Invariant nondegenerate (0,2)-tensors on nilradicals of parabolic subalgebras

Viviana del Barco

Instituto de Matemática, Estatística e Computação Científica
Universidade Estadual de Campinas

VI EGEO
VI Workshop on Differential Geometry

La Falda, Argentina - August 4, 2016

Let $\mathfrak{g}$ be a finite dimensional Lie algebra over $\mathbb{R}$ and consider a bilinear form $b : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{R}$. Notice

$$b \in \mathfrak{g}^* \otimes \mathfrak{g}^* = \Lambda^2\mathfrak{g}^* \oplus \mathcal{S}(\mathfrak{g}).$$

We say that b is an «invariant tensor» if

$$b([X, Y], Z) + b(Y, [X, Z]) = b(X, [Y, Z]) \quad \text{for all } X, Y, Z \in \mathfrak{g}. \quad (1)$$

For b skew-symmetric or symmetric: $Z^2(\mathfrak{g}) = \ker(d : \Lambda^2\mathfrak{g}^* \longrightarrow \Lambda^3\mathfrak{g}^*)$ and $\mathcal{S}^{inv}(\mathfrak{g}) = \{b \in \mathcal{S}(\mathfrak{g}) : \ b \text{ satisfies } (1)\}$.

Let $\mathfrak{g}$ be a finite dimensional Lie algebra over $\mathbb{R}$ and consider a bilinear form $b : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{R}$. Notice

$$b \in \mathfrak{g}^* \otimes \mathfrak{g}^* = \Lambda^2\mathfrak{g}^* \oplus \mathcal{S}(\mathfrak{g}).$$

We say that b is an «invariant tensor» if

$$b([X,Y],Z) + b(Y,[X,Z]) = b(X,[Y,Z]) \qquad \text{for all } X,Y,Z \in \mathfrak{g}. \quad (1)$$

For b skew-symmetric or symmetric: $Z^2(\mathfrak{g}) = \ker(d : \Lambda^2\mathfrak{g}^* \longrightarrow \Lambda^3\mathfrak{g}^*)$ and $\mathcal{S}^{inv}(\mathfrak{g}) = \{b \in \mathcal{S}(\mathfrak{g}) : \ b \text{ satisfies } (1)\}$.

Particular cases of interest when b is non-degenerate

- b is skew-symmetric $\rightsquigarrow$ $b = \omega$ is a *symplectic structure* on $\mathfrak{g}$.
- b is symmetric $\rightsquigarrow$ $b = \langle \ , \ \rangle$ is an *ad-invariant metric* and (1) is equivalent:

$$\langle [X,Y],Z\rangle + \langle Y,[X,Z]\rangle = 0 \qquad \text{for all } X,Y,Z \in \mathfrak{g}.$$

Assume $\mathfrak{g}$ admits a non-degenerate invariant tensor b. Then

$\rightsquigarrow$ if $b = \omega$ is symplectic then

- $\mathfrak{g}$ is even dimensional,
- $\mathfrak{g}$ is not semisimple,
- $\dim \mathfrak{g}^j + \dim \mathfrak{g}_j \leq \dim \mathfrak{g}$ for all $j \geq 0$
- $\mathfrak{g}$ nilpotent then $0 \neq [\omega] \in H^2(\mathfrak{g})$.

Assume $\mathfrak{g}$ admits a non-degenerate invariant tensor b. Then

⇝ if $b = \omega$ is symplectic then

- $\mathfrak{g}$ is even dimensional,
- $\mathfrak{g}$ is not semisimple,
- $\dim \mathfrak{g}^j + \dim \mathfrak{g}_j \leq \dim \mathfrak{g}$ for all $j \geq 0$
- $\mathfrak{g}$ nilpotent then $0 \neq [\omega] \in H^2(\mathfrak{g})$.

⇝ if $b = \langle\ ,\ \rangle$ is an ad-invariant metric then

- $\mathfrak{g}$ could be semisimple.
- $\dim \mathfrak{g}^j + \dim \mathfrak{g}_j = \dim \mathfrak{g}$ for all $j \geq 0$
- $\mathfrak{g}$ solvable then $\mathfrak{z}(\mathfrak{g}) \neq 0$.
- $\sigma(X, Y, Z) = \langle [X, Y], Z \rangle \in H^3(\mathfrak{g})$.
- $\bigcap_{X \in \mathfrak{g}} [C_{\mathfrak{g}}(X), \mathfrak{g}] = 0$

Plan

Focus on nilpotent Lie algebras $\mathfrak{n}$.

Objective identify Lie-algebraic structure of $\mathfrak{n}$ when it admits a non-degenerate invariant tensor.

Goal properties of irreducible factors of $Z^2(\mathfrak{g})$ and $H^2(\mathfrak{n})$ ($\mathcal{S}^{inv}(\mathfrak{n})$) under the action of $\mathfrak{g} \subset \mathrm{Der}(\mathfrak{n})$, when $\mathfrak{n}$ admits a skew-symmetric (symmetric) invariant tensor.

Particular case: $\mathfrak{n}$ is a nilradical of a parabolic subalgebra $\mathfrak{p}$ of a split simple real Lie algebra $\mathfrak{g}$.

For which $\mathfrak{p}$ and $\mathfrak{g}$ does $\mathfrak{n}$ admit a non-degenerate invariant tensor? Of both types?

Structure results

Let $\mathfrak{n}$ be a real nilpotent Lie algebra and let

$$\mathfrak{n} = \mathfrak{n}^0 \supset \mathfrak{n}^1 \supset \cdots \supset \mathfrak{n}^{k-1} \supset \mathfrak{n}^k = 0$$

be the lower central serie of $\mathfrak{n}$. Let $\mathfrak{g} \subseteq \mathrm{Der}(\mathfrak{n})$ be reductive. We have

$$\mathfrak{n}^* = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \cdots \oplus \mathfrak{m}_k$$

where each $\mathfrak{m}_i$ is a $\mathfrak{g}$-submodule such that $\mathfrak{m}_1 = (\mathfrak{n}^1)^\circ$, $\mathfrak{m}_1 \oplus \mathfrak{m}_2 = (\mathfrak{n}^2)^\circ$, etc.. For this gradation

$$\mathfrak{n}^* \otimes \mathfrak{n}^* = \bigoplus_{1 \leq i,j \leq k} \mathfrak{m}_i \otimes \mathfrak{m}_j, \quad \Lambda^2 \mathfrak{n}^* = \bigoplus_{1 \leq i \leq j \leq k} \mathfrak{m}_i \wedge \mathfrak{m}_j, \quad \mathcal{S}(\mathfrak{n}) = \bigoplus_{1 \leq i \leq j \leq k} \mathfrak{m}_i \odot \mathfrak{m}_j$$

For each $t = 1, \ldots, \lceil k/2 \rceil$ let $P_t : \mathfrak{n}^* \otimes \mathfrak{n}^* \to \mathfrak{m}_t \otimes \mathfrak{m}_{k-t+1}$.

For a (skew)-symmetric invariant tensor $b : \mathfrak{n} \times \mathfrak{n} \longrightarrow \mathbb{R}$ we can prove

$$b(X, Y) = 0 \text{ for all } X \in \mathfrak{n}^i, Y \in \mathfrak{n}^j \text{ such that } i + j \geq k.$$

Therefore:

$$\mathcal{S}^{inv}(\mathfrak{n}) \subset \mathfrak{m}_1 \odot \mathfrak{m}_k + (\mathfrak{m}_1 + \mathfrak{m}_2) \odot \mathfrak{m}_{k-1} + \ldots = \bigoplus_{i+j \leq k+1} \mathfrak{m}_i \odot \mathfrak{m}_j$$

$$Z^2(\mathfrak{n}) \subset \mathfrak{m}_1 \wedge \mathfrak{m}_k + (\mathfrak{m}_1 + \mathfrak{m}_2) \wedge \mathfrak{m}_{k-1} + \ldots = \bigoplus_{i+j \leq k+1} \mathfrak{m}_i \wedge \mathfrak{m}_j$$

For a (skew)-symmetric invariant tensor $b : \mathfrak{n} \times \mathfrak{n} \longrightarrow \mathbb{R}$ we can prove

$$b(X, Y) = 0 \text{ for all } X \in \mathfrak{n}^i, Y \in \mathfrak{n}^j \text{ such that } i + j \geq k.$$

Therefore:

$$\mathcal{S}^{inv}(\mathfrak{n}) \subset \mathfrak{m}_1 \odot \mathfrak{m}_k + (\mathfrak{m}_1 + \mathfrak{m}_2) \odot \mathfrak{m}_{k-1} + \ldots = \bigoplus_{i+j \leq k+1} \mathfrak{m}_i \odot \mathfrak{m}_j$$

$$Z^2(\mathfrak{n}) \subset \mathfrak{m}_1 \wedge \mathfrak{m}_k + (\mathfrak{m}_1 + \mathfrak{m}_2) \wedge \mathfrak{m}_{k-1} + \ldots = \bigoplus_{i+j \leq k+1} \mathfrak{m}_i \wedge \mathfrak{m}_j$$

b non-degenerate $\Rightarrow$ for all $0 \neq X \in \mathfrak{n}^{k-1}$ $\exists Y \in \mathfrak{n} \ominus \mathfrak{n}^1$ such that $b(X, Y) \neq 0 \Rightarrow P_1(b) \neq 0$.

For a (skew)-symmetric invariant tensor $b : \mathfrak{n} \times \mathfrak{n} \longrightarrow \mathbb{R}$ we can prove

$$b(X, Y) = 0 \text{ for all } X \in \mathfrak{n}^i, Y \in \mathfrak{n}^j \text{ such that } i + j \geq k.$$

Therefore:

$$\mathcal{S}^{inv}(\mathfrak{n}) \subset \mathfrak{m}_1 \odot \mathfrak{m}_k + (\mathfrak{m}_1 + \mathfrak{m}_2) \odot \mathfrak{m}_{k-1} + \ldots = \bigoplus_{i+j \leq k+1} \mathfrak{m}_i \odot \mathfrak{m}_j$$

$$Z^2(\mathfrak{n}) \subset \mathfrak{m}_1 \wedge \mathfrak{m}_k + (\mathfrak{m}_1 + \mathfrak{m}_2) \wedge \mathfrak{m}_{k-1} + \ldots = \bigoplus_{i+j \leq k+1} \mathfrak{m}_i \wedge \mathfrak{m}_j$$

b non-degenerate $\Rightarrow$ for all $0 \neq X \in \mathfrak{n}^{k-1}$ $\exists Y \in \mathfrak{n} \ominus \mathfrak{n}^1$ such that $b(X, Y) \neq 0 \Rightarrow P_1(b) \neq 0$.

What for $P_t(b)$, $t \geq 2$?

Theorem (Cagliero,-,'16)

If $\mathfrak{n}$ admits a non-degenerate (skew-)symmetric invariant tensor then for any $t = 1, \ldots, \lceil k/2 \rceil$ such that

$$\dim \mathfrak{n}^{k-t} + \dim \mathfrak{n}^{t-1} > \dim \mathfrak{n}, \tag{2}$$

there is an irreducible $\mathfrak{g}$-submodule V_t of $S^{inv}(\mathfrak{n})$ (resp. $Z^2(\mathfrak{n})$) such that $P_t(\sigma) \neq 0$ for any nonzero element σ in V_t.

Theorem (Cagliero,-,'16)

If $\mathfrak{n}$ admits a non-degenerate (skew-)symmetric invariant tensor then for any $t = 1, \ldots, \lceil k/2 \rceil$ such that

$$\dim \mathfrak{n}^{k-t} + \dim \mathfrak{n}^{t-1} > \dim \mathfrak{n}, \tag{2}$$

there is an irreducible $\mathfrak{g}$-submodule V_t of $S^{inv}(\mathfrak{n})$ (resp. $Z^2(\mathfrak{n})$) such that $P_t(\sigma) \neq 0$ for any nonzero element σ in V_t.

If either $\mathfrak{n}$ or $\mathbb{R} \oplus \mathfrak{n}$ is symplectic, there is an irreducible $\mathfrak{g}$-submodule U_t of $H^2(\mathfrak{n})$ $P_t(\sigma) \neq 0$ for any $\sigma \in \Lambda^2 \mathfrak{n}^*$ corresponding to a nonzero cohomology class in U_t.

Nilradicals

We consider a particular family of nilpotent Lie algebras.
Fix

- $\tilde{\mathfrak{g}}$ a real simple split Lie algebra,
- Δ root system of $\mathfrak{g}$, with positive roots Δ^+ and simple roots Π,
- $\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}^- + \mathfrak{h} + \tilde{\mathfrak{g}}^+$ the triangular decomposition and $\tilde{\mathfrak{b}} = \mathfrak{h} + \tilde{\mathfrak{g}}^+$ Borel subalgebra.
- $\mathfrak{p}$ parabolic subalgebra of $\tilde{\mathfrak{g}}$, $\mathfrak{p} = \mathfrak{g} \ltimes \mathfrak{n}$ with Levi factor $\mathfrak{g} = \mathfrak{g}^- \oplus \mathfrak{h} \oplus \mathfrak{g}^+$.

Recall: $\mathfrak{p}$ parabolic subalgebra of $\tilde{\mathfrak{g}} \leftrightarrow \Pi_0$ where $\Pi_0 \subseteq \Pi$.

We consider a particular family of nilpotent Lie algebras.
Fix

- $\tilde{\mathfrak{g}}$ a real simple split Lie algebra,
- Δ root system of $\mathfrak{g}$, with positive roots Δ^+ and simple roots Π,
- $\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}^- + \mathfrak{h} + \tilde{\mathfrak{g}}^+$ the triangular decomposition and $\tilde{\mathfrak{b}} = \mathfrak{h} + \tilde{\mathfrak{g}}^+$ Borel subalgebra.
- $\mathfrak{p}$ parabolic subalgebra of $\tilde{\mathfrak{g}}$, $\mathfrak{p} = \mathfrak{g} \ltimes \mathfrak{n}$ with Levi factor $\mathfrak{g} = \mathfrak{g}^- \oplus \mathfrak{h} \oplus \mathfrak{g}^+$.

Recall: $\mathfrak{p}$ parabolic subalgebra of $\tilde{\mathfrak{g}} \leftrightarrow \Pi_0$ where $\Pi_0 \subseteq \Pi$.

We apply the previous ideas to the particular case:
$\rightsquigarrow$ $\mathfrak{n}$ the nilradical of $\mathfrak{p}$,
$\rightsquigarrow$ $\mathfrak{g}$ acting reductively by derivations on $\mathfrak{n}$.

First results:
If $|\Pi_0| \geq 3$ then $P_1(b) = 0$ for any b non-degenerate (skew-) symmetric invariant tensor on $\mathfrak{n}$.
If $|\Pi_0| = 2$ then $P_1(\langle\ ,\ \rangle) = 0$ for any $\langle\ ,\ \rangle$ ad-invariant metric on $\mathfrak{n}$.

Main results [Cagliero, -,'16][-,16]:

First results:
If $|\Pi_0| \geq 3$ then $P_1(b) = 0$ for any b non-degenerate (skew-) symmetric invariant tensor on $\mathfrak{n}$.
If $|\Pi_0| = 2$ then $P_1(\langle\ ,\ \rangle) = 0$ for any $\langle\ ,\ \rangle$ ad-invariant metric on $\mathfrak{n}$.

Main results [Cagliero, -,'16][-,16]:

$\rightsquigarrow$ $\mathfrak{n}$ (or $\mathbb{R} \oplus \mathfrak{n}$) is symplectic if and only if $\mathfrak{n}$ is

- The abelian Lie algebras.
- The free 2-step nilpotent Lie algebra generated by 2 or 3 elements.
- The filiform Lie algebra of dimension 4: $[X_1, X_2] = X_3$, $[X_1, X_3] = X_4$.
- $\{X_i, Y_i, Z_{ij} : 1 \leq i, j \leq 2\}$, $[X_i, Y_j] = Z_{ij}$, $1 \leq i, j \leq 2$.
- $\{X_i, Y_i, Z_{ij} : 1 \leq i \leq j \leq n\}$, $[X_i, Y_j] = [X_j, Y_i] = Z_{ij}$, $1 \leq i \leq j \leq n$; $n = 2, 3$.
- $\{X_i, Y_i, Z_{ij} : 1 \leq i < j \leq 3\}$, $[X_i, Y_j] = -[X_j, Y_i] = Z_{ij}$, $1 \leq i < j \leq 3$.
- $\mathbb{R}X \ltimes_{\mathrm{ad}_X} (\mathbb{R}^{n-1} \oplus \mathbb{R}^{n-1})$ with $\mathrm{ad}_X = \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix}$.

First results:
If $|\Pi_0| \geq 3$ then $P_1(b) = 0$ for any b non-degenerate (skew-) symmetric invariant tensor on $\mathfrak{n}$.
If $|\Pi_0| = 2$ then $P_1(\langle\ ,\ \rangle) = 0$ for any $\langle\ ,\ \rangle$ ad-invariant metric on $\mathfrak{n}$.

Main results [Cagliero, -,'16][-,16]:

$\rightsquigarrow$ $\mathfrak{n}$ (or $\mathbb{R} \oplus \mathfrak{n}$) is symplectic if and only if $\mathfrak{n}$ is

- The abelian Lie algebras.
- The free 2-step nilpotent Lie algebra generated by 2 or 3 elements.
- The filiform Lie algebra of dimension 4: $[X_1, X_2] = X_3$, $[X_1, X_3] = X_4$.
- $\{X_i, Y_i, Z_{ij} : 1 \leq i, j \leq 2\}$, $[X_i, Y_j] = Z_{ij}$, $1 \leq i, j \leq 2$.
- $\{X_i, Y_i, Z_{ij} : 1 \leq i \leq j \leq n\}$, $[X_i, Y_j] = [X_j, Y_i] = Z_{ij}$, $1 \leq i \leq j \leq n$; $n = 2, 3$.
- $\{X_i, Y_i, Z_{ij} : 1 \leq i < j \leq 3\}$, $[X_i, Y_j] = -[X_j, Y_i] = Z_{ij}$, $1 \leq i < j \leq 3$.
- $\mathbb{R}X \ltimes_{\mathrm{ad}_X} (\mathbb{R}^{n-1} \oplus \mathbb{R}^{n-1})$ with $\mathrm{ad}_X = \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix}$.

$\rightsquigarrow$ $\mathfrak{n}$ admits ad-invariant metric if and only if $\mathfrak{n}$ is

- abelian Lie algebras.
- the free 2 step nilpotent Lie algebra on 3-generators,
- the free 3 step nilpotent Lie algebra on 2-generators.

Let $\mathfrak{n}$ be associated to $\Pi_0 \subset \Pi$, then

- for each $\gamma \in \Delta$, consider

$$o(\gamma) = \sum_{\alpha \in \Pi_0} \mathrm{coord}_\alpha(\gamma), \text{ and set } \mathfrak{n}_{(i)} = \bigoplus_{\substack{\gamma \in \Delta \\ o(\gamma) = i}} \tilde{\mathfrak{g}}_\gamma, \; i \in \mathbb{Z}$$

- the lower central serie is given by

$$\mathfrak{n}^j = \bigoplus_{i=j+1}^{k} \mathfrak{n}_{(i)},$$

and the nilpotency index of $\mathfrak{n}$ is $k = o(\gamma_{\max})$

Let $\mathfrak{n}$ be associated to $\Pi_0 \subset \Pi$, then

- for each $\gamma \in \Delta$, consider

$$o(\gamma) = \sum_{\alpha \in \Pi_0} \mathrm{coord}_\alpha(\gamma), \text{ and set } \mathfrak{n}_{(i)} = \bigoplus_{\substack{\gamma \in \Delta \\ o(\gamma) = i}} \tilde{\mathfrak{g}}_\gamma, \ i \in \mathbb{Z}$$

- the lower central serie is given by

$$\mathfrak{n}^j = \bigoplus_{i=j+1}^{k} \mathfrak{n}_{(i)},$$

  and the nilpotency index of $\mathfrak{n}$ is $k = o(\gamma_{\max})$

Via the Killing form of $\tilde{\mathfrak{g}}$ we identify

$$\mathfrak{n}^* \longleftrightarrow \mathfrak{n}^- = \bigoplus_{i=1}^{k} \mathfrak{n}_{(-i)} \quad \text{and} \quad \mathfrak{m}_j \longleftrightarrow \mathfrak{n}_{(-j)}, \, j = 1, \ldots, k.$$

So $P_1 : \mathfrak{n}^- \otimes \mathfrak{n}^- \to \mathfrak{n}_{(-1)} \otimes \mathfrak{n}_{(-k)}$.

AD-INVARIANT METRICS

LEMA

Assume $|\Pi_0| \geq 2$. *Let* $X_\gamma \in \mathfrak{n}^{k-1}$ *and* $\alpha \in \Pi_0$. *There exist* $\delta, \beta_1, \ldots, \beta_t \in \Delta_{\mathfrak{n}}^+$ *such that*

$$X_\gamma = [[[X_\delta, X_{\beta_t}], X_{\beta_{t-1}}], \cdots, X_{\beta_1}] \quad \textit{(up to non-zero multiple)}$$

where $\mathrm{coord}_\alpha(\delta) = 0$, $\mathrm{coord}_\alpha(\beta_j) \neq 0$ *for all* $j = 1, \ldots, t$ *and* $\delta + \beta_t + \beta_{t-1} + \ldots + \beta_{t-s}$, $s = 0, \ldots, t-1$ *is a root in* $\Delta_{\mathfrak{n}}^+$.

AD-INVARIANT METRICS

Lema

Assume $|\Pi_0| \geq 2$. *Let* $X_\gamma \in \mathfrak{n}^{k-1}$ *and* $\alpha \in \Pi_0$. *There exist* $\delta, \beta_1, \ldots, \beta_t \in \Delta_{\mathfrak{n}}^+$ *such that*

$$X_\gamma = [[[X_\delta, X_{\beta_t}], X_{\beta_{t-1}}], \cdots, X_{\beta_1}] \quad \textit{(up to non-zero multiple)}$$

where $\mathrm{coord}_\alpha(\delta) = 0$, $\mathrm{coord}_\alpha(\beta_j) \neq 0$ *for all* $j = 1, \ldots, t$ *and* $\delta + \beta_t + \beta_{t-1} + \ldots + \beta_{t-s}$, $s = 0, \ldots, t-1$ *is a root in* $\Delta_{\mathfrak{n}}^+$.

Proposition

If $|\Pi_0| \geq 2$ *then* $\mathfrak{n}$ *does not admit ad-invariant metrics.*

AD-INVARIANT METRICS

Lema

Assume $|\Pi_0| \geq 2$. *Let* $X_\gamma \in \mathfrak{n}^{k-1}$ *and* $\alpha \in \Pi_0$. *There exist* $\delta, \beta_1, \ldots, \beta_t \in \Delta_\mathfrak{n}^+$ *such that*

$$X_\gamma = [[[X_\delta, X_{\beta_t}], X_{\beta_{t-1}}], \cdots, X_{\beta_1}] \quad \textit{(up to non-zero multiple)}$$

where $\mathrm{coord}_\alpha(\delta) = 0$, $\mathrm{coord}_\alpha(\beta_j) \neq 0$ *for all* $j = 1, \ldots, t$ *and* $\delta + \beta_t + \beta_{t-1} + \ldots + \beta_{t-s}$, $s = 0, \ldots, t-1$ *is a root in* $\Delta_\mathfrak{n}^+$.

Proposition

If $|\Pi_0| \geq 2$ *then* $\mathfrak{n}$ *does not admit ad-invariant metrics.*

Proof. Let $X_\gamma \in \mathfrak{n}^{k-1}$ and X_μ s.t. $o(\mu) = 1$. Then

$$\begin{aligned} \langle X_\gamma, X_\mu \rangle &= \left\langle [[[X_\delta, X_{\beta_t}], X_{\beta_{t-1}}], \cdots, X_{\beta_1}], X_\mu \right\rangle \\ &= \langle X_\delta, [X_{\beta_t}, \cdots, [X_{\beta_2}, [X_{\beta_1}, X_\mu]]] \rangle = 0 \end{aligned}$$

AD-INVARIANT METRICS

Lema

Assume $|\Pi_0| \geq 2$. *Let* $X_\gamma \in \mathfrak{n}^{k-1}$ *and* $\alpha \in \Pi_0$. *There exist* $\delta, \beta_1, \ldots, \beta_t \in \Delta^+_{\mathfrak{n}}$ *such that*

$$X_\gamma = [[[X_\delta, X_{\beta_t}], X_{\beta_{t-1}}], \cdots, X_{\beta_1}] \quad \textit{(up to non-zero multiple)}$$

where $\mathrm{coord}_\alpha(\delta) = 0$, $\mathrm{coord}_\alpha(\beta_j) \neq 0$ *for all* $j = 1, \ldots, t$ *and* $\delta + \beta_t + \beta_{t-1} + \ldots + \beta_{t-s}$, $s = 0, \ldots, t-1$ *is a root in* $\Delta^+_{\mathfrak{n}}$.

Proposition

If $|\Pi_0| \geq 2$ *then* $\mathfrak{n}$ *does not admit ad-invariant metrics.*

Proof. Let $X_\gamma \in \mathfrak{n}^{k-1}$ and X_μ s.t. $o(\mu) = 1$. Then

$$\begin{aligned} \langle X_\gamma, X_\mu \rangle &= \left\langle [[[X_\delta, X_{\beta_t}], X_{\beta_{t-1}}], \cdots, X_{\beta_1}], X_\mu \right\rangle \\ &= \langle X_\delta, [X_{\beta_t}, \cdots, [X_{\beta_2}, [X_{\beta_1}, X_\mu]]] \rangle = 0 \end{aligned}$$

$\Pi_0 = \{\alpha\} \rightsquigarrow$ case by case study, gives only existence on B_3, $\alpha = \gamma_3$ and G_2, $\alpha = \gamma_1$ and abelian.

SYMPLECTIC STRUCTURES

We will show that $P_1(U) = 0$ for almost any irreducible submodule U of $H^2(\mathfrak{n})$.

Symplectic structures

We will show that $P_1(U) = 0$ for almost any irreducible submodule U of $H^2(\mathfrak{n})$.

A set of representatives of $\mathfrak{g}$-highest weight vectors of $H^2(\mathfrak{n})$ is given in terms of root vectors X_γ (Kostant):

$$H^2(\mathfrak{n})_{\mathsf{hwv}} = \{X_{-\alpha_1} \wedge X_{-\alpha_2} \in \Lambda^2\mathfrak{n}^- : \{\alpha_1, \alpha_2\} = w\Delta^- \cap \Delta^+,\, w \in W^{1,2}\}$$

An insight into those roots above gives

Proposition

If $s_\alpha \in W$ denotes the reflection associated to the simple root $\alpha \in \Pi$, then

$$\begin{aligned} H^2(\mathfrak{n})_{\mathrm{hwv}} = \{X_{-\alpha} \wedge X_{-\beta} : \alpha, \beta \in \Pi_0,\, (\alpha,\beta) = 0\} \cup \\ \{X_{-\alpha} \wedge X_{-s_\alpha(\beta)} : \alpha \in \Pi_0,\ \beta \in \Pi,\ (\beta,\alpha) < 0\}. \end{aligned}$$

Proposition

Assume $\mathfrak{n}$ is non abelian. If either $\mathfrak{n}$ or $\mathbb{R} \oplus \mathfrak{n}$ is symplectic, then there exists $\alpha \in \Pi_0$ and $\beta \in \Pi$ such that $\mathrm{coord}_\gamma(s_\alpha(\beta)) = \mathrm{coord}_\gamma(\gamma_{\mathrm{máx}})$ for all $\gamma \in \Pi_0$, and $(\beta, \alpha) < 0$.

Proposition

Assume $\mathfrak{n}$ *is non abelian. If either* $\mathfrak{n}$ *or* $\mathbb{R} \oplus \mathfrak{n}$ *is symplectic, then there exists* $\alpha \in \Pi_0$ *and* $\beta \in \Pi$ *such that* $\mathrm{coord}_\gamma(s_\alpha(\beta)) = \mathrm{coord}_\gamma(\gamma_{\mathrm{máx}})$ *for all* $\gamma \in \Pi_0$*, and* $(\beta, \alpha) < 0$.

Proof.

Proposition

Assume $\mathfrak{n}$ is non abelian. If either $\mathfrak{n}$ or $\mathbb{R} \oplus \mathfrak{n}$ is symplectic, then there exists $\alpha \in \Pi_0$ and $\beta \in \Pi$ such that $\mathrm{coord}_\gamma(s_\alpha(\beta)) = \mathrm{coord}_\gamma(\gamma_{\text{máx}})$ for all $\gamma \in \Pi_0$, and $(\beta, \alpha) < 0$.

Proof. Assume $\mathfrak{n}$ or $\mathbb{R} \oplus \mathfrak{n}$ is symplectic, then $P_1(\sigma) \neq 0$ for some $\sigma \in H^2(\mathfrak{n})_{\mathrm{hwv}}$; recall $P(\sigma) \in \mathfrak{n}_{(-1)} \wedge \mathfrak{n}_{(-k)}$.

Proposition

Assume $\mathfrak{n}$ *is non abelian. If either* $\mathfrak{n}$ *or* $\mathbb{R} \oplus \mathfrak{n}$ *is symplectic, then there exists* $\alpha \in \Pi_0$ *and* $\beta \in \Pi$ *such that* $\mathrm{coord}_\gamma(s_\alpha(\beta)) = \mathrm{coord}_\gamma(\gamma_{\text{máx}})$ *for all* $\gamma \in \Pi_0$*, and* $(\beta, \alpha) < 0$*.*

Proof. Assume $\mathfrak{n}$ or $\mathbb{R} \oplus \mathfrak{n}$ is symplectic, then $P_1(\sigma) \neq 0$ for some $\sigma \in H^2(\mathfrak{n})_{\mathrm{hwv}}$; recall $P(\sigma) \in \mathfrak{n}_{(-1)} \wedge \mathfrak{n}_{(-k)}$.

$\mathfrak{n}$ nonabelian $\Rightarrow P(X_{-\alpha} \wedge X_{-\beta}) = 0$ for all $\alpha, \beta \in \Pi_0$

Proposition

Assume $\mathfrak{n}$ *is non abelian. If either* $\mathfrak{n}$ *or* $\mathbb{R}\oplus\mathfrak{n}$ *is symplectic, then there exists* $\alpha\in\Pi_0$ *and* $\beta\in\Pi$ *such that* $\mathrm{coord}_\gamma(s_\alpha(\beta))=\mathrm{coord}_\gamma(\gamma_{\text{máx}})$ *for all* $\gamma\in\Pi_0$*, and* $(\beta,\alpha)<0$*.*

Proof. Assume $\mathfrak{n}$ or $\mathbb{R}\oplus\mathfrak{n}$ is symplectic, then $P_1(\sigma)\neq 0$ for some $\sigma\in H^2(\mathfrak{n})_{\mathrm{hwv}}$; recall $P(\sigma)\in\mathfrak{n}_{(-1)}\wedge\mathfrak{n}_{(-k)}$.

$\mathfrak{n}$ nonabelian $\Rightarrow P(X_{-\alpha}\wedge X_{-\beta})=0$ for all $\alpha,\beta\in\Pi_0$

$\Rightarrow P(X_{-\alpha}\wedge X_{-s_\alpha(\beta)})\neq 0$ for some $\alpha\in\Pi_0$, $\beta\in\Pi$ with $(\alpha,\beta)<0$.

Proposition

Assume $\mathfrak{n}$ is non abelian. If either $\mathfrak{n}$ or $\mathbb{R} \oplus \mathfrak{n}$ is symplectic, then there exists $\alpha \in \Pi_0$ and $\beta \in \Pi$ such that $\mathrm{coord}_\gamma(s_\alpha(\beta)) = \mathrm{coord}_\gamma(\gamma_{\mathrm{máx}})$ for all $\gamma \in \Pi_0$, and $(\beta, \alpha) < 0$.

Proof. Assume $\mathfrak{n}$ or $\mathbb{R} \oplus \mathfrak{n}$ is symplectic, then $P_1(\sigma) \neq 0$ for some $\sigma \in H^2(\mathfrak{n})_{\mathrm{hwv}}$; recall $P(\sigma) \in \mathfrak{n}_{(-1)} \wedge \mathfrak{n}_{(-k)}$.

$\mathfrak{n}$ nonabelian $\Rightarrow P(X_{-\alpha} \wedge X_{-\beta}) = 0$ for all $\alpha, \beta \in \Pi_0$

$\Rightarrow P(X_{-\alpha} \wedge X_{-s_\alpha(\beta)}) \neq 0$ for some $\alpha \in \Pi_0$, $\beta \in \Pi$ with $(\alpha, \beta) < 0$.

Also

$$
\begin{aligned}
P(X_{-\alpha} \wedge X_{-s_\alpha(\beta)}) \neq 0 \quad &\Leftrightarrow \quad X_{-s_\alpha(\beta)} \in \mathfrak{n}_{(-k)} \\
&\Leftrightarrow \quad X_{s_\alpha(\beta)} \in \mathfrak{n}^{k-1} \\
&\Leftrightarrow \quad \mathrm{coord}_\gamma(s_\alpha(\beta)) = \mathrm{coord}_\gamma(\gamma_{\mathrm{máx}}) \text{ for all } \gamma \in \Pi_0.
\end{aligned}
$$

Proposition

Assume $\mathfrak{n}$ *is non abelian. If either* $\mathfrak{n}$ *or* $\mathbb{R} \oplus \mathfrak{n}$ *is symplectic, then there exists* $\alpha \in \Pi_0$ *and* $\beta \in \Pi$ *such that* $\mathrm{coord}_\gamma(s_\alpha(\beta)) = \mathrm{coord}_\gamma(\gamma_{\text{máx}})$ *for all* $\gamma \in \Pi_0$*, and* $(\beta, \alpha) < 0$*.*

Proof. Assume $\mathfrak{n}$ or $\mathbb{R} \oplus \mathfrak{n}$ is symplectic, then $P_1(\sigma) \neq 0$ for some $\sigma \in H^2(\mathfrak{n})_{\mathrm{hwv}}$; recall $P(\sigma) \in \mathfrak{n}_{(-1)} \wedge \mathfrak{n}_{(-k)}$.

$\mathfrak{n}$ nonabelian $\Rightarrow P(X_{-\alpha} \wedge X_{-\beta}) = 0$ for all $\alpha, \beta \in \Pi_0$

$\Rightarrow P(X_{-\alpha} \wedge X_{-s_\alpha(\beta)}) \neq 0$ for some $\alpha \in \Pi_0$, $\beta \in \Pi$ with $(\alpha, \beta) < 0$.

Also

$$
\begin{aligned}
P(X_{-\alpha} \wedge X_{-s_\alpha(\beta)}) \neq 0 \quad &\Leftrightarrow \quad X_{-s_\alpha(\beta)} \in \mathfrak{n}_{(-k)} \\
&\Leftrightarrow \quad X_{s_\alpha(\beta)} \in \mathfrak{n}^{k-1} \\
&\Leftrightarrow \quad \mathrm{coord}_\gamma(s_\alpha(\beta)) = \mathrm{coord}_\gamma(\gamma_{\text{máx}}) \text{ for all } \gamma \in \Pi_0.
\end{aligned}
$$

Remark: for $\alpha \in \Pi_0$ and $\beta \in \Pi$, $s_\alpha(\beta) = \beta - \frac{(\beta,\alpha)}{||\alpha||^2}\alpha$.

Proposition

Assume $\mathfrak{n}$ is non abelian. If either $\mathfrak{n}$ or $\mathbb{R}\oplus\mathfrak{n}$ is symplectic, then there exists $\alpha\in\Pi_0$ and $\beta\in\Pi$ such that $\mathrm{coord}_\gamma(s_\alpha(\beta)) = \mathrm{coord}_\gamma(\gamma_{\text{máx}})$ for all $\gamma\in\Pi_0$, and $(\beta,\alpha)<0$.

Proof. Assume $\mathfrak{n}$ or $\mathbb{R}\oplus\mathfrak{n}$ is symplectic, then $P_1(\sigma)\neq 0$ for some $\sigma\in H^2(\mathfrak{n})_{\mathrm{hwv}}$; recall $P(\sigma)\in\mathfrak{n}_{(-1)}\wedge\mathfrak{n}_{(-k)}$.

$\mathfrak{n}$ nonabelian $\Rightarrow P(X_{-\alpha}\wedge X_{-\beta})=0$ for all $\alpha,\beta\in\Pi_0$

$$\Rightarrow P(X_{-\alpha}\wedge X_{-s_\alpha(\beta)})\neq 0 \text{ for some } \alpha\in\Pi_0,\ \beta\in\Pi \text{ with } (\alpha,\beta)<0.$$

Also

$$\begin{aligned} P(X_{-\alpha}\wedge X_{-s_\alpha(\beta)})\neq 0 \quad &\Leftrightarrow \quad X_{-s_\alpha(\beta)}\in\mathfrak{n}_{(-k)} \\ &\Leftrightarrow \quad X_{s_\alpha(\beta)}\in\mathfrak{n}^{k-1} \\ &\Leftrightarrow \quad \mathrm{coord}_\gamma(s_\alpha(\beta))=\mathrm{coord}_\gamma(\gamma_{\text{máx}}) \text{ for all } \gamma\in\Pi_0. \end{aligned}$$

Remark: for $\alpha\in\Pi_0$ and $\beta\in\Pi$, $s_\alpha(\beta)=\beta-\frac{(\beta,\alpha)}{||\alpha||^2}\alpha$.

First conclusion: If $\mathfrak{n}$ corresponds to $\Pi_0\subset\Pi$, where Π_0 has 3 or more roots, then neither $\mathfrak{n}$ nor $\mathbb{R}\oplus\mathfrak{n}$ are symplectic.

Proposition

Let $\mathfrak{n}$ be a nonabelian nilradical of a parabolic subalgebra of $\tilde{\mathfrak{g}}$ associated to Π_0. If either $\mathfrak{n}$ or $\mathbb{R} \oplus \mathfrak{n}$ is symplectic, then one of the following holds

1. $\Pi_0 = \{\alpha\}$ *and there exists* $\beta \in \Pi$ *such that* $\mathrm{coord}_\alpha(\gamma_{\text{máx}}) = -2\frac{(\beta,\alpha)}{||\alpha||^2}$.
2. $\Pi_0 = \{\alpha, \beta\}$, $\mathrm{coord}_\beta(\gamma_{\text{máx}}) = 1$ *and* $\mathrm{coord}_\alpha(\gamma_{\text{máx}}) = -2\frac{(\beta,\alpha)}{||\alpha||^2}$.

In either case, $(\beta, \alpha) < 0$.

Remark: abelian Lie algebras $\mathfrak{n}$ arise only when $\Pi_0 = \{\alpha\}$ and $\mathrm{coord}_\alpha(\gamma_{\text{máx}}) = 1$.

Proposition

Let $\mathfrak{n}$ be a nonabelian nilradical of a parabolic subalgebra of $\tilde{\mathfrak{g}}$ associated to Π_0. If either $\mathfrak{n}$ or $\mathbb{R} \oplus \mathfrak{n}$ is symplectic, then one of the following holds

1. $\Pi_0 = \{\alpha\}$ *and there exists* $\beta \in \Pi$ *such that* $\mathrm{coord}_\alpha(\gamma_{\text{máx}}) = -2\frac{(\beta,\alpha)}{||\alpha||^2}$.
2. $\Pi_0 = \{\alpha, \beta\}$, $\mathrm{coord}_\beta(\gamma_{\text{máx}}) = 1$ *and* $\mathrm{coord}_\alpha(\gamma_{\text{máx}}) = -2\frac{(\beta,\alpha)}{||\alpha||^2}$.

In either case, $(\beta, \alpha) < 0$.

Remark: abelian Lie algebras $\mathfrak{n}$ arise only when $\Pi_0 = \{\alpha\}$ and $\mathrm{coord}_\alpha(\gamma_{\text{máx}}) = 1$.

We search by type the sets Π_0 verifying 1. or 2. or $\mathfrak{n}$ abelian. Moreover, we filter with the condition $\dim \mathfrak{n}^j + \dim \mathfrak{n}_j \leq \dim \mathfrak{n}$.

Proposition

Let $\mathfrak{n}$ be a nonabelian nilradical of a parabolic subalgebra of $\tilde{\mathfrak{g}}$ associated to Π_0. If either $\mathfrak{n}$ or $\mathbb{R} \oplus \mathfrak{n}$ is symplectic, then one of the following holds

1. $\Pi_0 = \{\alpha\}$ *and there exists* $\beta \in \Pi$ *such that* $\mathrm{coord}_\alpha(\gamma_{\text{máx}}) = -2\frac{(\beta,\alpha)}{||\alpha||^2}$.
2. $\Pi_0 = \{\alpha, \beta\}$, $\mathrm{coord}_\beta(\gamma_{\text{máx}}) = 1$ *and* $\mathrm{coord}_\alpha(\gamma_{\text{máx}}) = -2\frac{(\beta,\alpha)}{||\alpha||^2}$.

In either case, $(\beta, \alpha) < 0$.

Remark: abelian Lie algebras $\mathfrak{n}$ arise only when $\Pi_0 = \{\alpha\}$ and $\mathrm{coord}_\alpha(\gamma_{\text{máx}}) = 1$.

We search by type the sets Π_0 verifying 1. or 2. or $\mathfrak{n}$ abelian. Moreover, we filter with the condition $\dim \mathfrak{n}^j + \dim \mathfrak{n}_j \leq \dim \mathfrak{n}$.

Finally we get...

The Lie algebra $\mathfrak{n}$ or $\mathbb{R} \oplus \mathfrak{n}$ is symplectic if and only if $\mathfrak{n}$ is one of the following ($\tilde{\mathfrak{g}}$ and Π_0 specified):

- The abelian Lie algebras (A, D, E, $\Pi_0 = \{\alpha\}$*).
- The free 2-step nilpotent Lie algebra generated by 2 or 3 elements (B_m, $m = 2, 3$, $\Pi_0 = \{\gamma_m\}$).
- The filiform Lie algebra of dimension 4: $[X_1, X_2] = X_3$, $[X_1, X_3] = X_4$ (B_2, $\Pi_0 = \Pi$).
- $\{X_i, Y_i, Z_{ij} : 1 \le i, j \le 2\}$, $[X_i, Y_j] = Z_{ij}$, $1 \le i, j \le 2$ (A_4, $\Pi_0 = \{\gamma_2, \gamma_3\}$).
- $\{X_i, Y_i, Z_{ij} : 1 \le i \le j \le n\}$, $[X_i, Y_j] = [X_j, Y_i] = Z_{ij}$, $1 \le i \le j \le n$; $n = 2, 3$ (C_m $\Pi_0 = \{\gamma_{m-1}\}$, $m = 3, 4$).
- $\{X_i, Y_i, Z_{ij} : 1 \le i < j \le 3\}$, $[X_i, Y_j] = -[X_j, Y_i] = Z_{ij}$, $1 \le i < j \le 3$ (D_4, $\Pi_0 = \{\gamma_3, \gamma_4\}$).
- $\mathbb{R}X \ltimes_{\mathrm{ad}_X} (\mathbb{R}^{n-1} \oplus \mathbb{R}^{n-1})$ with $\mathrm{ad}_X = \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix}$ (A_m, $\Pi_0 = \{\gamma_k, \gamma_{k+1}\}$, $k = 1, \ldots, m-1$).

Thanks for your attention ...

Bibliography I

- Alekseevsky, D. and Perelomov, A., *Poisson and symplectic structures on Lie algebras I*, J. Geom. Phys. **22** (3) (1997), 191–211.
- del Barco, V., *Symplectic structures on free nilpotent Lie algebras*, preprint, 2011. Available at arXiv:1111.3280.
- del Barco, V., *Symplectic structures on nilmanifolds: an obstruction for their existences*, J. Lie Theory, **24** (3) (2014), 889–908.
- Dotti, I. and Tirao, P., *Symplectic structures on Heisenberg-type nilmanifolds*, Manuscr. Math., **102** (3) (2000), 383–401.
- Humphreys, J., Introduction to Lie algebras and representation theory. Graduate Texts in Mathematics, 9. Springer-Verlag, New York-Berlin, 1978.
- Kostant, B., *Lie algebra cohomology and the generalized Borel-Weil theorem*, Ann. of Math. (2), **74** (1961), 329–387.