A half-space theorem for graphs of constant mean curvature $0 < H < \frac{1}{2}$ in $\mathbb{H}^2 \times \mathbb{R}$

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Introduction

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Two properly immersed minimal surfaces in $\mathbb{R}^3$ that do not intersect must be parallel planes.

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Half spaces theorems with respect to vertical minimal planes in Nil_3 and Sol_3 (2011-Daniel, William, Meeks and Rosenberg)

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In Euclidian spaces on dimension higher than 4, there is no half space theorem, since there exist rotational proper minimal hypersurfaces contained in a slab.

Graphs of a function u defined in a domain D whose boundary ∂D is composed of complet arcs $\{A_i\}$ and $\{B_j\}$, such that the curvatures of the arcs with respect to the domain are $\kappa(A_i) = 2H$ and $\kappa(B_j) = -2H$. These graphs will have constant mean curvature and u will assume the value $+\infty$ on each A_i and $-\infty$ on each B_j.

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D-Scherk type domains
u-Scherk type solution

Scherk type graphs

· Minimal

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· CMC $0 < H < \frac{1}{2}$
$H^2 \times \mathbb{R}$; H^2-Hyperbolic plane
Existence: Folha, Melo
Half-space theorem: ——, Mazet (2015)

Theorem (—,Mazet, 2015)

Let $D \subset \mathbb{H}^2$ be a Scherk type domain and u be a Scherk type solution over D. Denote by $\Sigma = Graph(u)$. If S is a properly immersed CMC $0 < H < \frac{1}{2}$ surface contained in $D \times \mathbb{R}$ and above Σ, then S is a vertical translate of Σ.

CMC Scherk type graphs

Let
H^2 - the hyperbolic plane
$H^2 \times \mathbb{R}$ - given by the product metric
D - simply connected domain in H^2
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The graph Σ has mean curvature H if u satisfies the equation

$$\mathcal{L}u := div \frac{\nabla u}{W} - 2H = 0$$

CMC Scherk type graphs

Let D be a Scherk type domain, under certain hypothesis, Folha and Melo prove the existence of a solution u for the following Dirichlet problem

$$\begin{aligned} \mathcal{L}u &= 0, \quad in \quad D \\ u &= +\infty, \quad on \quad A_i \\ u &= -\infty, \quad on \quad B_i. \end{aligned} \tag{1}$$

The details and proof of the theorem can be found in

A. Folha and S. Melo. The Dirichlet problem for constant mean curvature graphs in $H \times \mathbb{R}$ over unbounded domains. Pacific Journal of Mathematics, v. 251, n. 1, p. 37-65, 2011.

The Main Result

Theorem (—,Mazet, 2015)

Let $D \subset \mathbb{H}^2$ be a Scherk type domain and u be a Scherk type solution over D. Denote by $\Sigma = Graph(u)$. If S is a properly immersed CMC $0 < H < \frac{1}{2}$ surface contained in $D \times \mathbb{R}$ and above Σ, then S is a vertical translate of Σ.

Proof of Theorem

We know that S is a properly immersed CMC surface contained in $D \times \mathbb{R}$ above Σ. Then, let $y \in D$, $B_y \subset D$ and $\epsilon\prime$ as above. We have three cases to analyze

1) Suppose that there exists $p \in D$ such that $g(p) = u(p)$. In this case, by the maximum principle, we conclude that $u = g$ and $S = \Sigma$.

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2) Suppose that $g > u$ and $inf(g - u) = 0$. In this case, by the Proposition 1 there exists β solution of (6)-(7) defined over $B \setminus B_y$ such that $u \leq \beta \leq g$. Moreover, Proposition 2 assures that $\beta = u + \epsilon l$. This yields a contradiction, since we assume that $inf(g - u) = 0$.

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2) Suppose that $g > u$ and $inf(g - u) = 0$. In this case, by the Proposition 1 there exists β solution of (6)-(7) defined over $B \setminus B_y$ such that $u \leq \beta \leq g$. Moreover, Proposition 2 assures that $\beta = u + \epsilon I$. This yields a contradiction, since we assume that $inf(g - u) = 0$.
3) Finally, suppose that $g > u$ and $inf(g - u) = \alpha > 0$. Then, pushing up Σ by a vertical translations, that is, by considering $u + \alpha$ instead of u, we have now that $g \geq u + \alpha$ and $inf(g - \alpha - u) = 0$, this case reduces to cases (1) and (2) and we conclude that $\beta = u + \alpha$, where α is a constant.

Lemma

There exists a constant $\epsilon > 0$ *such that for all* $t \in [0, \epsilon)$ *there exists* $v \in C^2(B\prime_y \backslash B_y) \cap C^0(\partial(B\prime_y \backslash B_y))$ *such that v solves 3 and* $v = u$ *in* $\partial B\prime_y$ *and* $v = u + t$ *over* ∂B_y.

Proposition

There is a solution $\beta \in v \in C^2(D \backslash \overline{B_y}) \cap C^0(D \backslash B_y)$ *for the Dirichlet problem (6)-(7) such that* $max(u, v) \leq \beta \leq min(u + \epsilon I, g)$.

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Claim 3. *The function* β *is continuos up to the boundary* ∂B_y *and takes the value* $u + \epsilon l$ *on it.*

Proposition

Let $\beta \in v \in C^1(D \backslash \overline{B_y}))$ *solution of the Dirichlet problem (3) such that* $u \leq \beta \leq u + \epsilon l$. *Then,* $\beta = u + \epsilon l$

Bibliography

- L. Mazet, G.A. Wanderley. *A Half-Space Theorem for graphs of constant mean curvature* $0 < H < \frac{1}{2}$ *in* $\mathbb{H}^2 \times \mathbb{R}$. Illinois J. of Math. no1, 43-53 (2015).
- A. Folha and S. Melo. *The Dirichlet problem for constant mean curvature graphs in* $H \times \mathbb{R}$ *over unbounded domains.* Pacific Journal of Mathematics, v. 251, n. 1, p. 37-65, 2011.
- D. Hoffman and William H. Meeks, III. *The strong halfspace theorem for minimal surfaces.* Inventiones Mathematicae, v. 101, n. 1, p. 373-377, 1990.
- A. Menezes. *A half-space theorem for ideal Scherk graphs in* $M \times \mathbb{R}$. Michigan Math. J. 63, 675-685, 2014.

Thank you