For every j, the number m has $(2j+1)$ allowed values: $m = -j, -j+1, \ldots, j-1, j$.

Since j_1 and j_2 are usually fixed, *we will be using, throughout the rest of this chapter, the shorthand notation* $\mid j,\ m\rangle$ *to abbreviate* $\mid j_1,\ j_2;\ j,\ m\rangle$. The set of vectors $\{\mid j,\ m\rangle\}$ form a complete and orthonormal basis:

$$\sum_j \sum_{m=-j}^{j} \mid j,\ m\rangle\langle j,\ m \mid = 1, \tag{7.119}$$

$$\langle j',\ m' \mid j,\ m\rangle = \delta_{j,j'}\delta_{m',m}. \tag{7.120}$$

The space where the total angular momentum $\hat{\vec{J}}$ operates is spanned by the basis $\{\mid j,\ m\rangle\}$; this space is known as a ~~*product*~~ sum *space*. It is important to know that this space is the same as the one spanned by $\{\mid j_1, j_2;\ m_1, m_2\rangle\}$; that is, the space which includes both subspaces 1 and 2. So the dimension of the space which is spanned by the basis $\{\mid j,\ m\rangle\}$ is also equal to $N = (2j_1+1)\times(2j_2+1)$ as specified by (7.106).

The issue now is to find the transformation that connects the bases $\{\mid j_1, j_2;\ m_1, m_2\rangle\}$ and $\{\mid j,\ m\rangle\}$.

7.3.1.1 Transformation between Bases: Clebsch–Gordan Coefficients

Let us now return to the addition of $\hat{\vec{J}}_1$ and $\hat{\vec{J}}_2$. This problem consists in essence of obtaining the eigenvalues of $\hat{\vec{J}}^2$ and $\hat{J}_z$ and of expressing the states $\mid j,\ m\rangle$ in terms of $\mid j_1, j_2;\ m_1, m_2\rangle$. We should mention that $\mid j,\ m\rangle$ is the state in which $\hat{\vec{J}}^2$ and $\hat{\vec{J}}_z$ have fixed values, $j(j+1)$ and m, but in general not a state in which the values of $\hat{\vec{J}}_{1z}$ and $\hat{\vec{J}}_{2z}$ are fixed; as for $\mid j_1, j_2;\ m_1, m_2\rangle$, it is the state in which $\hat{\vec{J}}_1^2$, $\hat{\vec{J}}_2^2$, $\hat{\vec{J}}_{1z}$, and $\hat{\vec{J}}_{2z}$ have fixed values.

The $\{\mid j_1, j_2;\ m_1, m_2\rangle\}$ and $\{\mid j,\ m\rangle\}$ bases can be connected by means of a transformation as follows. Inserting the identity operator as a sum over the complete basis $\mid j_1, j_2;\ m_1, m_2\rangle$, we can write

$$\begin{aligned}\mid j,\ m\rangle &= \left(\sum_{m_1=-j_1}^{j_1}\sum_{m_2=-j_2}^{j_2} \mid j_1, j_2;\ m_1, m_2\rangle\langle j_1, j_2;\ m_1, m_2 \mid\right)\mid j,\ m\rangle \\ &= \sum_{m_1 m_2}\langle j_1, j_2;\ m_1, m_2 \mid j,\ m\rangle \mid j_1, j_2;\ m_1, m_2\rangle,\end{aligned} \tag{7.121}$$

where we have used the normalization condition (7.104); since the bases $\{\mid j_1, j_2;\ m_1, m_2\rangle\}$ and $\{\mid j,\ m\rangle\}$ are both *normalized*, this transformation must be *unitary*. The coefficients $\langle j_1, j_2;\ m_1, m_2 \mid j,\ m\rangle$, which depend only on the quantities j_1, j_2, j, m_1, m_2, and m, *are the matrix elements of the unitary transformation which connects the* $\{\mid j,\ m\rangle\}$ and $\{\mid j_1, j_2;\ m_1, m_2\rangle\}$ bases. These coefficients are called the *Clebsch–Gordan coefficients*.

The problem of angular momentum addition reduces then to finding the Clebsch–Gordan coefficients $\langle j_1, j_2;\ m_1, m_2 \mid j,\ m\rangle$. These coefficients are taken to be *real by convention*; hence

$$\boxed{\langle j_1, j_2;\ m_1, m_2 \mid j,\ m\rangle = \langle j,\ m \mid j_1, j_2;\ m_1, m_2\rangle.} \tag{7.122}$$

Using (7.104) and (7.120) we can infer the orthonormalization relation for the Clebsch–Gordan coefficients:

$$\sum_{m_1 m_2}\langle j',\ m' \mid j_1, j_2;\ m_1, m_2\rangle\langle j_1, j_2;\ m_1, m_2 \mid j,\ m\rangle = \delta_{j',j}\delta_{m',m}, \tag{7.123}$$