Gas de Fermi-Dirac (Reichl-Huang y Schwabl)

fermiones : *exclusión* $(S=1)$ $$Z_\mu^{(\mathrm{FD})}(T,V) = \sum_{n_o=0}^{1} \cdots \sum_{n_\infty=0}^{1} e^{-\beta \sum_{j=0}^{\infty} n_j(\epsilon_j-\mu)}$$

ϵ_j depende solo de $\boldsymbol{p}_\ell$ $\rightarrow$ $j=(\ell, m_s)$ $\quad m_s = -s, -s+1, \ldots, s$

$g_s = 2s+1$ proyecciones del espín s

$$Z_\mu^{\mathrm{FD}}(T,V) = \prod_{\ell=0}^{\infty} \left(1 + z e^{-\beta\,\epsilon_\ell}\right)^{g_s}$$

($g_s \neq 1$ siempre)

$$\hookrightarrow \quad \Omega_{\mathrm{FD}}(T,V,\mu) = -kT g_s \sum_{\ell=0}^{\infty} \ln\left(1 + z e^{-\beta\,\epsilon_\ell}\right)$$

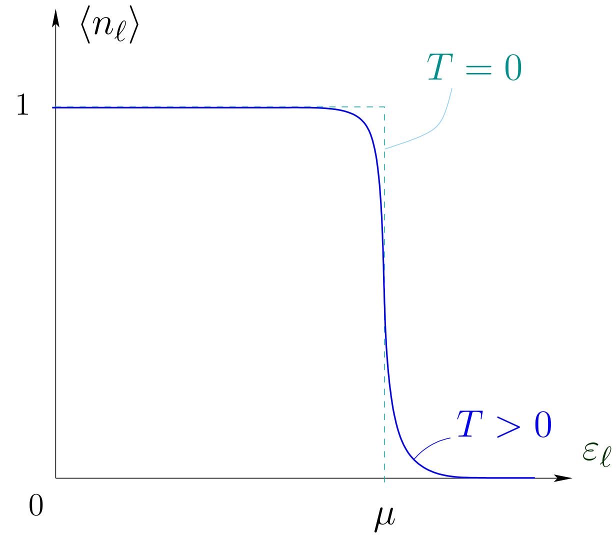
$$\langle N \rangle = -\left(\frac{\partial \Omega_{\mathrm{FD}}}{\partial \mu}\right)_{T,V} = g_s \sum_{\ell=0}^{\infty} \frac{1}{z^{-1} e^{\beta\,\epsilon_\ell} + 1} = \sum_{\ell=0}^{\infty} \frac{g_s}{e^{\beta(\epsilon_\ell - \mu)} + 1} = \sum_{\ell=0}^{\infty} g_s \langle n_\ell \rangle$$

$\hookrightarrow$ siempre se cumple $g_s\langle n_\ell\rangle \leq g_s$ (menos mal : exclusión)

no hay condensación

Gas de Fermi-Dirac

$$g_s\langle n_\ell\rangle = \frac{g_s}{z^{-1}\, e^{\beta\,\epsilon_\ell} + 1} = \frac{g_s}{e^{\beta(\epsilon_\ell-\mu)} + 1}$$



$T = 0 \quad \leftrightarrow \quad \beta = \infty$:

todos los niveles se despueblan

excepto cuando $\epsilon_\ell < \mu$

$T \gtrsim 0$, la distribución se aparta suavemente

de la correspondiente a $T = 0$

límite termodinámico : ϵ (individual) se torna continua

$$\hookrightarrow \quad n(\epsilon) = \frac{1}{e^{\beta\,(\epsilon-\mu)} + 1}$$

$$\sum_{\boldsymbol{\ell}} f(p_{\boldsymbol{\ell}})\,\Delta\ell_x\,\Delta\ell_y\,\Delta\ell_z \;\rightarrow\; \frac{4\pi V}{h^3}\int \mathrm{d}p\; p^2\; f(p) = \frac{4\sqrt{2}\pi m^{3/2} V}{h^3}\int \mathrm{d}\epsilon\,\sqrt{\epsilon}\; f(\epsilon)$$

Gas de Fermi-Dirac

$$\Omega_{\mathrm{FD}}(T,V,\mu) = -kT g_s \sum_{\ell=0}^{\infty} \ln\left(1 + z e^{-\beta\,\epsilon_\ell}\right) \quad \rightarrow \quad \Omega_{\mathrm{FD}}(T,V,\mu) = -\frac{g_s kTV}{\lambda^3}\, f_{5/2}(z)$$

$$f_{5/2}(z) \equiv \frac{4}{\sqrt{\pi}} \int_0^{\infty} \mathrm{d}x\, x^2\, \ln\left(1 + z\, e^{-x^2}\right) \quad \left[= \sum_{j=1}^{\infty} \frac{(-1)^{j+1}\, z^j}{j^{5/2}} \quad \text{si } z < 1 \right]$$

sustitución $x^2 = p^2/(2mkT)$

desarrollo en serie del logaritmo

$(T,V,\mu) \quad \leftrightarrow \quad (\beta, V, z)$ variables independientes

(ejercicio)

$$U = -\left(\frac{\partial \ln Z(\beta,V,z)}{\partial \beta}\right)_{V,z}$$

$\leftarrow$ OJO: dice V, z y no V, μ

$$\hookrightarrow \quad U = \frac{3}{2}\, g_s\, \frac{kTV}{\lambda^3}\, f_{5/2}(z)$$

bosones : $U = \frac{3}{2}\, g_s'\, \frac{kTV}{\lambda^3}\, g_{5/2}(z)$

Gas de Fermi-Dirac

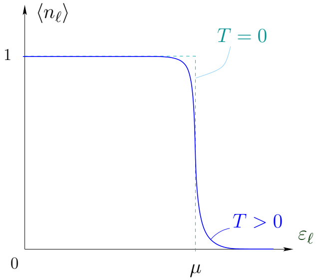
$$U = \frac{3}{2}\, g_s \, \frac{kTV}{\lambda^3}\, f_{5/2}(z) \qquad \text{bosones}: \; U = \frac{3}{2}\, g_s' \, \frac{kTV}{\lambda^3}\, g_{5/2}(z)$$

como $\Omega = -PV \quad \rightarrow \quad U = \frac{3}{2}\, PV$ (igual que Bose-Einstein)

$$\langle N \rangle = z \left(\frac{\partial \ln Z(\beta, V, z)}{\partial z} \right)_{\beta, V} = \frac{g_s V}{\lambda^3}\, f_{3/2}(z)$$

$$f_{3/2}(z) \equiv \frac{4}{\sqrt{\pi}} \int_0^\infty \mathrm{d}x\, x^2\, \frac{1}{z^{-1}\, e^{x^2} + 1} = z\, \frac{\partial f_{5/2}(z)}{\partial z} \qquad \left[= \sum_{j=1}^{\infty} \frac{(-1)^{j+1}\, z^j}{j^{3/2}} \quad \text{si } z < 1 \right]$$

$f_{3/2}(z)$ creciente con z (ejercicio) : dado $\frac{\langle N \rangle}{V}$, cuando crece T debe decrecer z

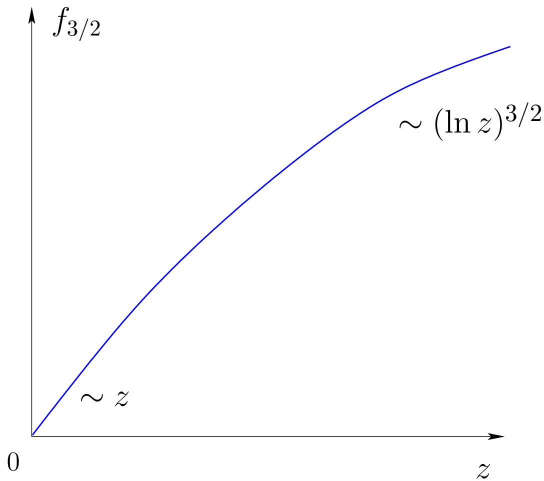


$\hookrightarrow$ para $T \rightarrow \infty$, $z \rightarrow 0 \quad \leftrightarrow \quad \mu \rightarrow -\infty$

$\hookrightarrow$ para $T \rightarrow 0$, $z \rightarrow \infty \quad \leftrightarrow \quad \frac{\mu}{T} \rightarrow +\infty \qquad \mu \underset{T \rightarrow 0}{>} 0$ acotado

Gas de Fermi-Dirac

$f_{3/2}(z)$ creciente con z $\quad\rightarrow\quad$ $e^{\beta\mu} = \underset{T\to\infty}{z \rightarrow 0} \quad \leftrightarrow \quad \underset{T\to\infty}{\mu \rightarrow -\infty}$



$\underset{T\to 0}{z \rightarrow \infty} \quad \leftrightarrow \quad \dfrac{\mu}{T} \rightarrow +\infty$: $\quad \underset{T\to 0}{\mu > 0}$ acotado

análisis para $T \rightarrow 0$:

$f_{3/2}(z)$ para z grandes

$$U = \sum_{\ell} g_s \,\langle n_\ell \rangle \epsilon_\ell \;\rightarrow\; \frac{4\sqrt{2}\,\pi\, g_s m^{3/2}\, V}{h^3} \int_0^\infty \mathrm{d}\epsilon\, \sqrt{\epsilon}\; \epsilon\, n(\epsilon) = \int_0^\infty \mathrm{d}\epsilon\; g(\epsilon)\, \epsilon\; n(\epsilon)$$

$$\langle N \rangle = \sum_{\ell} g_s \,\langle n_\ell \rangle \rightarrow \int_0^\infty \mathrm{d}\epsilon\; g(\epsilon)\; n(\epsilon)$$

densidad de estados $g(\epsilon)$

Gas de Fermi-Dirac - T bajas

$$U = \int_0^\infty \mathrm{d}\epsilon \; g(\epsilon) \, \epsilon \, n(\epsilon) \qquad\qquad \langle N \rangle = \int_0^\infty \mathrm{d}\epsilon \; g(\epsilon) \; n(\epsilon)$$

en general :

$$I = \int_0^\infty \mathrm{d}\epsilon \; f(\epsilon) \; n(\epsilon)$$

desarrollo de Sommerfeld : (Schwabl)

$$I = \int_0^\mu \mathrm{d}\epsilon \; f(\epsilon) + \int_0^\infty \mathrm{d}\epsilon \; f(\epsilon) \;\; \big[n(\epsilon) - \Theta(\mu - \epsilon) \big]$$

$\uparrow$ función escalón de Heaviside

$\nearrow$ primer término : límite $T \to 0$

definiendo $f(-\epsilon) = f(\epsilon)$ para $\epsilon < 0$

$n(\epsilon)$ sólo difiere mucho de $\Theta(\mu - \epsilon)$ cerca de $\epsilon = \mu$

$$\epsilon < 0 < \mu(T \to 0) : \; n(\epsilon) = \frac{1}{1 + e^{-\beta(\mu - \epsilon)}} \approx 1 + \mathcal{O}(e^{-\beta(\mu - \epsilon)})$$

$$I \approx \int_0^\mu \mathrm{d}\epsilon \; f(\epsilon) + \int_{-\infty}^\infty \mathrm{d}\epsilon \; f(\epsilon) \;\; \big[n(\epsilon) - \Theta(\mu - \epsilon) \big]$$

Gas de Fermi-Dirac - T bajas

$$I \approx \int_0^{\mu} \mathrm{d}\epsilon\, f(\epsilon) + \int_{-\infty}^{\infty} \mathrm{d}\epsilon\, f(\epsilon)\, \underbrace{\big[n(\epsilon) - \Theta(\mu - \epsilon)\big]}$$

impar alrededor de $\epsilon = \mu$ (ejercicio)

expansión de Taylor de $f(\epsilon)$ alrededor de μ ... $x = \dfrac{\epsilon - \mu}{kT}$

(potencias pares de x no aportan a la integral)

$$I = \int_0^{\mu} \mathrm{d}\epsilon\, f(\epsilon) + \int_{-\infty}^{\infty} \mathrm{d}x \left[\frac{1}{e^x + 1} - \Theta(x)\right] \times \left(f'(\mu)\,(kT)^2\, x + \frac{f'''(\mu)}{3!}\,(kT)^4\, x^3 + \ldots \right)$$

$$\hookrightarrow \quad I = \int_0^{\mu} \mathrm{d}\epsilon\, f(\epsilon) + \frac{\pi^2}{6}\,(kT)^2\, f'(\mu) + \frac{7\pi^4}{360}\,(kT)^4\, f'''(\mu) + \ldots$$

(ejercicio)

expansión de Sommerfeld

$$\int_0^{\infty} \mathrm{d}u\, \frac{u^{y-1}}{e^u + 1} = \left(1 - 2^{1-y}\right)\, \zeta(y)\, \Gamma(y)$$

ζ de Riemann : $\zeta(2) = \dfrac{\pi^2}{6}$ $\quad \zeta(4) = \dfrac{\pi^4}{90}$

función Γ

Gas de Fermi-Dirac - T bajas

$$I = \int_0^\infty d\epsilon\, f(\epsilon)\, n(\epsilon) = \int_0^\mu d\epsilon\, f(\epsilon) + \frac{\pi^2}{6}(kT)^2 f'(\mu) + \frac{7\pi^4}{360}(kT)^4 f'''(\mu) + \ldots$$

expansión de Sommerfeld

la aplicamos a

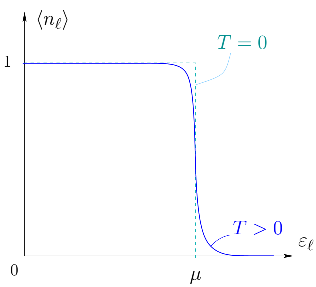
$$\langle N \rangle = \sum_\ell g_s \langle n_\ell \rangle \;\rightarrow\; \frac{g_s m^{3/2} V}{\sqrt{2}\,\pi^2 \hbar^3} \int_0^\infty d\epsilon\, \sqrt{\epsilon}\, n(\epsilon) = \frac{g_s V}{\lambda^3} f_{3/2}(z)$$

sustituyendo ... despejamos $f_{3/2}(z)$ para z grandes $\big(T$ chicas$\big)$

$$f_{3/2}(z) = \frac{4}{3\sqrt{\pi}} (\ln z)^{3/2} \left[1 + \frac{\pi^2}{8}(\ln z)^{-2} + \ldots + \mathcal{O}\big((\ln z)^{-4}\big)\right]$$

(ejercicio)

bajas temperaturas o altas densidades $\leftrightarrow$ $\lambda^3/v \gg 1$, aproximamos la ec. anterior



$$\frac{1}{v} = \frac{4}{3\sqrt{\pi}} (\ln z)^{3/2} \frac{g_s}{\lambda^3} \left[1 + \frac{\pi^2}{8}(\ln z)^{-2} + \ldots\right]$$

$$\hookrightarrow \quad \mu(T=0) \equiv \epsilon_F = \frac{\hbar^2}{2m} \left(\frac{6\,\pi^2}{g_s\, v}\right)^{2/3}$$

energía de Fermi

Gas de Fermi-Dirac - T bajas

$$\epsilon_F = \frac{p_F^2}{2m} \qquad \rightarrow \qquad p_F \text{ impulso de Fermi}$$

$$kT_F = \epsilon_F \qquad \rightarrow \qquad \textit{temperatura de Fermi } \; T_F \equiv \epsilon_F / k$$

$T \ll T_F$: gas “degenerado” , las partículas tienden a ocupar los estados de menor energía (hasta donde permite g_s)

puede “invertirse” $(T \ll T_F \;\leftrightarrow\; z \gg 1)$

$$f_{3/2}(z) = \frac{4}{3\sqrt{\pi}} (\ln z)^{3/2} \left[1 + \frac{\pi^2}{8} (\ln z)^{-2} + \ldots \right]$$

$$\hookrightarrow \quad \mu \; (= kT \, \ln z) = \epsilon_F \left[1 - \frac{\pi^2}{12} \left(\frac{kT}{\epsilon_F} \right)^2 + \ldots \right]$$ (ejercicio)

expansión de Sommerfeld para la energía interna :

$$U = \frac{3}{5} \langle N \rangle \, \epsilon_F \left[1 + \frac{5}{12} \pi^2 \left(\frac{kT}{\epsilon_F} \right)^2 + \ldots \right]$$ (ejercicio)

Gas de Fermi-Dirac - T bajas

$$U = \frac{3}{5}\langle N\rangle\,\epsilon_F\left[1 + \frac{5}{12}\pi^2\left(\frac{kT}{\epsilon_F}\right)^2 + \ldots\right]$$

energía del estado fundamental

se cumple

$$\sum_{|\boldsymbol{p}|<p_F}\frac{p^2}{2m} = \frac{3}{5}\langle N\rangle\,\epsilon_F$$

(ejercicio)

$$c_v \approx \frac{\pi^2}{2}\frac{k^2}{\epsilon_F}T \qquad\qquad c_v \underset{T\to 0}{\longrightarrow} 0$$

$$U = \frac{3}{2}PV \quad\rightarrow\quad P \approx \frac{2}{5}\frac{\epsilon_F}{v}\left[1 + \frac{5\pi^2}{12}\left(\frac{kT}{\epsilon_F}\right)^2 + \cdots\right]$$

$$\hookrightarrow P(T=0)\neq 0$$

(exclusión)

Gas de Fermi-Dirac - T altas

T altas o bajas densidades $\rightarrow \quad \lambda^3/v \ll 1$ (no se notaría la cuántica)

$\hookrightarrow \quad z \rightarrow 0$

$$\frac{\lambda^3}{g_s v} = f_{3/2}(z) = z - \frac{z^2}{2^{3/2}} + \ldots$$

"invirtiendo"

$$z = \frac{\lambda^3}{g_s v} + \frac{1}{2^{3/2}} \left(\frac{\lambda^3}{g_s v} \right)^2 + \ldots$$

(ejercicio)

$\rightarrow$ si $\lambda^3/(g_s v)$ muuuuy pequeño conservamos sólo el primer término : Maxwell-Boltzmann

$$\frac{Pv}{kT} = \frac{4\pi\, g_s\, v}{h^3} \int_0^\infty \mathrm{d}p\, p^2\, \ln\left(1 + z\, e^{-\frac{\beta p^2}{2m}}\right) = \frac{g_s v}{\lambda^3} f_{5/2}(z)$$

$$\hookrightarrow \quad \frac{Pv}{kT} = \frac{g_s v}{\lambda^3} \left(z - \frac{z^2}{2^{5/2}} + \ldots \right) = 1 + \frac{1}{2^{5/2}} \frac{\lambda^3}{g_s v} + \ldots$$

Maxwell-Boltzmann