# Gas de Bose-Einstein

$$Z_\mu^{(\mathrm{BE})}(T,V) = \sum_{n_o=0}^{\infty} \cdots \sum_{n_\infty=0}^{\infty} e^{-\beta \sum_{j=0}^{\infty} n_j(\epsilon_j-\mu)} = \prod_{j=0}^{\infty} \left[\frac{1}{1-e^{-\beta(\epsilon_j-\mu)}}\right]$$

$\epsilon_j$ depende solo de $\boldsymbol{p}_\ell$ $\quad\rightarrow\quad j=(\ell, m_s) \qquad m_s = -s, -s+1, \ldots, s$

$g_s = 2s+1$ proyecciones del espín $s$

$$\hookrightarrow \quad \Omega_{\mathrm{BE}}(T,V,\mu) = kT\, g_s \sum_{\ell=0}^{\infty} \ln\left[1-e^{-\beta(\epsilon_\ell-\mu)}\right]$$

$$\langle N\rangle = -\left(\frac{\partial\Omega_{\mathrm{BE}}}{\partial\mu}\right)_{T,V} = \sum_{\ell=0}^{\infty} \frac{1\; g_s}{e^{\beta(\epsilon_\ell-\mu)}-1} = \sum_{\ell=0}^{\infty} \langle n_\ell\rangle$$

$$\langle n_\ell\rangle = \frac{z\, g_s}{e^{\beta\epsilon_\ell}-z} \geq 0 \quad\Rightarrow\quad e^{\beta\epsilon_\ell} > z \;\; \forall\, \ell \quad\rightarrow\quad \mathbf{0 < z < 1} \quad\Rightarrow\quad \boldsymbol{\mu < 0}$$

$\rightarrow$ estado fundamental : $\quad \langle n_o\rangle = \dfrac{z\, g_s}{1-z}$

# Gas de Bose-Einstein

$$\Omega_{\mathrm{BE}}(T,V,\mu) = kT\, g_s \sum_{\ell=0}^{\infty} \ln\left[1 - e^{-\beta(\epsilon_\ell - \mu)}\right] \qquad \langle N\rangle = \sum_{\ell=0}^{\infty} \frac{1\; g_s}{e^{\beta(\epsilon_\ell-\mu)} - 1}$$

límite termodinámico : $\langle N\rangle$ , $V$ muy grandes $\Rightarrow$ $\boldsymbol{p} = \hbar \boldsymbol{k}$ y $\epsilon = \dfrac{\boldsymbol{p}^2}{2m}$ continuas

$$\sum_{\ell} f(\boldsymbol{\ell})\, \Delta\ell_x\, \Delta\ell_y\, \Delta\ell_z \rightarrow \frac{4\pi V}{h^3} \int \mathrm{d}p\; p^2\; f(p) = \frac{4\sqrt{2}\pi m^{3/2} V}{h^3} \int \mathrm{d}\epsilon\; \sqrt{\epsilon}\; f(\epsilon)$$

$$\frac{\Omega_{\mathrm{BE}}(T,V,\mu)}{V} = \frac{kT\; g_s}{V} \ln(1-z) - \frac{kT\; g_s}{\lambda^3}\, g_{5/2}(z)$$

$$\frac{1}{v} \equiv \frac{\langle N\rangle}{V} = \frac{\langle n_o\rangle}{V} + \frac{1\; g_s}{\lambda^3}\, g_{3/2}(z) \qquad \langle n_o\rangle = \frac{z\; g_s}{1-z}$$

$$g_{5/2}(z) = -\frac{4}{\sqrt{\pi}} \int_0^{\infty} \mathrm{d}x\; x^2\, \ln(1 - z e^{-x^2}) = \sum_{j=1}^{\infty} \frac{z^j}{j^{5/2}}$$

$$g_{3/2}(z) = \frac{4}{\sqrt{\pi}} \int_0^{\infty} \mathrm{d}x\; \frac{x^2}{z^{-1} e^{x^2} - 1} = z\frac{\partial}{\partial z} g_{5/2}(z) = \sum_{j=1}^{\infty} \frac{z^j}{j^{3/2}}$$

# Gas de Bose-Einstein

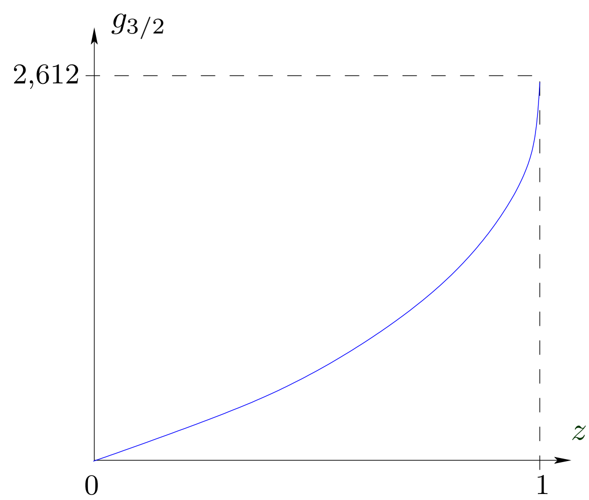
$$\frac{1}{v} \equiv \frac{\langle N \rangle}{V} = \frac{\langle n_o \rangle}{V} + \frac{1}{\lambda^3}\, g_s \, g_{3/2}(z) \qquad \langle n_o \rangle = \frac{z\, g_s}{1-z} = \frac{1\, g_s}{e^{-\beta\mu}-1}$$

$T$ suficientemente altas o $v$ grandes (bajas densidades)

término $\ell = 0$ aporta un *diferencial*

$\hookrightarrow$ innecesario separarlo

$g_{3/2}(z) \geq 0$ $\quad \left(g'_{3/2}(z) \geq 0\right)$ acotada




$T$ ó $v$ suficientemente bajos :

$g_{3/2}$ llega al máximo valor $(= 2,612)$

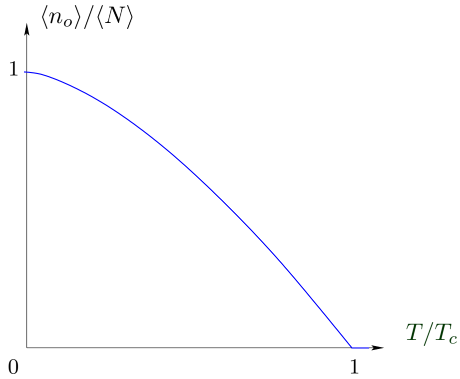
para $z$ máximo

$\hookrightarrow$ ¡ el término $g_{3/2}(z)/\lambda^3$ no alcanza !

***condensación de Bose-Einstein***

el aporte de $\boldsymbol{p} = 0$ deja de ser diferencial $\qquad \langle n_o \rangle / V \neq 0$

# Gas de Bose-Einstein



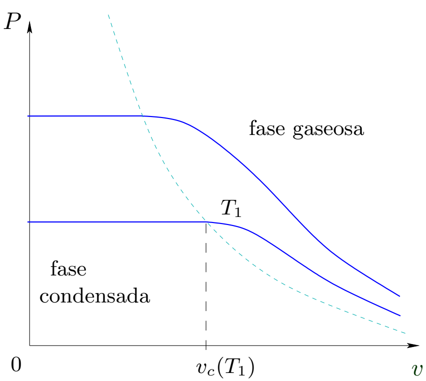

fracción de partículas en $\boldsymbol{p} = 0$

$$\frac{\langle n_o \rangle}{\langle N \rangle} = 1 - \frac{2,612\, g_s}{\lambda^3} \frac{V}{\langle N \rangle} = 1 - \left( \frac{T}{T_c} \right)^{3/2}$$

(si $T > T_c$ esta fracción se anula)

$$\begin{cases} P^{>} = \dfrac{kT\, g_s}{\lambda^3}\, g_{5/2}(z) & (T > T_c \quad \text{ó} \quad v > v_c) \\ P^{<} = \dfrac{kT\, g_s}{\lambda^3}\, g_{5/2}(1) & (T < T_c \quad \text{ó} \quad v < v_c) \end{cases}$$

↘ *independiente* de $v$




$$P^{<} \underset{T \to 0}{\longrightarrow} 0 \qquad \boldsymbol{p} = 0 \quad \text{(fase condensada)}$$

**no** aporta a la presión

*coexistencia*

# Gas de Bose-Einstein

ec. Clausius-Clapeyron :

$g_s = 1$

$$\frac{\mathrm{d}P_o}{\mathrm{d}T} = \frac{5}{2}\,\frac{k\,g_{5/2}(1)}{\lambda^3} = \frac{1}{T\,v_c}\left[\frac{5}{2}\,kT\frac{g_{5/2}(1)}{g_{3/2}(1)}\right] = \frac{\ell}{T\,\Delta v}$$

$\Delta v = v_c \nearrow \qquad (v_{\mathrm{cond}} = 0)$

calor latente de la transformación (por partícula) :

$$\ell = \frac{5}{2}\,kT\,\frac{g_{5/2}(1)}{g_{3/2}(1)}$$

discontinuidades en derivadas *primeras* de energía libre de Gibbs

$\rightarrow$ **transición de fase de *primer orden***

$$\Omega = -PV = -\frac{kT\,V}{\lambda^3}\,g_{5/2}(z)$$

$$S = -\left(\frac{\partial\Omega}{\partial T}\right)_{V,\mu} \rightarrow \begin{cases} S^{>} = \frac{5}{2}\,\frac{k\,V}{\lambda^3}\,g_{5/2}(z) - \langle N\rangle k\,\ln z \\ S^{<} = \underbrace{\frac{5}{2}\,\frac{k\,V}{\lambda^3}\,g_{5/2}(1)} \end{cases}$$

solo aporta la fase gaseosa $\quad (\boldsymbol{s}_{\mathrm{cond}} = \mathbf{0})$

# Gas de Fermi-Dirac (Reichl-Huang y Schwabl)

**fermiones** : *exclusión* $\quad (S=1) \qquad Z_\mu^{(\mathrm{FD})}(T,V) = \sum_{n_o=0}^{1} \cdots \sum_{n_\infty=0}^{1} e^{-\beta \sum_{j=0}^{\infty} n_j(\epsilon_j - \mu)}$

$\epsilon_j$ depende solo de $\boldsymbol{p}_\ell \quad \rightarrow \quad j = (\ell, m_s) \qquad m_s = -s, -s+1, \ldots, s$

$g_s = 2s+1$ proyecciones del espín $s$

$$Z_\mu^{\mathrm{FD}}(T,V) = \prod_{\ell=0}^{\infty} \left(1 + z e^{-\beta\,\epsilon_\ell}\right)^{g_s}$$

**($g_s \neq 1$ siempre)**

$$\hookrightarrow \quad \Omega_{\mathrm{FD}}(T,V,\mu) = -kT g_s \sum_{\ell=0}^{\infty} \ln\left(1 + z e^{-\beta\,\epsilon_\ell}\right)$$

$$\langle N \rangle = -\left(\frac{\partial \Omega_{\mathrm{FD}}}{\partial \mu}\right)_{T,V} = g_s \sum_{\ell=0}^{\infty} \frac{1}{z^{-1} e^{\beta\,\epsilon_\ell} + 1} = \sum_{\ell=0}^{\infty} \frac{g_s}{e^{\beta(\epsilon_\ell - \mu)} + 1} = \sum_{\ell=0}^{\infty} g_s \langle n_\ell \rangle$$

$\hookrightarrow$ siempre se cumple $g_s \langle n_\ell \rangle \leq g_s$ (menos mal : exclusión)

***no hay condensación***

# Gas de Fermi-Dirac

$$g_s\langle n_\ell\rangle = \frac{g_s}{z^{-1}\,e^{\beta\,\epsilon_\ell}+1} = \frac{g_s}{e^{\beta(\epsilon_\ell-\mu)}+1}$$

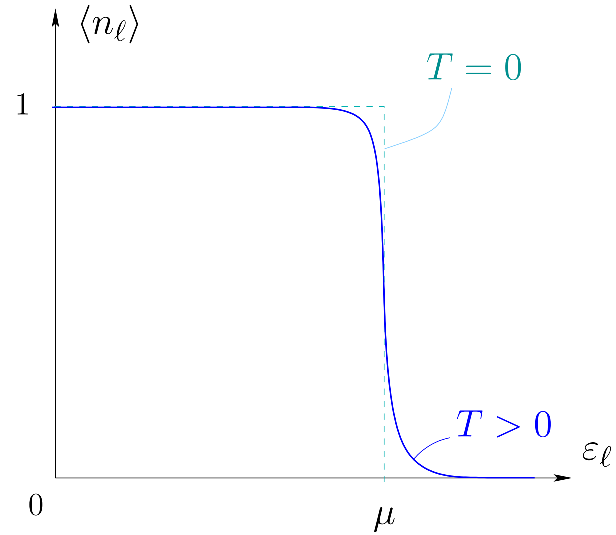
$T=0 \quad \leftrightarrow \quad \beta=\infty$ :

*todos* los niveles se despueblan

excepto cuando $\epsilon_\ell < \mu$

$T \gtrsim 0$ , la distribución se aparta suavemente

de la correspondiente a $T=0$




límite termodinámico : $\epsilon$ (individual) se torna continua

$$\hookrightarrow \quad n(\epsilon) = \frac{1}{e^{\beta\,(\epsilon-\mu)}+1}$$

$$\sum_{\boldsymbol{\ell}} f(p_{\boldsymbol{\ell}})\,\Delta\ell_x\,\Delta\ell_y\,\Delta\ell_z \;\rightarrow\; \frac{4\pi V}{h^3}\int \mathrm{d}p\;p^2\;f(p) = \frac{4\sqrt{2}\pi m^{3/2}V}{h^3}\int \mathrm{d}\epsilon\;\sqrt{\epsilon}\;f(\epsilon)$$

# Gas de Fermi-Dirac

$$\Omega_{\rm FD}(T,V,\mu) = -kT g_s \sum_{\ell=0}^{\infty} \ln\left(1 + z e^{-\beta\,\epsilon_\ell}\right) \quad \to \quad \Omega_{\rm FD}(T,V,\mu) = -\frac{g_s kTV}{\lambda^3}\, f_{5/2}(z)$$

$$f_{5/2}(z) \equiv \frac{4}{\sqrt{\pi}} \int_0^\infty \mathrm{d}x\, x^2\, \ln\left(1 + z\, e^{-x^2}\right) \quad \left[= \sum_{j=1}^{\infty} \frac{(-1)^{j+1}\, z^j}{j^{5/2}} \quad \text{si } z < 1\right]$$

sustitución $x^2 = p^2/(2mkT)$

desarrollo en serie del logaritmo

$(T,V,\mu) \quad \leftrightarrow \quad (\beta,V,z)$ variables independientes

(ejercicio) $$U = -\left(\frac{\partial}{\partial\beta} \ln Z(\beta,V,z)\right)_{V,z}$$

$\leftarrow$ OJO: dice $V, z$ y no $V, \mu$

$$\hookrightarrow \quad U = \frac{3}{2}\, g_s\, \frac{kTV}{\lambda^3}\, f_{5/2}(z)$$

bosones : $U = \frac{3}{2}\, g_s'\, \frac{kTV}{\lambda^3}\, g_{5/2}(z)$

# Gas de Fermi-Dirac

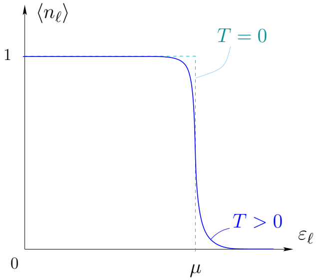
$$U = \frac{3}{2}\, g_s\, \frac{kTV}{\lambda^3}\, f_{5/2}(z) \qquad \text{bosones}: \ U = \frac{3}{2}\, g'_s\, \frac{kTV}{\lambda^3}\, g_{5/2}(z)$$

como $\Omega = -PV \quad \rightarrow \quad U = \frac{3}{2}\, PV$ (igual que Bose-Einstein)

$$\langle N \rangle = z \left( \frac{\partial \ln Z(\beta, V, z)}{\partial z} \right)_{\beta, V} = \frac{g_s V}{\lambda^3}\, f_{3/2}(z)$$

$$f_{3/2}(z) \equiv \frac{4}{\sqrt{\pi}} \int_0^\infty \mathrm{d}x\, x^2\, \frac{1}{z^{-1}\, e^{x^2} + 1} = z\, \frac{\partial f_{5/2}(z)}{\partial z} \qquad \left[ = \sum_{j=1}^{\infty} \frac{(-1)^{j+1}\, z^j}{j^{3/2}} \quad \text{si } z < 1 \right]$$

$f_{3/2}(z)$ creciente con $z$ (ejercicio) : dado $\frac{\langle N \rangle}{V}$ , cuando crece $T$ debe decrecer $z$




$\hookrightarrow$ para $T \rightarrow \infty$ , $z \rightarrow 0 \quad \leftrightarrow \quad \mu \rightarrow -\infty$

$\hookrightarrow$ para $T \rightarrow 0$ , $z \rightarrow \infty \quad \leftrightarrow \quad \frac{\mu}{T} \rightarrow +\infty \qquad \underset{T \rightarrow 0}{\mu > 0}$ acotado