Homogeneity for Riemannian Quotient Manifolds

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- Example: if $\Gamma\backslash\mathbb{R}^n$ is homogeneous then Γ consists of pure translations so $\Gamma\backslash\mathbb{R}^n$ is the product of a torus and an euclidean space.

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- Then the following are equivalent.
  - $\Gamma\backslash M$ is homogeneous
  - Every $\gamma \in \Gamma$ is an isometry of constant displacement
  - Every $\gamma \in \Gamma$ is an isometry of bounded displacement
  - Every $\gamma \in \Gamma$ is just a pure translation along the Euclidean factor (M_1, ds_1^2) of (M, ds^2)

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- (1) If $\mathbb{A} = \mathbb{R}$: $\Gamma \subset \{\pm 1\}$
  (2) If $\mathbb{A} = \mathbb{C}$: Γ is cyclic of order > 2
  (3) If $\mathbb{A} = \mathbb{H}$: Γ is binary dihedral, binary tetrahedral, binary octahedral or binary icosahedral

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- For $K = 0$: this is covered by the trivial example
- For $K > 0$: this involves some nontrivial finite group theory based on (i) $\gamma \neq \pm I$ has constant displacement if and only if it has eigenvalues $\{\lambda, \overline{\lambda}; \dots ; \lambda, \overline{\lambda}\}$ and (ii) an induction involving binary polyhedral and $SL(2;\mathbb{Z}_p)$ groups

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- Unbounded: here the evidence is that isometries of bounded displacement are ordinary translations along the Euclidean factor
  - Riemannian manifolds of sectional curvature $\leqq 0$
  - Riemannian manifolds without focal points
  - Riemannian manifolds homogeneous under a semisimple group with no compact factor
  - Riemannian manifolds homogeneous under an exponential solvable Lie group of isometries

Dichotomy – Bounded Cases

- Bounded: here much of the progress on the conjecture has been case by case verification
  - Riemannian or Finsler symmetric spaces
  - Compact homogeneous with a certain Weyl group condition, e.g. Stieffel manifolds
  - Twistor bundles over Grassmann manifolds, hermitian or quaternionic symmetric spaces, nearly-Kähler (3–symmetric) spaces, the 5–symmetric E_8/A_4A_4 , . . .

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- Example: $M = G/K_1$ fibered over $N = G/K_1K_2$.
  - M and N carry normal Riemannian metrics from G
  - Γ: finite subgroup of $Z_G K_2$
  - Then Γ acts on M: by $(z, k_2)(gK_1) = zgk_2^{-1}K_1$
  - This is isometric and centralizes the (transitive) isometric action of G on M so $\Gamma\backslash M$ is homogeneous

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- Theorem. Suppose that M is homogeneous and $M \to \Gamma \backslash M$ is a Riemannian covering. Then $\Gamma \backslash M$ is homogeneous if and only if every $\gamma \in \Gamma$ is an isometry of constant displacement. In that case Γ is a discrete group of ordinary translations along the euclidean factor of M.

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- If $M = G/K$, G exponential solvable, and γ is a bounded isometry then $\alpha = 1$. This uses some basic unipotent group theory, and includes the case of nilpotent G.

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- Theorem. Let $M \to \Gamma\backslash M$ be a Riemannian covering. Then $\Gamma\backslash M$ is homogeneous if and only if every $\gamma \in \Gamma$ is an isometry of constant displacement.

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- Examples:
  - $G/[K,K] \to G/K$ hermitian symmetric base
  - $G/K_1 \to G/Sp(1)K_1$ quaternion–Kaehler symm. base
  - $G/K_1 \to G/SU(3)K_1$ nearly–Kaehler 3–symmetric base
  - $E_8/SU(5) \to E_8/SU(5)SU(5)$ 5–symmetric base
  - $SO(k+\ell)/SO(k) \to SO(k+\ell)/[SO(k) \times SO(\ell)]$ real Stieffel manifold over real Grassmann manifold
  - $Sp(k+\ell)/Sp(k) \to Sp(k+\ell)/[Sp(k) \times Sp(\ell)]$ quaternion Stieffel manifold over quaternion Grassmann manifold

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- found all isometries of constant displacement of $\widetilde{M}$,
- proved the Homogeneity Conjecture for those cases:
- Theorem. Let $\pi : \widetilde{M} \to M$ be one of the examples listed above of isotropy–split fibration $G/K_1 \to G/K_1K_2$. Let $\widetilde{M} \to \Gamma\backslash\widetilde{M}$ be a Riemannian covering. Then these are equivalent: (1) $\gamma \in \Gamma$ is of constant displacement (2) $\Gamma \subset Z_G \times r(K_2)$, (3) $\Gamma\backslash\widetilde{M}$ is homogeneous.

Thank you for your attention