# On the cohomology of filiform Lie algebras over $\mathbb{Z}_2$

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**Theorem (M. Vergne, 1970)**

*Every filiform nilpotent Lie algebra over* $\mathbb{R}$ *has a basis* $e_1, \ldots, e_n$ *such that*

- $[e_1, e_i] = e_{i+1}$, *for all* $2 \leq i \leq n-1$,
- $[e_i, e_j] \in \mathfrak{g}^{i+j}$, *for all* $i, j$ *with* $i+j \neq n+1$,
- $\exists a \in \mathbb{R}$ *such that* $[e_i, e_{n-i+1}] = (-1)^i a e_n$ *for* $2 \leq i \leq n-1$. *If* $n$ *is odd,* $a = 0$,

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### Example

$[e_1, e_i] = e_{i+1}$ for $2 \leq i \leq 6$, and $[e_3, e_4] = e_6, [e_2, e_5] = e_6, [e_3, e_5] = e_7$.

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$[e_1, e_i] = e_{i+1}$ for $2 \leq i \leq 6$, and $[e_3, e_4] = e_6, [e_2, e_5] = e_6, [e_3, e_5] = e_7$. The derived algebra of $\mathfrak{g}' = \mathrm{Span}(e_3, \ldots, e_7)$ and $\dim([\mathfrak{g}', \mathfrak{g}']) = 2$. If $\mathfrak{g}$ was following Vergne's theorem, we should have had that $\dim([\mathfrak{g}', \mathfrak{g}']) \leq 1$.

## Definition

Let $\mathfrak{g}$ be an $n$-dimensional nilpotent Lie algebra. If there is an element $x \in \mathfrak{g}$ such that $\mathrm{ad}^{n-2}(x) \neq 0$, then we say that $\mathfrak{g}$ is *filiform*.

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### Example

Consider the 8-dimensional Lie algebra $\mathfrak{g} = \mathrm{Span}(e_1, \ldots, e_8)$ over $\mathbb{Z}_2$ with nontrivial bracket relations

$$
\begin{aligned}
&[e_1, e_2] = e_3 \\
&[e_1, e_3] = e_4, \quad [e_2, e_3] = e_4 \\
&[e_1, e_4] = e_5, \quad [e_2, e_4] = e_5 \\
&[e_1, e_5] = e_6, \quad [e_3, e_4] = e_6 \\
&[e_1, e_6] = e_7, \quad [e_2, e_6] = e_7, \quad [e_3, e_5] = e_7 \\
&[e_2, e_7] = e_8, \quad [e_3, e_6] = e_8, \quad [e_4, e_5] = e_8.
\end{aligned}
$$

### Definition

Let $\mathfrak{g}$ be an $n$-dimensional filiform Lie algebra over a field $\mathbb{F}$ of characteristic two. We say that $\mathfrak{g}$ is of *Vergne type* if there is a basis $e_1, \ldots, e_n$ for $\mathfrak{g}$ such that $[e_1, e_i] = e_{i+1}$ for all $2 \leq i \leq n-1$ and $[e_i, e_j] = c_{i,j}e_{i+j}$ for some $c_{i,j} \in \mathbb{F}$ for all $i, j \geq 2$ with $i + j \leq n$.

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$$\begin{aligned} \mathfrak{m}_0(n) &= \mathrm{Span}(e_1, \ldots, e_n), \quad [e_1, e_i] = e_{i+1}, \ 1 < i < n, \\ \mathfrak{m}_2(n) &= \mathrm{Span}(e_1, \ldots, e_n), \quad [e_1, e_i] = e_{i+1}, \ 1 < i < n, \\ & \qquad\qquad\qquad\qquad\quad\ [e_2, e_j] = e_{j+2}, \ 2 < j < n-1. \end{aligned}$$

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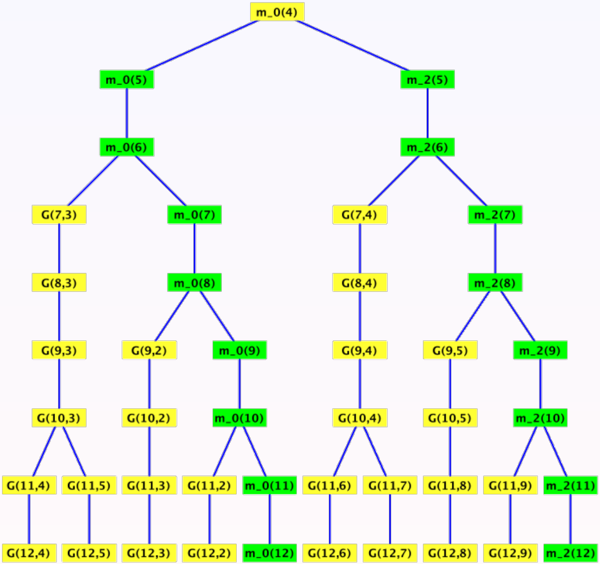
\mathfrak{m}_0 &= \mathrm{Span}(e_1, e_2, \ldots), \quad && [e_1, e_i] = e_{i+1}, \ i > 1, \\
\mathfrak{m}_2 &= \mathrm{Span}(e_1, e_2, \ldots), \quad && [e_1, e_i] = e_{i+1}, \ i > 1, \\
& && [e_2, e_j] = e_{j+2}, \ j > 2.
\end{aligned}
$$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m_0(12)
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m_2(12)

**Theorem (Nikolayevsky, T, 2015)**

*The first three Betti numbers of the Lie algebra $\mathfrak{m}_0(n)$ over $\mathbb{Z}_2$ are given by*

1. $b_1(\mathfrak{m}_0(n)) = 2$,
2. $b_2(\mathfrak{m}_0(n)) = \lfloor \frac{1}{2}(n+1) \rfloor$, *where* $\lfloor . \rfloor$ *denotes the integer part,*
3. $b_3(\mathfrak{m}_0(n)) = \frac{1}{3}(2^p - 1)(2^{p-1} - 1) + \frac{1}{2}m(m-1) + \lfloor \frac{1}{2}(n-1) \rfloor$, *where* $n = 2^p + m$ *and* $0 < m \leq 2^p$.

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**Theorem (Nikolayevsky, T, 2015)**

*The cohomology classes of the cocycles*

$$e^1,\ e^2,\ \sum_{l=0}^{\infty} D^l(e^{i_1} \wedge e^{i_2} \wedge \cdots \wedge e^{i_q}) \wedge e^{q+l+1},$$

*where* $q \geq 1,\ 2 \leq i_1 < i_2 < \ldots < i_q$, *form a basis for* $H^*(\mathfrak{m}_0)$ *over the field* $\mathbb{Z}_2$.

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*where* $q \geq 1,\ 2 \leq i_1 < i_2 < \ldots < i_q$, *form a basis for* $H^*(\mathfrak{m}_0)$ *over the field* $\mathbb{Z}_2$.

$D_1(e^1) = D_1(e^2) = 0,\ \ D_1(e^i) = e^{i-1},\ i \geq 3,$
$D(\xi \wedge \zeta) = D(\xi) \wedge \zeta + \xi \wedge D(\zeta).$

**Theorem (T, 2015)**

*For $n \in \mathbb{N}$, the Lie algebras $\mathfrak{m}_0(n)$ and $\mathfrak{m}_2(n)$ over a field of characteristic two have the same Betti numbers.*

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**Theorem (T, 2015)**

*For any nilpotent Lie algebra of Vergne type of dimension at least 5 over a field of characteristic two, there exists a non-isomorphic Lie algebra of Vergne type having the same Betti numbers.*

Thanks for your attention!

Any Questions?