S-representation.

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Definition

The first normal space ν_p^1 of M at p is the linear span of $\{\alpha(X,Y) \mid X,Y \in T_pM\} \subset \nu_pM$, that is, the orthogonal complement in ν_pM of the $\xi \in \nu_pM$ for which $A_\xi = 0$. This is because the equation $\langle \alpha(x,y), \xi \rangle = \langle A_\xi X, Y \rangle$.

Proposition 1. 2005-Olmos

Let M^n, $n \geq 2$, be a homogeneous irreducible full submanifold of the euclidean space such that the normal holonomy group in each irreducible factor (of the normal holonomy theorem) acts non-transitively on the unity sphere. Then the first normal space of M coincides with the normal space.

Remarks:

- i. In b) the $codim(M) \leq \frac{n(n+1)}{2}$ because $A : \xi \to A_\xi$ is an injective map of the normal space in the symmetric matrices $n \times n$; if the codimension is maximal the A is bijective.
- ii. In b) the normal space like sphere submanifold is $\nu_p(\bar{M}) = \nu_p(M) \cap \{p\}^\perp$

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The first normal space ν_p^1 of M at p is the linear span of $\{\alpha(X,Y) \mid X, Y \in T_pM\} \subset \nu_pM$, that is, the orthogonal complement in ν_pM of the $\xi \in \nu_pM$ for which $A_\xi = 0$. This is because the equation $\langle \alpha(x,y), \xi \rangle = \langle A_\xi X, Y \rangle$.

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Let M^n, $n \geq 2$, be a homogeneous irreducible full submanifold of the euclidean space such that the normal holonomy group in each irreducible factor (of the normal holonomy theorem) acts non-transitively on the unity sphere. Then the first normal space of M coincides with the normal space.

Remarks:

- i. In b) the $codim(M) \leq \frac{n(n+1)}{2}$ because $A : \xi \to A_\xi$ is an injective map of the normal space in the symmetric matrices $n \times n$; if the codimension is maximal the A is bijective.
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remember the part b) Let M be a homogeneous and full submanifolds of the sphere such that the normal holonomy acts irreducibly and is non-transitive. The M is an orbit of an S-representation.

If we include the hypothesis that M is an irreducible submanifold then we have the following results:

In maximal dimension (2015-Olmos-Riaño)

Let M^n, $n \geq 2$, homogeneous irreducible full submanifold of the euclidean space and the sphere $S^{\frac{n(n+1)}{2}+n-1}$, such that the normal holonomy group acts non-transitively and irreducible in $\nu_p(\bar{M})$ then M is an orbit of an S-representation.

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Definition

Let M an Homogeneous submanifold of $\mathbb{S}^N$. Let $\nu_p M$ the normal at p we call $\mathfrak{a}$ subspace of $\nu_p M$ maximal abelian if $\mathfrak{a}$ is a maximal subspace of $\nu_p M$ such that $[A_\xi, A_\eta] = 0, \forall \xi, \eta \in \mathfrak{a}$. Then the normal holonomy of M has sections of the **compact type** if $\exp(iA_{\mathfrak{a}})$ ($\exp$ the matrix exponential) is compact for all maximal abelian.

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Let M^n, $n \geq 2$, be a homogeneous submanifold of the $\mathbb{S}^N$ such that the normal holonomy acts irreducibly and non-transitively. Then M is an orbit of an irreducible S-representation if and only if the normal holonomy of M has sections of compact type.

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