Negative Ricci curvature on Lie groups with a compact Levi factor

Cynthia Will,
FaMAF and CIEM, Córdoba, Argentina

VI Workshop on Differential Geometry,
Córdoba
August, 2016

When a differentiable manifold admits a Riemannian metric with a particular sign of the curvature?

Homogeneous manifolds: (left invariant metrics)

Sectional curvature $\surd$

Ricci Curvature: positive, zero, $\surd$ negative ???.

Known Results:

- Bochner: The isometry group of a compact Riemannian manifold with negative Ricci curvature is discrete. $\rightsquigarrow$ A homogeneous space with negative Ricci curvature is non-compact.
- Any unimodular Lie group which admits a left-invariant metric with negative Ricci curvature is (non-compact) semisimple [DLM]. Moreover, such semisimple Lie group can not have compact factors [JP].
- Any non-flat Einstein solvmanifold.
- There is no any left-invariant metric on $\mathrm{SL}(2,\mathbb{R})$ with negative Ricci curvature [Mil], though there exist negative Ricci left-invariant metrics on $\mathrm{SL}(n,\mathbb{R})$ for every $n \geq 3$ [DL] and on most non-compact simple groups [DLM].

- $\mathfrak{g}$ = solvable Lie algebra, with nilradical $\mathfrak{n}$, $\mathfrak{z}$ is the center of $\mathfrak{n}$. If $\mathfrak{g}$ admits an inner product with $\mathrm{Rc} < 0$, then there exists an element $Y \in \mathfrak{g}$ such that all the eigenvalues of $\operatorname{ad} Y|_{\mathfrak{z}}$ have positive real part. On the other hand, if there exists $Y \in \mathfrak{g}$ such that all the eigenvalues of $\operatorname{ad} Y|_{\mathfrak{n}}$ have positive real part, then $\mathfrak{g}$ admits an inner product of negative Ricci curvature [NN]. $\rightsquigarrow$ characterized when $\mathfrak{n}$ is the Heisenberg Lie algebra or the (standard) filiform Lie algebra.
- $\mathfrak{u}(2) \ltimes_\theta \mathbb{R}^4$ admits an inner product with negative Ricci curvature [LW]. Also $\mathfrak{gl}(2) \ltimes_\theta \mathbb{R}^4$. Is it somehow general?

Theorem

Let (V, π) *be a non-trivial real representation of* $\mathfrak{su}(2)$ *extended to* $\mathfrak{u}(2)$ *by letting the center act as multiples of the identity, then the Lie algebra* $\mathfrak{u}(2) \ltimes V$ *admits an inner product with negative Ricci curvature.*

Theorem

Let (V, π) be a non-trivial real representation of $\mathfrak{su}(2)$ extended to $\mathfrak{u}(2)$ by letting the center act as multiples of the identity, then the Lie algebra $\mathfrak{u}(2) \ltimes V$ admits an inner product with negative Ricci curvature.

Theorem

Let (V_n, π_n) be the usual real representation of $\mathfrak{su}(m)$ on the space of complex homogeneous polynomials of degree n in m variables $\mathcal{P}_{m,n}(\mathbb{C})$ extended to $\mathfrak{u}(m)$ by letting the center act as multiples of the identity. Hence the Lie algebra $\mathfrak{u}(m) \ltimes V_n$ admits a inner product with negative Ricci curvatures for all $n \geq 2$.

Theorem

Let (V_n, π_n) be the usual real representation of $\mathfrak{sl}(m)$ on the space of complex homogeneous polynomials of degree n in m variables $\mathcal{P}_{m,n}(\mathbb{C})$, extended to $\mathfrak{gl}(m)$ by letting the center act as multiples of the identity. Hence the Lie algebra $\mathfrak{gl}(m) \ltimes V_n$ admits an inner product with negative Ricci curvature for any m and $n \geq 2$.

Theorem

Let $\mathfrak{h}$ be a non-compact semisimple Lie algebra acting on an abelian Lie algebra $\mathfrak{n}$. Consider the Lie algebra

$$\mathfrak{g} = (\mathbb{R}Z \oplus \mathfrak{h}) \ltimes \mathfrak{n}$$

where $\operatorname{ad} Z|_{\mathfrak{h}} = 0$ *and* $\operatorname{ad} Z|_{\mathfrak{n}} = Id$*. If there exists an inner product on $\mathfrak{h}$ such that the corresponding Ricci operator preserves a Cartan decomposition and it is negative definite then there exist an inner product on $\mathfrak{g}$ such that the corresponding Ricci operator is negative definite.*

Dotti: $\mathfrak{h} \ltimes \mathfrak{n}$ admits an inner product with non-positive Ricci operator.

Some background:

$\mathfrak{g} = (\mathbb{R}^m, [\cdot,\cdot])$ Lie algebra of dimension m, $\leftrightsquigarrow$ $[\cdot,\cdot] = \mu$,
$\mu \in \mathcal{L}_m \subset \Lambda^2(\mathbb{R}^m)^* \otimes \mathbb{R}^m$, given by

$\{\mu : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m : \mu$ bilinear, skew-symmetric and satisfies Jacobi$\}$.

$\mathrm{GL}_m(\mathbb{R})$ acts on $\mathcal{L}_m$ by change of basis $\rightsquigarrow$

$$(g \cdot \mu)(X, Y) = g\mu(g^{-1}X, g^{-1}Y),$$

$(\mathbb{R}^m, \mu) \simeq (\mathbb{R}^m, g \cdot \mu)$, $g \in \mathrm{GL}_m(\mathbb{R})$ though, may not to $(\mathbb{R}^m, \mu_o)$
for $\mu_o \in \overline{\mathrm{GL}_m(\mathbb{R}) \cdot \mu}$.

Proposition

Suppose μ, $\lambda \in \mathcal{L}_m$ and that λ is in the closure of the orbit $\mathrm{GL}_m(\mathbb{R}) \cdot \mu$. If the Lie algebra $(\mathbb{R}^m, \lambda)$ admits an inner product of negative Ricci curvature, then so does the Lie algebra $(\mathbb{R}^m, \mu)$.

fix an inner product on $\mathfrak{g} = (\mathbb{R}^m, \mu)$, then $\mathrm{GL}(\mathfrak{g}) \cdot \mu$ parameterizes the set of all inner products on $\mathfrak{g}$.

$$(\mathfrak{g}, g \cdot \mu, \langle \cdot, \cdot \rangle) \simeq (\mathfrak{g}, \mu, \langle g \cdot, g \cdot \rangle), \quad g \in \mathrm{GL}(\mathfrak{g}).$$

Rc of $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$ is given by

$$\mathrm{Rc} = M - \tfrac{1}{2}B - S(\mathrm{ad}\, H),$$

Where

$B =$ Killing form,

$H \in \mathfrak{g} : \langle H, X\rangle = \mathrm{tr}\,\mathrm{ad}\, X \qquad$ for any $X \in \mathfrak{g}$,

$S(\mathrm{ad}\, H) = \tfrac{1}{2}(\mathrm{ad}\, H + (\mathrm{ad}\, H)^t)$,

$$\begin{aligned} M(X, Y) &= -\tfrac{1}{2}\sum\langle[X, X_i], X_j\rangle\langle[Y, X_i], X_j\rangle \\ &\quad +\tfrac{1}{4}\sum\langle[X_i, X_j], X\rangle\langle[X_i, X_j], Y\rangle. \end{aligned}$$

$\mathfrak{su}(2)$,

$$H = \begin{bmatrix} \mathbf{i} & \\ & -\mathbf{i} \end{bmatrix}, \quad X = \begin{bmatrix} & 1 \\ -1 & \end{bmatrix}, \quad Y = \begin{bmatrix} & \mathbf{i} \\ \mathbf{i} & \end{bmatrix},$$

$$[H, X] = 2Y, \qquad [H, Y] = -2X, \qquad [X, Y] = 2H.$$

Irreducible Representations ($n \geq 2$):
$\mathcal{P}_{2,n}(\mathbb{C})$: the space of homogeneous polynomials in two variables of degree n.
The group acts

$$(g \cdot P)(z_1, z_2) = P(g^{-1} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}).$$

and we get the action of the algebra by derivations.

Irreducible real representations (for $n \geq 2$):

$\mathcal{P}_{2,n}(\mathbb{C})$ as real:

$$\{z_1^n, \mathbf{i}\, z_1^n, \ldots, z_1^{n-j} z_2^j, \mathbf{i}\, z_1^{n-j} z_2^j, \ldots, z_2^n, \mathbf{i}\, z_2^n\},$$

$\rightsquigarrow \dim V_n = 2 \dim \mathcal{P}_{2,n}(\mathbb{C})$.

For n even $J : \mathcal{P}_{2,n}(\mathbb{C}) \to \mathcal{P}_{2,n}(\mathbb{C})$

$$J(z_1^{n-j} z_2^j) = (-1)^j z_1^j z_2^{n-j}$$

is a real structure $\rightsquigarrow \dim V_n^o = \dim \mathcal{P}_{2,n}(\mathbb{C}) = n + 1$ odd.

Extend to $\mathfrak{u}(2) = \mathfrak{su}(2) \oplus \mathbb{R}Z$, $Z = Id$, by the identity.

Idea of the proof for $\mathfrak{u}(2)$:

First do it for the irreducible ones.

Even dimensional case:

$\mathfrak{g} = \mathfrak{u}(2)$, $n \in \mathbb{N}$, $n \geq 2$,
$\mathfrak{h}_n = \mathfrak{g} \ltimes V_n$ is a non-solvable nor semisimple Lie algebra with levi factor $\mathfrak{su}(2)$. $\dim \mathfrak{h}_n = (2n+6)$ with orthonormal basis

$$\beta = \{Z, H, X, Y, z_1^n, \mathbf{i}\, z_1^n, \ldots, z_1^{n-j} z_2^j, \mathbf{i}\, z_1^{n-j} z_2^j, \ldots, z_2^n, \mathbf{i}\, z_2^n\}.$$

recall the action:

$$\pi_n(H) = \left[\begin{array}{cc|cc|c|cc}
0 & n & & & & & \\
-n & 0 & & & & & \\
\hline
 & & 0 & n-2 & & & \\
 & & -n+2 & 0 & & & \\
\hline
 & & & & \ddots & & \\
\hline
 & & & & & 0 & -n \\
 & & & & & n & 0
\end{array}\right].$$

$$\pi_n(X) = \left[\begin{array}{cc|cc|cc|c|cc}
0 & & 1 & & & & & & \\
 & 0 & & 1 & & & & & \\
\hline
-n & & 0 & & 2 & & & & \\
 & -n & & 0 & & 2 & & & \\
\hline
 & & -n+1 & & & & & & \\
 & & & -n+1 & & & & & \\
\hline
 & & & & & & \ddots & & \\
\hline
 & & & & & & & n & \\
 & & & & & & & & n \\
 & & & & & & -1 & 0 & \\
 & & & & & & & -1 & 0
\end{array}\right], \quad
\pi_n(Y) = \left[\begin{array}{cc|cc|cc|c|cc}
0 & & & 1 & & & & & \\
 & 0 & -1 & & & & & & \\
\hline
 & n & 0 & & & 2 & & & \\
-n & & & 0 & -2 & & & & \\
\hline
 & & & n-1 & 0 & & & & \\
 & & -n+1 & & & 0 & & & \\
\hline
 & & & & & & \ddots & & \\
\hline
 & & & & & & & & n \\
 & & & & & & & -n & \\
 & & & & & & 1 & 0 & \\
 & & & & & & -1 & & 0
\end{array}\right]$$

For each $t > 0 \quad \phi_t \in \mathrm{GL}(\mathfrak{h}_n)$

$$\phi_t|_{\mathfrak{g}} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & t & \\ & & & t \end{bmatrix}, \quad \phi_t|_{V_n} = \begin{bmatrix} t & & & & & & \\ & t & & & & & \\ & & t^2 & & & & \\ & & & \ddots & & & \\ & & & & t^2 & & \\ & & & & & t & \\ & & & & & & t \end{bmatrix}.$$

$$\lambda_n = \lim_{t \to \infty} \phi_t \cdot [\cdot, \cdot].$$

solvable Lie algebra, $\mathfrak{a} = Span\{Z, H\}$, $\mathfrak{n} = Span\{X, Y, V_n\}$

$$\lambda_n(H,X) = 2Y, \qquad \lambda_n(H,Y) = -2X,$$

$$\lambda_n(Z,sp_k) = sp_k, \qquad \lambda_n(H,sp_k) = [H,sp_k], \ \ \forall\, k,$$

$$\lambda_n(X,sp_n) = [X,sp_n] = -n\, sp_{n-1}, \ \ \lambda_n(X,sp_0) = [X,sp_0] = n\, sp_1,$$

$$\lambda_n(Y,p_n) = [Y,p_n] = -n\,\mathbf{i}p_{n-1}, \ \ \lambda_n(Y,\mathbf{i}p_n) = [Y,\mathbf{i}p_n] = n\, p_{n-1},$$

$$\lambda_n(Y,p_0) = [Y,p_0] = -n\,\mathbf{i}p_1, \ \ \lambda_n(Y,\mathbf{i}p_0) = [Y,\mathbf{i}p_0] = n\, p_1.$$

That is:

$$\mathrm{ad}_{\lambda_n}(X)|_{\mathfrak{n}} = \left[\begin{array}{c|c} \begin{smallmatrix}0 & \\ & 0\end{smallmatrix} & \\ \hline & \begin{array}{cc} \begin{array}{c|c|c} & \begin{smallmatrix}0 & \\ & 0\end{smallmatrix} & \\ \hline \begin{smallmatrix}-n & \\ & -n\end{smallmatrix} & & \\ \hline & \begin{smallmatrix}0 & \\ & 0\end{smallmatrix} & \end{array} & \\ & \ddots \\ & \begin{array}{c|c} & \begin{smallmatrix}n & \\ & n\end{smallmatrix} \\ \hline \begin{smallmatrix}0 & \\ & 0\end{smallmatrix} & \end{array} \end{array} \end{array}\right],$$

$$\mathrm{ad}_{\lambda_n}(Y)|_{\mathfrak{n}} = \left[\begin{array}{c|c} \begin{smallmatrix}0 & \\ & 0\end{smallmatrix} & \\ \hline & \begin{array}{cc} \begin{array}{c|c|c} & \begin{smallmatrix} & 0 \\ 0 & \end{smallmatrix} & \\ \hline \begin{smallmatrix} & n \\ -n & \end{smallmatrix} & & \begin{smallmatrix} & 0 \\ 0 & \end{smallmatrix} \\ \hline & \begin{smallmatrix} & 0 \\ 0 & \end{smallmatrix} & \end{array} & \\ & \ddots \\ & \begin{array}{c|c} & \begin{smallmatrix} & n \\ -n & \end{smallmatrix} \\ \hline \begin{smallmatrix} & 0 \\ 0 & \end{smallmatrix} & \end{array} \end{array} \end{array}\right],$$

Note that the mean curvature vector $= \dim V_n Z$.

$$\langle \mathrm{Rc}_{\lambda_n} Z, Z\rangle = -2(n+1), \qquad \langle \mathrm{Rc}_{\lambda_n} H, H\rangle = -\operatorname{tr} S(\operatorname{ad} H|_{\mathfrak{n}})^2 = 0,$$

$$\langle \mathrm{Rc}_{\lambda_n} X, X\rangle = \langle \mathrm{Rc}\, Y, Y\rangle = -2n^2,$$

$$\langle \mathrm{Rc}_{\lambda_n} sp_k, sp_k\rangle = -n^2 - 2(n+1), \quad \text{for } k = 0, n, s = 1, \mathbf{i},$$

$$\langle \mathrm{Rc}_{\lambda_n} sp_k, sp_k\rangle = n^2 - 2(n+1), \quad \text{for } k = n-1, 1, s = 1, \mathbf{i},$$

$$\langle \mathrm{Rc}_{\lambda_n} sp_k, sp_k\rangle = -2(n+1), \quad \text{for } k \neq n, n-1, 1, 0, s = 1, \mathbf{i},$$

$$\{Z, H, X, Y, a\,p_n, b\,\mathbf{i}p_n, p_{n-1}, \ldots, \mathbf{i}p_1, a\,p_0, b\,\mathbf{i}p_0\} \leftrightsquigarrow f$$

$$\langle \mathrm{Rc}_{f\cdot\lambda_n} Z, Z\rangle = -2(n+1), \qquad \langle \mathrm{Rc}_{f\cdot\lambda_n} H, H\rangle = -n^2(\tfrac{b}{a} - \tfrac{a}{b})^2$$

$$\langle \mathrm{Rc}_{f\cdot\lambda_n} X, X\rangle = \langle \mathrm{Rc}_{f.\lambda_n} Y, Y\rangle = -n^2(a^2 + b^2),$$

$$\langle \mathrm{Rc}_{f\cdot\lambda_n} sp_k, sp_k\rangle = -2(n+1), \quad \text{for } k \neq n, n-1, 1, 0, s = 1, \mathbf{i},$$

$$\langle \mathrm{Rc}_{f\cdot\lambda_n} sp_k, sp_k\rangle = \tfrac{1}{2}(a^2 + b^2)n^2 - 2(n+1), \text{ for } k = 1, n-1, s = 1, \mathbf{i},$$

$$\langle \mathrm{Rc}_{f\cdot\lambda_n} p_k, p_k\rangle = -n^2a^2 + \tfrac{n^2}{2}\left(\left(\tfrac{b}{a}\right)^2 - \left(\tfrac{a}{b}\right)^2\right) - 2(n+1), \quad k = 0, n,$$

$$\langle \mathrm{Rc}_{f\cdot\lambda_n} \mathbf{i}p_k, \mathbf{i}p_k\rangle = -n^2b^2 + \tfrac{n^2}{2}\left(\left(\tfrac{a}{b}\right)^2 - \left(\tfrac{b}{a}\right)^2\right) - 2(n+1), \quad k = 0, n.$$

there exist $a \neq b$ such that every term is negative.
Very similar with the odd dimensional representations.

Theorem

Let (V, π) *be a non-trivial real representation of* $\mathfrak{su}(2)$ *extended to* $\mathfrak{u}(2)$ *by letting the center act as multiples of the identity, then the Lie algebra* $\mathfrak{u}(2) \ltimes V$ *admits an inner product with negative Ricci curvature.*

Idea: Decompose the representation in irreducible components and use the previous results with the same a and b for all of them.

THANK YOU!