7.2.2 Finite Rotations

NO: estos operadores actúan sobre estados de momento angular

The operator $\hat{R}_z(\phi)$ corresponding to a rotation (of the coordinates of a spinless particle) over a *finite* angle ϕ about the z-axis can be constructed in terms of the infinitesimal rotation operator (7.20) as follows. We divide the angle ϕ into N infinitesimal angles $\delta\phi$: $\phi = N\,\delta\phi$. The rotation over the finite angle ϕ can thus be viewed as a series of N consecutive infinitesimal rotations, each over the angle $\delta\phi$, about the z-axis, applied consecutively one after the other:

$$\hat{R}_z(\phi) = \hat{R}_z(N\delta\phi) = (R_z(\delta\phi))^N = \left(1 - i\frac{\delta\phi}{\hbar}\hat{L}_z\right)^N. \tag{7.27}$$

Since $\delta\phi = \phi/N$, and if $\delta\phi$ is infinitesimally small, we have

$$\hat{R}_z(\phi) = \lim_{N\to\infty}\prod_{k=1}^{N}\left(1 - \frac{i}{\hbar}\frac{\phi}{N}\hat{n}\cdot\vec{L}\right) = \lim_{N\to\infty}\left(1 - \frac{i}{\hbar}\frac{\phi}{N}\hat{L}_z\right)^N, \tag{7.28}$$

or

$$\boxed{\hat{R}_z(\phi) = e^{-i\phi\hat{L}_z/\hbar}}. \tag{7.29}$$

We can generalize this result to infer the rotation operator $\hat{R}_n(\phi)$ corresponding to a rotation over a finite angle ϕ around an axis $\vec{n}$:

$$\boxed{\hat{R}_n(\phi) = e^{-i\phi\vec{n}\cdot\hat{\vec{L}}/\hbar}}, \tag{7.30}$$

where $\vec{L}$ is the orbital angular momentum. This operator represents the rotation of the coordinates of a spinless particle over an angle ϕ about an axis $\vec{n}$.

The discussion that led to (7.30) was carried out for a spinless system. A more general study for a system with spin would lead to a relation similar to (7.30):

$$\boxed{\hat{R}_n(\phi) = e^{-\frac{i}{\hbar}\phi\vec{n}\cdot\hat{\vec{J}}}}, \tag{7.31}$$

where $\hat{\vec{J}}$ is the total angular momentum operator; this is known as the *rotation operator*. For instance, the rotation operator $\vec{R}_x(\phi)$ of a rotation through an angle ϕ about the x-axis is given by

$$\hat{R}_x(\phi) = e^{-i\phi\hat{J}_x/\hbar}. \tag{7.32}$$

The properties of $\hat{R}_n(\phi)$ are determined by those of the operators $\hat{J}_x$, $\hat{J}_y$, $\hat{J}_z$.

Remark

The Hamiltonian of a particle in a *central potential*, $\hat{H} = \hat{P}^2/(2m) + \hat{V}(r)$, is *invariant under spatial rotations* since, as shown in Chapter 6, it commutes with the orbital angular momentum:

$$[\hat{H},\ \hat{\vec{L}}] = 0 \qquad \Longrightarrow \qquad \left[\hat{H},\ e^{-i\phi\vec{n}\cdot\hat{\vec{L}}/\hbar}\right] = 0. \tag{7.33}$$

Due to this symmetry of space isotropy or rotational invariance, the *orbital angular momentum is conserved*[1]. So, in the case of particles moving in central potentials, the orbital angular momentum is a constant of the motion.

[1] In classical physics when a system is invariant under rotations, its total angular momentum is also conserved.