Leaf Spaces of Singular Riemannian Foliations and Applications to Spectral Geometry

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The Plan

- Leaf Spaces of Singular Riemannian Foliations (SRF)
- Isometries of SRF Leaf Spaces and Spectral Properties
- Results and Implications

Definition

A *singular Riemannian foliation* on a manifold M is a partition $\mathcal{F}$ of M by connected immersed submanifolds (known as the leaves) that satisfy the following two conditions:

1. The module $\Xi_{\mathcal{F}}$ of smooth vector fields that are tangent to the leaves is transitive on each leaf in the sense that there exist a collection of smooth vector fields $\{X_i\}$ on M such that for each $x \in M$ the tangent space to the leaf L_x through x is spanned by the vectors X_i. Note that the dimension of the leaves may vary over the manifold.

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2. There exists a Riemannian metric g on M that is adapted to $\mathcal{F}$ in the sense that every geodesic that is perpendicular at one point to a leaf remains perpendicular to every leaf that it meets. In other words, the normal distribution to the leaves is totally geodesic.

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Examples of SRF with closed leaves:

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3. Any isometric action on M by a closed group G with the orbits as the (closed) leaves. These are said to be *homogenous*.
4. If one has a proper Lie groupoid G, one can define a transversally invariant Riemannian metric on the space of objects G_0 of G. If the orbits are closed we then have an SRF with closed leaves. (Pflaum, et al.)

Definition

A smooth function f on a SRF $(M, \mathcal{F})$ is said to be *basic* if f is constant on the leaves.

Definition

A smooth structure on $M/\mathcal{F}$ is the algebra $C^\infty(M/\mathcal{F})$ consisting of functions $f : M/\mathcal{F} \to \mathbb{R}$ whose pullback via $\pi : M \to M/\mathcal{F}$ is a smooth basic function on M in the above sense.

Definition

A map $\varphi : M_1/\mathcal{F}_1 \to M_2/\mathcal{F}_2$ is called *smooth* if the pullback of every smooth function $f \in C^\infty(M_2/\mathcal{F}_2)$ by φ is a smooth function in $C^\infty(M_1/\mathcal{F}_1)$. If, in addition, φ is a metric space isometry between the leaf spaces above and has a smooth inverse, then it is a *smooth SRF leaf space isometry*.

In the case of manifolds this definition reduces to the usual notion of smooth isometry between manifolds, and similarly for orbifolds.

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- If $(M, \mathcal{F})$ is a homogenous SRF where $\mathcal{F}$ partitions the space into orbits of G, then the basic spectrum is the G-invariant spectrum.
- If $(M, \mathcal{F})$ is a regular Riemannian foliation, then the basic spectrum is the basic spectrum of a regular Riemannian foliation*.

Theorem [Adelstein-S.] If $\varphi : M_1/\mathcal{F}_1 \to M_2/\mathcal{F}_2$ is a smooth SRF leaf space isometry, then the $\mathcal{F}_1$-basic spectrum on M_1 is equivalent to $\mathcal{F}_2$-basic spectrum on M_2.

Immediate consequence: The smooth SRF leaf space isometry is the correct equivalence relation for spectral problems.

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So let's look at the equivalence classes.

Let Q be the leaf space of a singular Riemannian foliation with closed leaves. We define the isometry class of Q to be the following:

$$[Q] = \{Q' = M/\mathcal{F} \,|\, (M, \mathcal{F}) \text{ is an SRF with closed leaves and } \exists \varphi : Q \to Q', \text{ a smooth SRF leaf space isometry}\}.$$

If $(M, \mathcal{F})$ is a singular Riemannian foliation whose leaf space $M/\mathcal{F}$ belongs to $[Q]$, then we shall call $M/\mathcal{F}$ a representation of $[Q]$.

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- If an isometry class contains only trivial (singular) Riemannian foliations, we will call it a *trivial class*.

Note: Not every ambient manifold M admits a non-trivial SRF. For example, no compact M with negative curvature admits a singular Riemannian foliation that is non-trivial, vias the work of A. Lytchak. Further, no complete, simply connected manifold without conjugate points can admit a singular Riemannian foliation, again via the work of A. Lytchak. This includes closed domains with non-convex boundary, for example.

For compact leaf spaces, we can choose an ambient space M that is compact.

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Definition

If Σ is a stratum, then the quotient codimension of Σ is defined to be:

$$qcodim(\Sigma) = dim\big(\pi(M_{reg})\big) - dim\big(\pi(\Sigma)\big).$$

Lemma

If Q and Q' belong to the same isometry class, then $dim(Q) = dim(Q')$ and the quotient codimensions of the strata of Q are equal to the quotient codimensions of the corresponding strata in Q'.

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Definition

The boundary of $Q = M/\mathcal{F}$ is defined to be the closure of set of strata that have quotient codimenion equal to one. This corresponds to the notion of boundary for Aleksandrov spaces. Note: the boundary need not be smooth. Conversely, $Q = M/\mathcal{F}$ has no boundary if every singular stratum Σ has $qcodim(\Sigma) > 1$.

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Corollary

The property of having boundary in the above sense is preserved under smooth SRF isometries.

When does an SRF leaf space class contain an orbifold?

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Definition

$(M, \mathcal{F})$ is said to be *infinitesimally polar* at x if the induced foliation $(T_xM, T_x\mathcal{F})$ is polar. If $\mathcal{F}$ is infinitesimally polar at all $x \in M$ then $\mathcal{F}$ is inifinitesimally polar. Polar SRFs are always infinitesimally polar.

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Theorem (Lytchak, Thorbergsson)

Q is a manifold or an orbifold, if and only if every SRF in the class of $[Q]$ is infinitesimally polar.

Implications for Spectral Geometry

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As we know from Ian Adelstein's talk earlier this week, we have a pair of isometry classes which demonstrate that an orbifold class may be isospectral to an orbit space class, and also that constant sectional curvature is not audible.

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For example, cardinality of the isotropy subgroups is not preserved throughout the equivalence class.

Results from the literature: the work of M. Alexandrino, A. Lytchak, and M. Radeschi, can be combined into the following:

Theorem (Alexandrino, Lytchak, Radeschi)

If Q satisfies any of the conditions below, then every metric space isometry $\varphi : M/\mathcal{F} \to Q$ is a smooth SRF leaf space isometry:

1. *Q is a manifold (Classical result due to Myers-Steenrod)*
2. *Q is an orbifold,*
3. *Q is has dim ≤ 3*
4. *Q has no boundary.*

The question of whether or not a metric space isometry is smooth is a significant open question.

Corollary

In the previous cases, we have that the metric space structure of Q determines the basic spectrum.

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Another Simple Remark : As a consequence of the previous, we note that if two leaf spaces differ by some property but have the same metric space structure, then that property will not be audible for the classes listed in the last theorem.

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Another Simple Remark : As a consequence of the previous, we note that if two leaf spaces differ by some property but have the same metric space structure, then that property will not be audible for the classes listed in the last theorem.

Let's look at an application of this remark...

Theorem (Adelstein, S.)

If G_1 acts on M by isometries, and the action is orbit equivalent to an isometric action by G_2 on M, then the orbit equivalence is a smooth SRF leaf space isometry, and hence the quotients M/G_1 and M/G_2 have equivalent invariant Laplace spectra. In particular, the orbit space of an isometric group action on a manifold is always isospectral to the orbit space of the effectivization of the group action on the original manifold.

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Corollary

The property of effectiveness of an action on an orbit space is inaudible to the spectrum. Similarly, an ineffective orbifold will be isometric, and hence isospectral, to its effectivization.

Remark: There are also many examples in the literature of polar actions with infinite principal isotropy subgroups, which produce singular Riemannian foliations that are polar, and hence, infinitesimally polar, thus the quotients of these actions are orbifolds. We deduce that the finiteness of isotropy is also inaudible to the spectrum.

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Application: If Q is a bad orbifold, one may analyze its spectrum via any representative in its SRF isometry class. The lack of the existence of a smooth cover is not an obstacle to the analysis of its spectrum.

Another nice result from the literature regarding smooth leaf space isometries:

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Theorem (Alexandrino, Radeschi)

Let M_1 and M_2 be complete Riemannian manifolds and suppose $(M_1, \mathcal{F}_1)$, $(M_2, \mathcal{F}_2)$ are singular Riemannian foliations with closed leaves. Assume that there exists a metric space isometry $\varphi : M_1/\mathcal{F}_1 \to M_2/\mathcal{F}_2$ that preserves the codimension of the leaves. Then φ is a smooth SRF leaf space isometry.

Orbit spaces and Reductions of Actions

An action by a group G_1 on a manifold M_1 can often be exchanged for a typically simpler group action by G_2 on a possibly different manifold M_2 via a reduction of the action.

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There are several different reductions of this type, including the principal isotropy reduction, the minimal reduction, the effectivization of an action, and so forth, that induce metric space isometries on the leaf space. Many of these reductions are, in fact, smooth SRF leaf space isometries either because they preserve leaf codimension or because they are covered by the four cases when such isometries are known to be smooth.

Leaf spaces that arise from regular Riemannian foliations:

Theorem

$[Q]$ is an orbit space class if and only if $[Q]$ contains a representation as the leaf-closure space of a regular Riemannian foliation.

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$[Q]$ *is an orbit space class if and only if* $[Q]$ *contains a representation as the leaf-closure space of a regular Riemannian foliation.*

It is essentially this fact that has allowed the derivation of the heat trace formula for orbit spaces (K. Richardson) and the derivation of wave trace formula for the same (Sandoval).

Leaf spaces with boundary

Theorem

If Q has smooth boundary, then its eigenfunctions satisfy the Neumann boundary condition (vanishing normal derivative) and thus the spectrum of Q is the Neumann spectrum. If the boundary is not smooth, then the eigenfunctions satisfy the Neumann boundary condition on an open dense set of the boundary. On the complement of this set, the boundary condition is more complicated and depends on the geometry of the singular strata on the boundary set.

Nice boundary geometry:

Observation: If Q has boundary and has non-trivial representations, then there are no geodesic of mixed type.

Conjecture: There is a notion of a Laplacian and a spectrum for Aleksandrov spaces, via the work of K. Kuwae, Y. Machigashira, and T. Shioya. When Q is an Aleksandrov space, we conjecture that the Aleksandrov spectrum is the basic spectrum, or closely related to it, at least in the cases covered by the Theorem due to Alexandrino, Lytchak, and Radeschi.

Thank you!

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