Some interesting applications of (7.280) correspond to the cases where the vector operator $\hat{\vec{A}}$ is either the angular momentum, the position, or the linear momentum operator. Let us consider these three cases separately. First, substituting $\hat{\vec{A}} = \hat{\vec{J}}$ into (7.280), we recover the usual angular momentum commutation relations:

$$[\hat{J}_x, \hat{J}_y] = i\hbar \hat{J}_z, \qquad [\hat{J}_y, \hat{J}_z] = i\hbar \hat{J}_x, \qquad [\hat{J}_z, \hat{J}_x] = i\hbar \hat{J}_y. \tag{7.285}$$

Second, in the case of a spinless particle (i.e., $\hat{\vec{J}} = \hat{\vec{L}}$), and if $\hat{\vec{A}}$ is equal to the position operator, $\hat{\vec{A}} = \vec{R}$, then (7.280) will yield the following relations:

$$\left[\hat{x}, \hat{L}_x\right] = 0, \qquad \left[\hat{x}, \hat{L}_y\right] = i\hbar\hat{z}, \qquad \left[\hat{x}, \hat{L}_z\right] = -i\hbar\hat{y}, \tag{7.286}$$

$$\left[\hat{y}, \hat{L}_y\right] = 0, \qquad \left[\hat{y}, \hat{L}_z\right] = i\hbar\hat{x}, \qquad \left[\hat{y}, \hat{L}_x\right] = -i\hbar\hat{z}, \tag{7.287}$$

$$\left[\hat{z}, \hat{L}_z\right] = 0, \qquad \left[\hat{z}, \hat{L}_x\right] = i\hbar\hat{y}, \qquad \left[\hat{z}, \hat{L}_y\right] = -i\hbar\hat{x}. \tag{7.288}$$

Third, if $\hat{\vec{J}} = \hat{\vec{L}}$ and if $\hat{\vec{A}}$ is equal to the momentum operator, $\hat{\vec{A}} = \hat{\vec{P}}$, then (7.280) will lead to

$$\left[\hat{P}_x, \hat{L}_x\right] = 0, \qquad \left[\hat{P}_x, \hat{L}_y\right] = i\hbar\hat{P}_z, \qquad \left[\hat{P}_x, \hat{L}_z\right] = -i\hbar\hat{P}_y, \tag{7.289}$$

$$\left[\hat{P}_y, \hat{L}_y\right] = 0, \qquad \left[\hat{P}_y, \hat{L}_z\right] = i\hbar\hat{P}_x, \qquad \left[\hat{P}_y, \hat{L}_x\right] = -i\hbar\hat{P}_z, \tag{7.290}$$

$$\left[\hat{P}_z, \hat{L}_z\right] = 0, \qquad \left[\hat{P}_z, \hat{L}_x\right] = i\hbar\hat{P}_y, \qquad \left[\hat{P}_z, \hat{L}_y\right] = -i\hbar\hat{P}_x. \tag{7.291}$$

Now, introducing the operators

$$\hat{A}_{\pm} = \hat{A}_x \pm i\hat{A}_y, \tag{7.292}$$

and using the relations (7.282) to (7.284), we can show that

$$\left[\hat{J}_x,\ \hat{A}_{\pm}\right] = \mp\hbar\hat{A}_z, \qquad \left[J_y,\ \hat{A}_{\pm}\right] = -i\hbar\hat{A}_z, \qquad \left[J_z,\ \hat{A}_{\pm}\right] = \pm\hbar\hat{A}_{\pm}. \tag{7.293}$$

These relations in turn can be shown to lead to

$$\left[\hat{J}_{\pm},\ \hat{A}_{\pm}\right] = 0, \qquad \left[\hat{J}_{\pm},\ \hat{A}_{\mp}\right] = \pm 2\hbar\hat{A}_z. \tag{7.294}$$

Let us introduce the *spherical components* $\hat{A}_{-1}$, $\hat{A}_0$, $\hat{A}_1$ of the vector operator $\hat{A}$; they are defined in terms of the Cartesian coordinates $\hat{A}_x$, $\hat{A}_y$, $\hat{A}_z$ as follows:

$$\hat{A}_{\pm 1} = \mp\frac{1}{\sqrt{2}}(\hat{A}_x \pm i\hat{A}_y), \qquad \hat{A}_0 = \hat{A}_z. \tag{7.295}$$

For the particular case where $\hat{A}$ is equal to the position vector $\hat{R}$, we can express the components $\hat{R}_q$ (where $q = -1,\ 0,\ 1$),

$$\hat{R}_{\pm 1} = \mp\frac{1}{\sqrt{2}}(\hat{x} \pm i\hat{y}), \qquad \hat{R}_0 = \hat{z}, \tag{7.296}$$

in terms of the spherical coordinates (recall that $\hat{R}_1 = \hat{x} = r\sin\theta\cos\phi$, $\hat{R}_2 = \hat{y} = r\sin\theta\sin\phi$, and $\hat{R}_3 = \hat{z} = r\cos\theta$) as follows:

$$\hat{R}_{\pm 1} = \mp\frac{1}{\sqrt{2}} r e^{\pm i\phi}\sin\theta, \qquad \hat{R}_0 = r\cos\theta. \tag{7.297}$$

Using the relations (7.282) to (7.284) and (7.292) to (7.294), we can ascertain that

$$\boxed{\left[\hat{J}_z, \hat{A}_q\right] = \hbar q \hat{A}_q \qquad (q = -1, 0, 1),} \tag{7.298}$$

$$\boxed{\left[\hat{J}_\pm, \hat{A}_q\right] = \hbar\sqrt{2 - q(q \pm 1)}\hat{A}_{q\pm 1} \qquad (q = -1, 0, 1).} \tag{7.299}$$

7.4.3 Tensor Operators: Reducible and Irreducible Tensors

In general, a tensor of rank k has 3^k components, where 3 denotes the dimension of the space. For instance, a tensor such as

$$\hat{T}_{ij} = A_i B_j \qquad (i, j = x, y, z), \tag{7.300}$$

which is equal to the product of the components of two vectors $\vec{\hat{A}}$ and $\vec{B}$, is a *second-rank* tensor; this tensor has 3^2 components.

7.4.3.1 Reducible Tensors

A Cartesian tensor $\hat{T}_{ij}$ can be decomposed into three parts:

$$\hat{T}_{ij} = \hat{T}_{ij}^{(0)} + \hat{T}_{ij}^{(1)} + \hat{T}_{ij}^{(2)}, \tag{7.301}$$

with

$$\hat{T}_{ij}^{(0)} = \frac{1}{3}\delta_{i,j}\sum_{i=1}^{3}\hat{T}_{ii}, \tag{7.302}$$

$$\hat{T}_{ij}^{(1)} = \frac{1}{2}(\hat{T}_{ij} - \hat{T}_{ji}) \qquad (i \neq j), \tag{7.303}$$

$$\hat{T}_{ij}^{(2)} = \frac{1}{2}(\hat{T}_{ij} + \hat{T}_{ji}) - \hat{T}_{ij}^{(0)}. \tag{7.304}$$

Notice that if we add equations (7.302), (7.303), and (7.304), we end up with an identity relation: $\hat{T}_{ij} = \hat{T}_{ij}$.

The term $\hat{T}_{ij}^{(0)}$ has only one component and transforms like a scalar under rotations. The second term $\hat{T}_{ij}^{(1)}$ is an antisymmetric tensor of rank 1 which has three independent components; it transforms like a vector. The third term $\hat{T}_{ij}^{(2)}$ is a symmetric second-rank tensor with zero trace, and hence has five independent components; $\hat{T}_{ij}^{(2)}$ cannot be reduced further to tensors of lower rank. These five components define an irreducible second-rank tensor.

In general, any tensor of rank k can be decomposed into tensors of lower rank that are expressed in terms of linear combinations of its 3^k components. However, there always remain

$(2k+1)$ components that behave as a tensor of rank k which cannot be reduced further. These $(2k+1)$ components are symmetric and traceless with respect to any two indices; they form the components of an *irreducible tensor* of rank k.

Equations (7.301) to (7.304) show how to decompose a Cartesian tensor operator, $\hat{T}_{ij}$, into a sum of irreducible *spherical* tensor operators $\hat{T}_{ij}^{(0)}$, $T_{ij}^{(1)}$, $T_{ij}^{(2)}$. Cartesian tensors are not very suitable for studying transformations under rotations, because they are reducible whenever their rank exceeds 1. In problems that display spherical symmetry, such as those encountered in subatomic physics, spherical tensors are very useful simplifying tools. It is therefore interesting to consider irreducible spherical tensor operators.

7.4.3.2 Irreducible Spherical Tensors

Let us now focus only on the representation of irreducible tensor operators in spherical coordinates. An irreducible spherical tensor operator of rank k (k is integer) is a set of $(2k+1)$ operators $T_q^{(k)}$, with $q = -k, \ldots, k$, which transform in the same way as angular momentum under a rotation of axes. For example, the case $k = 1$ corresponds to a vector. The quantities $T_q^{(1)}$ are related to the components of the vector $\vec{\hat{A}}$ as follows (see (7.295)):

$$\hat{T}_{\pm 1}^{(1)} = \mp \frac{1}{\sqrt{2}}(\hat{A}_x \pm i\hat{A}_y), \qquad \hat{T}_0^{(1)} = \hat{A}_z. \tag{7.305}$$

In what follows we are going to study some properties of spherical tensor operators and then determine how they transform under rotations.

First, let us look at the various commutation relations of spherical tensors with the angular momentum operator. Since a vector operator is a tensor of rank 1, we can rewrite equations (7.298) to (7.299), respectively, as follows:

$$\left[\hat{J}_z,\ \hat{T}_q^{(1)}\right] = \hbar q \hat{T}_q^{(1)} \qquad (q = -1, 0, 1), \tag{7.306}$$

$$\left[\hat{J}_\pm,\ \hat{T}_q^{(1)}\right] = \hbar\sqrt{1(1+1) - q(q\pm 1)}\,\hat{T}_{q\pm 1}^{(1)}, \tag{7.307}$$

where we have adopted the notation $\hat{A}_q = \hat{T}_q^{(1)}$. We can easily generalize these two relations to any spherical tensor of rank k, $\hat{T}_q^{(k)}$, and obtain these commutators:

$$\boxed{\left[\hat{J}_z,\ \hat{T}_q^{(k)}\right] = \hbar q \hat{T}_q^{(k)} \qquad (q = -k, -k+1, \ldots, k-1, k),} \tag{7.308}$$

$$\boxed{\left[\hat{J}_\pm,\ \hat{T}_q^{(k)}\right] = \hbar\sqrt{k(k+1) - q(q\pm 1)}\,\hat{T}_{q\pm 1}^{(k)}.} \tag{7.309}$$

Using the relations

$$\langle k,\ q' \mid \hat{J}_z \mid k,\ q\rangle = \hbar q \langle k,\ q' \mid k,\ q\rangle = \hbar q \delta_{q',q}, \tag{7.310}$$

$$\langle k,\ q' \mid \hat{J}_\pm \mid k,\ q\rangle = \hbar\sqrt{k(k+1) - q(q\pm 1)}\delta_{q',q\pm 1}, \tag{7.311}$$

along with (7.308) and (7.309), we can write

$$\sum_{q'=-k}^{k} \hat{T}_{q'}^{(k)} \langle k,\ q' \mid \hat{J}_z \mid k,\ q\rangle = \hbar q \hat{T}_q^{(k)} = \left[\hat{J}_z,\ \hat{T}_q^{(k)}\right], \tag{7.312}$$

Now, taking the matrix elements of (7.309) between $| j,\ m\rangle$ and $|j',\ m'\rangle$, we obtain

$$\begin{aligned}
&\sqrt{(j' \pm m')(j' \mp m' + 1)}\,\langle j', m' \mp 1|\hat{T}_q^{(k)} \mid j,\ m\rangle \\
&\quad = \sqrt{(j \mp m)(j \pm m + 1)}\,\langle j', m'|\hat{T}_q^{(k)} \mid j,\ m \pm 1\rangle \\
&\quad\quad + \sqrt{(k \mp q)(k \pm q + 1)}\,\langle j', m'|\hat{T}_{q\pm 1}^{(k)} \mid j,\ m\rangle. \qquad (7.322)
\end{aligned}$$

This equation has a structure which is identical to the recursion relation (7.150). For instance, substituting $j = j', m = m', j_1 = j, m_1 = m, j_2 = k, m_2 = q$ into (7.150), we end up with

$$\begin{aligned}
&\sqrt{(j' \pm m')(j' \mp m' + 1)}\,\langle j', m' \mp 1 \mid j, k;\ m, q\rangle \\
&\quad = \sqrt{(j \mp m)(j \pm m + 1)}\,\langle j', m' \mid j, k;\ m \pm 1, q\rangle \\
&\quad\quad + \sqrt{(k \mp q)(k \pm q + 1)}\,\langle j', m' \mid j, k;\ m, q \pm 1\rangle. \qquad (7.323)
\end{aligned}$$

A comparison of (7.320) with (7.321) and (7.322) with (7.323) suggests that the dependence of $\langle j', m'|T_q^{(k)} \mid j,\ m\rangle$ on m', m, q is through a Clebsch–Gordan coefficient. The dependence, however, of $\langle j', m'|T_q^{(k)} \mid j,\ m\rangle$ on j', j, k has yet to be determined.

We can now state the **Wigner–Eckart theorem:** *The matrix elements of spherical tensor operators $\hat{T}_q^{(k)}$ with respect to angular momentum eigenstates* $| j,\ m\rangle$ are given by

$$\boxed{\langle j', m'|\hat{T}_q^{(k)} \mid j,\ m\rangle = \langle j, k;\ m, q|j',\ m'\rangle\langle j' \parallel \hat{T}^{(k)} \parallel j\rangle.} \qquad (7.324)$$

The factor $\langle j' \parallel \hat{T}^{(k)} \parallel j\rangle$, which depends only on j', j, k, is called the *reduced matrix element* of the tensor $\hat{T}_q^{(k)}$ (note that the double bars notation is used to distinguish the reduced matrix elements, $\langle j' \parallel \hat{T}^{(k)} \parallel j\rangle$, from the matrix elements, $\langle j',\ m'|\hat{T}_q^{(k)} \mid j,\ m\rangle$). The theorem implies that the matrix elements $\langle j',\ m'|\hat{T}_q^{(k)} \mid j,\ m\rangle$ are written as the product of two terms: a Clebsch–Gordan coefficient $\langle j, k;\ m, q|j',\ q'\rangle$—which depends on the geometry of the system (i.e., the orientation of the system with respect to the z-axis), but not on its dynamics (i.e., j', j, k)—and a dynamical factor, the reduced matrix element, which does not depend on the orientation of the system in space (m', q, m). The quantum numbers m', m, q—which specify the projections of the angular momenta $\hat{\vec{J}}\,'$, $\hat{\vec{J}}$, and $\vec{k}$ onto the z-axis—give the orientation of the system in space, for they specify its orientation with respect to the z-axis. As for j', j, k, they are related to the dynamics of the system, not to its orientation in space.

Wigner–Eckart theorem for a scalar operator

The simplest application of the Wigner–Eckart theorem is when dealing with a scalar operator $\hat{B}$. As seen above, a scalar is a tensor of rank $k = 0$; hence $q = 0$ as well; thus, equation (7.324) yields

$$\langle j', m'|\hat{B} \mid j,\ m\rangle = \langle j, 0;\ m, 0|j',\ m'\rangle\langle j' \parallel \hat{B} \parallel j\rangle = \langle j' \parallel \hat{B} \parallel j\rangle\delta j' j\,\delta m' m, \qquad (7.325)$$

since $\langle j, 0;\ m, 0|j',\ m'\rangle = \delta j' j\,\delta m' m$.

Wigner–Eckart theorem for a vector operator

As shown in (7.305), a vector is a tensor of rank 1: $T^{(1)} = A^{(1)} = \hat{\vec{A}}$, with $A_0^{(1)} = A_0 = A_z$ and $A_{\pm 1}^{(1)} = A_{\pm 1} = \mp(\hat{A}_x \pm i\hat{A}_y)/\sqrt{2}$. An application of (7.324) to the q-component of a vector