particle and the scattering potential is much smaller than the particle's incident kinetic energy, the scattered wave can be considered to be a plane wave.

Example 11.2

(a) Calculate the differential cross section in the first Born approximation for a Coulomb potential $V(r) = Z_1 Z_2 e^2/r$, where $Z_1 e$ and $Z_1 e$ are the charges of the projectile and target particles, respectively.

(b) To have a quantitative idea about the cross section derived in (a), consider the scattering of an alpha particle (i.e., a helium nucleus with $Z_1 = 2$ and $A_1 = 4$) from a gold nucleus ($Z_2 = 79$ and $A_2 = 197$). (i) If the scattering angle of the alpha particle in the Lab frame is $\theta_1 = 60°$, find its scattering angle θ in the CM frame. (ii) If the incident energy of the alpha particle is 8 MeV, find a numerical estimate for the cross section derived in (a).

Solution

In the case of a Coulomb potential, $V(r) = Z_1 Z_2 e^2/r$, equation (11.70) becomes

$$\frac{d\sigma}{d\Omega} = \frac{4Z_1^2 Z_2^2 e^4 \mu^2}{\hbar^4 q^2} \left| \int_0^\infty \sin(qr)\, dr \right|^2, \tag{11.75}$$

where

¡no existe!

$$\int_0^\infty \sin(qr)\, dr = \lim_{\lambda \to 0} \int_0^\infty e^{-\lambda r} \sin(qr)\, dr = \frac{1}{2i} \lim_{\lambda \to 0} \left[\int_0^\infty e^{-(\lambda - iq)r}\, dr - \int_0^\infty e^{-(\lambda + iq)r}\, dr \right]$$
$$= \frac{1}{2i} \lim_{\lambda \to 0} \left[\frac{1}{\lambda - iq} - \frac{1}{\lambda + iq} \right] = \frac{1}{q}. \tag{11.76}$$

Now, since $q = 2k \sin(\theta/2)$, an insertion of (11.76) into (11.75) leads to

$$\frac{d\sigma}{d\Omega} = \left(\frac{2Z_1 \mu Z_2 e^2}{\hbar^2 q^2} \right)^2 = \left(\frac{Z_1 Z_2 \mu e^2}{2\hbar^2 k^2} \right)^2 \sin^{-4}\left(\frac{\theta}{2}\right) = \frac{Z_1^2 Z_2^2 e^4}{16E^2} \sin^{-4}\left(\frac{\theta}{2}\right), \tag{11.77}$$

where $E = \hbar^2 k^2/2\mu$ is the kinetic energy of the incident particle. This relation is known as the *Rutherford formula* or the Coulomb cross section.

(b) (i) Since the mass ratio of the alpha particle to the gold nucleus is roughly equal to the ratio of their atomic masses, $m_1/m_2 = A_1/A_2 = \frac{4}{197} = 0.0203$, and since $\theta_1 = 60°$, equation (11.14) yields the value of the scattering angle in the CM frame:

$$\tan 60° = \frac{\sin\theta}{\cos\theta + 0.0203} \qquad \Longrightarrow \qquad \theta = 61°. \tag{11.78}$$

(ii) The numerical estimate of the cross section can be made easier by rewriting (11.77) in terms of the fine structure constant $\alpha = e^2/\hbar c = \frac{1}{137}$ and $\hbar c = 197.33$ MeV fm:

$$\frac{d\sigma}{d\Omega} = \frac{Z_1^2 Z_2^2}{16E^2} \left(\frac{e^2}{\hbar c}\right)^2 (\hbar c)^2 \sin^{-4}\left(\frac{\theta}{2}\right) = \left(\frac{Z_1 Z_2 \alpha}{4}\right)^2 \left(\frac{\hbar c}{E}\right)^2 \sin^{-4}\left(\frac{\theta}{2}\right). \tag{11.79}$$

Since $Z_1 = 2$, $Z_2 = 79$, $\theta = 61°$, $\alpha = \frac{1}{137}$, $\hbar c = 197.33$ MeV fm, and $E = 8$ MeV, we have

$$\frac{d\sigma}{d\Omega} = \left(\frac{2 \times 79}{4 \times 137}\right)^2 \left(\frac{197.33\text{ MeV fm}}{8\text{ MeV}}\right)^2 \sin^{-4}(30.5°)$$
$$= 30.87\text{ fm}^2 = 0.31 \times 10^{-28}\text{ m}^2 = 0.31\text{ barn}, \tag{11.80}$$