Submanifolds and Holonomy in Complex Space Forms

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VI Workshop on Differential Geometry
La Falda, August 2nd., 2016

Complex and Real Space Forms

- A **real space form** is a complete manifold with constant sectional curvature κ. Standard (simply connected) models of space forms are:

$$\mathbb{R}^n \quad (\kappa = 0), \quad S^n \quad (\kappa > 0), \quad H^n \quad (\kappa < 0).$$

- A **complex space form** is a complete Hermitian manifold with constant holomorphic sectional curvature $K(X, JX) = c$, $\forall X \in \mathfrak{X}^1(M)$.
  The standard simply connected models of complex space forms are
  1. the complex Euclidean space $\mathbb{C}^n$ $(c = 0)$;
  2. the complex projective space $\mathbb{C}P^n$, with the Fubini-Study metric $(c = 4)$;
  3. the complex hyperbolic space $\mathbb{C}H^n$ $(c = -4)$.

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Submanifolds

- Let M^n be a Riemannian submanifold of a real or complex space form Q. Let $\overline{\nabla}$ denote the L-C. connection of Q.
- Then $T_pQ = T_pM \oplus \nu_pM, \quad p \in M.$

  and $\nu M = \bigcup_{p \in M} \nu_p M$ is a fiber bundel with a natural connection $\nabla^\perp$ called **normal connection** defined by

  $$\nabla^\perp_X \xi = (\overline{\nabla}_X \xi)^\perp$$

- The (restricted) holonomy group associated to $\nabla^\perp$ is called (restricted) **normal holonomy group** and denoted by $\Phi_0^\perp$ and $\Phi^\perp$ resp.
- The normal bundle decomposes as

  $$\nu(M) = \nu_0 M \oplus \nu_s M$$

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Some questions

1. How does the normal holonomy group act? Can we classify them?
2. Which properties of the submanifold can we obtain from the normal holonomy group?
3. Can we expect to obtain a Berger-type theorem? (i.e., if a submanifold has a non transitive normal holonomy group, does it belong to some particular family of submanifolds?)

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The Normal Holonomy Theorem

Normal Holonomy Theorem [OI90]

Let M be a connected Rimannian submanifold of a real space form Q. Then $\Phi_0^\perp$ is compact and its representation on each normal space is the direct sum of a trivial representation and the isotropy representation of a semisimple symmetric space (s-representation).

This means that

$$\nu_p M = \nu_0 M \oplus \nu_s M.$$

- $\Phi_0^\perp$ acts trivialy on $\nu_0 M$;
- there is a semisimple symmetric space G/K such that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, $\nu_s M \equiv \mathfrak{p}$ and the action of $\Phi_0^\perp$ on $\nu_s M$ is equivalent to the action of $Ad(K)$ on $\mathfrak{p}$.

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The Normal Holonomy Theorem

The Normal Holonomy Theorem has many important applications in submanifold geometry, see [BCO16].

It tells us where to look for normal holonomy representations;

but we actually don't know yet if all s-representations can be realized as normal holonomy representations of submanifolds of real space forms (see [HO92]).

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Changing the ambient space

Is the NHT valid in complex space forms?

- The proof of the NHT is based on the existence of an algebraic curvature tensor $\mathcal{R}^\perp$ on νM with **non vanishing scalar curvature**, so one can apply Simon's theory of symmetric holonomy systems (see [Si62]).
- This is possible thanks to the simplicity of the Ricci identity for submanifolds of real space forms:

$$\langle R^\perp_{X,Y}\xi, \zeta\rangle = \langle [A_\xi, A_\zeta]X, Y\rangle$$

where $R^\perp$ is the curvature tensor of the normal connection of the submanifold.

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- If $M \subset Q$, with Q a complex space form, the Ricci identity is given by
$$\left\langle \overline{R}_{X,Y}\xi, \zeta \right\rangle = \left\langle R^{\perp}_{X,Y}\xi, \zeta \right\rangle - \left\langle [A_\xi, A_\zeta]X, Y \right\rangle .$$
with
$$\overline{R}_{X,Y} = \frac{1}{4}c(X \wedge Y + JX \wedge JY - 2\left\langle JX, Y \right\rangle J)$$
where $X \wedge Y(Z) = \langle Y, Z \rangle X - \langle X, Z \rangle Y$, J is the complex strucutre of Q.

So IN GENERAL, the Normal Holonomy Theorem does not need to hold in Complex Space Forms.

For example, the normal holonomy group of a codimension 2 totally geodesic

$$\mathbb{C}P^n \subset \mathbb{C}P^{n+2}$$

is the diagonal action of $U(1)$ on $\mathbb{C}^2$, which is not an s-representation, see [BCO16].

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Complex submanifolds

Theorem 1 (Alekseevsky, Di Scala [AD04] - Di Scala, V. [DV17])

Let M be a full connected submanifold of a complex space form. Then $\Phi_0^\perp$ acts on each normal space as the isotropy representation of a **Hermitian symmetric space** *without flat factor.*

- For submanifolds of $\mathbb{C}^n$ the result appears in [Di00].
- For $c \neq 0$, a first proof appears in [AD04] under the stronger hypothesis that $\Phi_0^\perp$ acts irreducibly or M has no relative nullity.
- As we mentioned before, the hypothesis of the submanifold being full can not be dropped.
- For complex submanifolds of $\mathbb{C}^n$ there is a de Rham-type Theorem: The normal holonomy group of a locally irreducible and full submanifold of $\mathbb{C}^n$ acts irreducibly on the normal space [Di00]. This is **false in real space forms**.

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Towards a classification problem

From the NHT for complex submanifolds we get that if the normal holonomy is transitive in the unit sphere of the normal space it must be the whole unitary group $U(k)$, with $k = dim(\nu M)$.

For complete complex submanifolds of $\mathbb{C}^n$ or $\mathbb{C}P^n$, the complete classification problem was solved by Console, Di Scala and Olmos in 2011:

Theorem 2 (Console, Di Scala, Olmos [CDO11])

- *If M is a complete, full and irreducible complex submanifold of $\mathbb{C}^n$, then the normal holonomy is transitive and $\Phi_0^\perp = U(\nu_p M)$.*
- *If M is a complete and full complex submanifolds of $\mathbb{C}P^n$, the normal holonomy of M is non transitive if and only if M is the complex orbit in $\mathbb{C}P^n$ of the isotropy representation of an irreducible Hermitian symmetric space of rank greater than or equal to 3.*

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The general case

Theorem 3 (Di Scala - V. [DV2])

The normal holonomy of an irreducible complex submanifold of $\mathbb{C}^n$ or $\mathbb{CP}^n$ is non transitive if and only if the submanifold is an open subset of one of the (cones over a) j^{th}- Mok's characteristic variety.

- Mok's characteristic varieties are important algebraic varieties studied by N. Mok in his work on rigidity theorems on Hermitian locally symmetric spaces [Mo89].
- The complex orbits that appears in the result of Console, Di Scala and Olmos are actually first Mok's characteristic varieties.
- They are explicitely constructed using **Jordan Triple Systems**, which allows us to describe some important geometric aspects of the submanifolds using very simple algebraic tools .

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Using the classification given by Console and Di Scala in [CD09] of normal holonomies of complex submanifolds with parallel second fundamental form, we can compute ALL the normal holonomies of these varieties, completely solving the classification problem for complex submanifolds of $\mathbb{C}^n$ or $\mathbb{C}P^n$.

They are the isotropy representations of the following compact irreducible Hermitian symmetric spaces G/K:

G/K
$SU(p+q)/S(U(p)\times U(q))$, $p,q>1$
$SO(2n)/U(n)$, $n>3$
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Some other families of Submanifolds

There are other families of submanifolds of complex space forms for which the NHT holds:

Isotropic, coisotropic and Lagrangian submanifolds

A submanifold M of a complex space form is called **isotropic** or **totally real** if $J(T_pM) \subset \nu_pM \ , \ \ \forall\, p \in M$.
It is called **coisotropic** if $J(\nu_pM) \subset T_pM \ , \ \ \forall\, p \in M$.
If $J(\nu_pM) = T_pM$, M is called a **Lagrangian** submanifold.

Theorem 4 (Di Scala, V. [DV17])

Let M be a coisotropic or Lagrangian submanifold of a complex space form. Then the restricted normal holonomy group of M acts on the normal space as the holonomy representation of a Riemannian symmetric space i.e. a flat factor plus a s-representation.

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Lagrangian Submanifolds

Let M be a Lagrangian submanifold.

1. $J : T_pM \to \nu_pM$ is an isomorphism which commutes with parallel transport. Then it induces a natural isomorphism between the normal and the Riemannian holonomy groups.
2. So the Riemannian holonomy group also acts on the tangent space as an s-representation.
3. As a consequence, a Ricci flat Lagrangian submanifold of a complex space form has non-exceptional Riemannian holonomy, i.e., it is either flat or the restricted holonomy group of its Levi-Civita connection is $SO(TM)$.

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Lagrangian Submanifolds

Let M be a Lagrangian submanifold.

1. $J : T_pM \to \nu_pM$ is an isomorphism which commutes with parallel transport. Then it induces a natural isomorphism between the normal and the Riemannian holonomy groups.
2. So the Riemannian holonomy group also acts on the tangent space as an s-representation.
3. As a consequence, a Ricci flat Lagrangian submanifold of a complex space form has non-exceptional Riemannian holonomy, i.e., it is either flat or the restricted holonomy group of its Levi-Civita connection is $SO(TM)$.

Totally real submanifolds

Extrinsically symmetric submanifolds of complex space forms are complex submanifolds, totally real submanifolds contained in a totally real totally geodesic submanifold, or lagrangian submanifolds of a totally geodesic complex submanifold (cf. [Na83]).

Theorem 5 (Di Scala, V. [DV17])

Let M be a totally real submanifold of a complex space form. Then:

- *If $J(TM)$ is a $\nabla^\perp$-parallel sub-bundle of νM, then the restricted normal holonomy group acts on each normal space as the holonomy representation of a symmetric space (i.e. a flat factor plus an s-representation).*
- *If M is contained in a totally real totally geodesic submanifold N of Q, $\nu M = \nu_N M \oplus \nu N_{|M}$. $\Phi_0^\perp$ is compact and it acts as the isotropy representation of a symmetric space in $\nu_N M$. Moreover, $\nu N_{|M} = W \oplus W^\perp$, with $J(TM) \subset W$, $\Phi_0^\perp$ acts trivially on W and transitively on $W^\perp$.*

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Final comments and open problems

1. The classification problem of the normal holonomy of complex submanifolds of $\mathbb{C}^n$ and $\mathbb{C}P^n$ is completely solved (in contrast with what happens with real space forms).
2. The problem of the classification of complex submanifolds of $\mathbb{C}H^n$ with non transitive normal holonomy is still open (and so far very difficult).
3. We don't know either if the normal holonomy of full totally real submanifolds is an s-representation (actually, there are not even many examples of full totally real submanifolds).

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