Self-dual Maxwell fields on curved space–times

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We present a manifestly conformally invariant formulation of Maxwell equations on asymptotically flat space–times. It is shown how to construct regular self-dual and antiself-dual fields from suitable radiation data, and the general solution as a sum of fields with both types of duality. The basic variable in this formalism is a scalar field F defined as the phase of the parallel propagator (associated with the Maxwell potential) from interior points to future null infinity along null geodesics. Field equations equivalent to the source free Maxwell's equations are derived for F. A perturbative solution based on Huygens' principle is proposed. Exact solutions are found for H-spaces. The use of these results on gravitational lensing is discussed. © *1996 American Institute of Physics.* [S0022-2488(96)00507-5]

I. INTRODUCTION

The global behavior of electromagnetic radiation on curved space–times is used in astrophysics to study several interesting phenomena. In gravitational lensing one studies the global behavior of null geodesics in a gravitational field produced by sources with compact support. Several authors have also used parallel propagation of vectors on null geodesics to study the behavior of polarized light. Recently, a non-local formalism for general relativity (GR) was presented[1,2] where the fundamental variable contains all the information of the null geodesics of the given space–time, thus providing a useful tool to study electromagnetic radiation in the geometrical optics limit.

It has also been emphasized that when the wavelength of the radiation is of the same order of magnitude of the gravitational radius of an intervening compact object, diffraction effects have to be taken into account. That is, in this case one needs wave optics on a curved background. However, it is difficult to find global solutions of Maxwell's equations on a general space–time since a Green function for this problem is generally not available. In practice, one either tries to find a Green function perturbatively, or assumes a flat background except on a small region of interest, or considers space–times with high degrees of symmetry.

We present here a new formulation of Maxwell theory where the basic variable,

$$F \equiv \int_\gamma A_a dx^a, \tag{I.1}$$

a line integral of the Maxwell potential along a specific path γ, is a non-local object and where regularity of this variable is equivalent to regularity of the Maxwell field on a global scale. We derive field equations for this variable whose regular solutions automatically yield global solutions of the Maxwell's equations. Even the perturbed solutions of these non-local equations are, by construction, regular fields on a global scale. The formalism has another interesting feature, namely, the background geometry enters the field equations only through its conformal structure.[1,2] Thus, this non-local formulation of Maxwell theory is manifestly conformally invariant. It is also well adapted to discuss different global problems concerning electromagnetic radia-

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tion. In the geometrical optics limit one recovers the description of the null geodesics via the non-local formulation of GR. In the full theory the formalism defines in a precise way Huygens or non-Huygens propagation of electromagnetic radiation on a curved background.

Before presenting technical details we briefly review some results already obtained using the basic variable in our formalism. In recent years this variable has received considerable attention not only for Maxwell Theory, but also for Yang–Mills Theory and General Relativity.[3–6] In 1975 Wu and Yang[7] suggested that the parallel propagator of a Maxwell potential A_a along a path γ,

$$\exp\left(-i\int_\gamma A_a dx^a\right)=\exp(-iF), \tag{I.2}$$

was better suited than the Maxwell field or potential to describe electromagnetism. [Note that our variable F is the phase of the propagator introduced in (I.2).]

In 1980 Sparling[8,9] obtained the field equations for the phase of the parallel propagator associated with a self-dual Maxwell field along null geodesics in Minkowski space. A brief description of his method follows.

Denoting by x points in the space–time, $(u,\zeta,\bar{\zeta})$ points in the future null boundary $\mathscr{I}^+$, the Sparling equation has the simple form

$$\eth F=-A_R(x,\zeta,\bar{\zeta}), \tag{I.3}$$

where F, the phase of the parallel propagator (I.2), is the line integral of the Maxwell potential along a null geodesic that begins at a point x and ends at null infinity intersecting the generator $(\zeta,\bar{\zeta})$ of $\mathscr{I}^+$; the "eth operator" $\eth$ is essentially $\partial/\partial\zeta$, and A_R, the "restricted" free data at $\mathscr{I}^+$, is obtained by evaluating the data $A(u,\zeta,\bar{\zeta})$ at the intersection of the future light cone of x with $\mathscr{I}^+$, the "light cone cut". In a general, asymptotically flat (in future null directions) space–time, the light cone cuts are 2-surfaces embedded in $\mathscr{I}^+$, described parametrically as

$$u=Z(x,\zeta,\bar{\zeta}). \tag{I.4}$$

In Minkowski space–time the Z function has a simple form in standard Lorentz coordinates x^a,

$$Z(x,\zeta,\bar{\zeta})=x^a l_a(\zeta,\bar{\zeta}), \tag{I.5}$$

where $l_a(\zeta,\bar{\zeta})$ spans the sphere of null covectors at x. Its explicit form in Minkowski coordinates is given in Appendix B.

Using the Green function of the eth operator one obtains the regular solution of (I.3) which, after a reconstruction procedure, yields the general solution of the self-dual Maxwell equations in Minkowski space.

Notice that in a similar way [starting from the complex conjugate of (I.3)] one obtains the field equations for the antiself-dual case. Furthermore, adding both solutions yields a general real Maxwell field satisfying the source free equations. Thus (I.3), a single equation for a scalar field, is equivalent to the full set of Maxwell equations on Minkowski space.

Is it possible to generalize the Sparling equation to curved space–times? Since one knows that, due to the linearity of the theory, self-dual data must produce a self-dual field, in principle it is possible to find a suitable generalization of Sparling's formalism for a "self-dual" parallel propagator on a curved space–time.

In this paper we present this generalization. We introduce the phase of the parallel propagator for a self-dual field associated with a special set of open curves, i.e., null geodesics that start at an interior point of the space–time and end at $\mathscr{I}^+$. We obtain the field equation for this propagator equivalent to the self-dual Maxwell's equations on curved space–times. This is a single equation on a complex non-local function where the free data enters as a source term.

The cut function Z (I.4) that describes the light cone cuts of $\mathscr{I}^+$ plays a fundamental role in our approach. As shown in Refs. 1 and 2, all the information about the conformal geometry of space–time is encoded in Z. Since Maxwell's equations are conformally invariant, the function Z contains precisely the needed information of the background space–time geometry. As a result we obtain a manifestly conformally invariant formulation of regular Maxwell's fields on space–times which are asymptotically flat at future null infinity.

In section II we review the geometrical meaning of Z, show its relationship to the underlying conformal metric and obtain some results that are used in the derivation of the field equation for our variable. In section III, we present the non-local variable F and give its kinematical relation with the Maxwell field. We derive field equations for F equivalent to the self-dual and antiself-dual Maxwell's equations and show how to recover a general Maxwell field from knowledge of F. A perturbative method for solving the field equation is presented and then used to show the non-Huygens nature of Maxwell fields propagating on curved space–times. A variation of this method to study fields on space–times which are small deviations from Minkowski space is suggested. In section IV the formalism is applied to obtain exact solutions in H spaces. The zeroth and first order solution to the field equation on small deviations from Minkowski space are also calculated. Section V contains a summary of the results and a brief discussion on the use of this formalism to approach the problems of scattering and gravitational lensing. In Appendix A we derive the equation for F in the general case (i.e., complex Maxwell fields without a definite duality), show the equivalence with the self-dual formulation and show how to recover the field strength and potential from F. Appendix B contains some review material on differential equations involving the eth operator and their Green functions, necessary for the article to be self-contained. Appendices C and D contain auxiliary calculations.

II. GEOMETRICAL PRELIMINARIES

A. The notion of duality

On the space–time (M,g_{ab}) a volume form ϵ_{abcd} is chosen satisfying the normalization condition

$$\epsilon_{abcd}\epsilon^{abcd}=-4!, \tag{II.1}$$

where, as usual, indices are raised using the inverse of the metric. Up to a sign (i.e., a choice of orientation), (II.1) singles out a unique volume form, which, together with the inverse of the metric, is used to construct the dual operator, a conformally invariant linear operator acting on 2-forms as

$$W_{ab}\rightarrow {}^*W_{ab}\equiv \tfrac{1}{2}\,\epsilon_{abcd}g^{ce}g^{df}W_{ef}. \tag{II.2}$$

From (II.1) and (II.2) it follows that $^{**}W_{ab}=-W_{ab}$, and so the possible eigenvalues of the dual operator are $\pm i$. A 2-form is called self-dual (SD) [antiself-dual (ASD)] if it is an eigenvector corresponding to the eigenvalue $i[-i]$. Under the inner product $g^{ac}g^{bd}W_{ab}Q_{cd}$ SD and ASD 2-forms are orthogonal to each other. It is important to note that every 2-form can be written as a sum of eigenvectors of the dual operator:

$$W_{ab}=W^+_{ab}+W^-_{ab}, \quad W^{\pm}\equiv \tfrac{1}{2}\,[W_{ab}\mp i^*W_{ab}]. \tag{II.3}$$

$W^+_{ab}[W^-_{ab}]$ is commonly referred to as "the SD [ASD] part of W_{ab}." This decomposition allows a compact form for (source free) Maxwell's equations, explicitly exhibiting their conformal invariance,

$$\partial_{[a}F^+_{bc]}=0, \quad \partial_{[a}F^-_{bc]}=0. \tag{II.4}$$

For real fields, the second equation is the complex conjugate of the first one, and Maxwell's equations reduce to a first order equation on a complex potential A_b^+,

$$\frac{1}{2}\,\epsilon_{ab}{}^{cd}\partial_{[c}A^+_{d]}=i\partial_{[a}A^+_{b]}, \tag{II.5}$$

instead of the usual second order equation on a real potential. It is clear from (II.4) and (II.3) that SD [ASD] closed 2-forms satisfy Maxwell's source free equations, and also that any solution of these equations is the sum of a SD closed 2-form and an ASD one. It follows that the problem of solving Maxwell's equations on (M,g_{ab}) amounts to finding SD and ASD closed 2-forms satisfying appropriate "initial conditions". The first step is to construct the dual operator (II.2), which, being conformally invariant, can be readily obtained from the Z function, as it is shown in the following subsection.

B. A non-local description of space–time

The theory of light cone cuts of null infinity[1,2] offers a completely different approach to general relativity. Instead of using a local field, the metric g_{ab}, to describe the geometry of the space–time, one introduces a non-local function that plays an equally important role, as it contains all the information of the conformal structure of space–time. In particular, it yields the null geodesics. A brief description of the kinematical features of this theory follows. The dynamics as well as other properties of this non-local variable are not presented here (in our formulation of Maxwell theory the background geometry is assumed fixed) but they can be found in (Refs. 1 and 2) and references therein.

Assume the space–time (M,g_{ab}) is asymptotically flat at future null infinity, $\mathscr{I}^+$. The intersection of the future null cone from $x^a\in M$ with $\mathscr{I}^+$ is a 2-surface called a light cone cut of null infinity. Introducing on $\mathscr{I}^+$ Bondi coordinates $(u,\zeta,\bar{\zeta})$ this cut can be locally described as

$$u=Z(x^a,\zeta,\bar{\zeta}). \tag{II.6}$$

Assuming the smooth function Z is given, we introduce the following scalars:

$$u=Z(x^a,\zeta,\bar{\zeta})\quad \omega=\eth\dot{Z},\quad \bar{\omega}=\bar{\eth}Z,\quad r=\eth\bar{\eth}Z, \tag{II.7}$$

where the $\eth$ (eth) and $\bar{\eth}$ (eth-bar) operators, defined in (B1) and (B2), are essentially partial derivatives with respect to ζ and $\bar{\zeta}$ respectively. For each $(\zeta,\bar{\zeta})\in S^2$ the scalars (II.7) define a coordinate system. An alternative notation found in the literature is

$$(u,\omega,\bar{\omega},r)=(\theta^0,\theta^+,\theta^-,\theta^1)=\theta^i,$$

with its gradient and dual vector basis denoted by $\theta^i{}_{,a}$, and θ^a_i, respectively. In past references, however, the following associated vector basis has been used:

$$\hat{L}^a=\theta^a_1,\quad \hat{M}^a=-\theta^a_{-1},\quad \hat{\bar{M}}^a=-\theta^a_+,\quad \hat{N}^a=\theta^a_0, \tag{II.8}$$

and we will keep the above convention in this work.

The coordinates (II.7) have an interesting geometrical interpretation: the points x^a satisfying $u=Z(x^a,\zeta,\bar{\zeta})=$const. form the past light cone of $(u,\zeta,\bar{\zeta})$ at $\mathscr{I}^+$, ω and $\bar{\omega}$ determine the direction angle of a geodesic on that cone and τ uniquely locates a point on that geodesic.[10,11] It follows that any null geodesic is characterized by the parameters $(u,\omega,\bar{\omega},\zeta,\bar{\zeta})$, and that the tangent at x to the affinely parametrized geodesic $[u=Z(x,\zeta,\bar{\zeta}),\ \omega=\eth Z(x,\zeta,\bar{\zeta}),\ \bar{\omega}=\bar{\eth}Z(x,\zeta,\bar{\zeta});\zeta,\bar{\zeta}]$ is the null vector

$$Z^a(x,\zeta,\bar{\zeta})\equiv g^{ab}Z_{,b}(x,\zeta,\bar{\zeta}). \tag{II.9}$$

By letting $(\zeta,\bar{\zeta})$ range over the sphere, this vector ranges over the -future- cone of null directions at x^a. Thus, the function Z determines the conformal structure of the space–time. As done in Ref. 1, we exploit the fact that Z depends smoothly on $(\zeta,\bar{\zeta})$ to obtain the components $g^{ij}(x,\zeta,\bar{\zeta})\equiv g^{ab}\theta^i_{,a}\theta^j_{,b}$ of the metric in the coordinate systems (II.7). The starting point is the fact that (II.9) is null, which immediately implies that

$$g^{00}=g^{ab}(x)\,\theta^0_{,a}\theta^0_{,b}=0. \tag{II.10}$$

Note that this result is true for any value of $(\zeta,\bar{\zeta})$. Thus, ð and $\bar{ð}$ of (II.10) are also equal to zero. However, explicitly taking ð and $\bar{ð}$ of (II.10) plus the fact that the metric does not depends on $(\zeta,\bar{\zeta})$ yields

$$ð_g^{00}=2g^{ab}(x)\,\theta^0_{,a}ð\theta^0_{,b}=2g^{0+}=0, \quad \bar{ð}g^{00}=2g^{0-}=0. \tag{II.11}$$

Thus, we have obtained three (trivial) components of the conformal metric. As the above equations suggest, the metric components in this coordinate system are obtained by taking a sufficient number of ð and $\bar{ð}$ derivatives of (II.10).

Taking $ð\bar{ð}$ of (II.10) we obtain

$$g^{-+}=-g^{01}. \tag{II.12}$$

Taking $ð^2$ of (II.10) we get

$$g^{++}+g^{ab}Z_{,a}\Lambda_{,b}=0,$$

where $\Lambda\equiv ð^2 Z$. Assuming Λ is expressed in the θ^i coordinates as $\Lambda(\theta^i,\zeta,\bar{\zeta})$ one has $\Lambda_{,b}=\Lambda_{,i}\theta^i_{,b}$. Using the previous results we immediately obtain

$$g^{++}=-g^{01}\Lambda_{,r}. \tag{II.13}$$

We can continue this procedure until all the components (up to a choice of g^{01}) are obtained [1]. It is worth mentioning that knowledge of Λ determines all the non-trivial $g^{ij}(\theta^i,\zeta,\bar{\zeta})$. The final form of the metric is given by

$$g^{01}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -\Lambda_{,r} & -1 & h^{1+} \\ 0 & -1 & -\bar{\Lambda}_{,r} & h^{1-} \\ 1 & h^{1+} & h^{1-} & h^{11} \end{pmatrix}, \tag{II.14}$$

where the h^{1j} depend explicitly on Λ [1]. (In the derivation of our field equations, however, they will not be needed since we are only looking for the projection of the metric on the null surfaces Z=const).

Thus, in this approach the function Z serves a dual purpose: $ð^2 Z\equiv\Lambda$ determines the conformal metric and (II.7) are natural coordinates that allow simple expressions for many relevant tensor fields, as the metric itself. In the particular case $\Lambda=0$ we obtain conformal Minkowski space and, for each $(\zeta,\bar{\zeta})$, (II.7) becomes a null coordinate system (Appendix B). In general, θ^0 is a null coordinate, and simple expressions for both the null cone congruence from a point x^a and the geodesic deviation vectors of this congruence are obtained using the θ^i coordinates. Since these results will be used in following sections, we present them now. The detailed calculations can be found in Appendix C.

If we denote by $x^a(x_0^b,\zeta,\bar{\zeta},r)$ the parametric form of the null cone with apex at a point $x_0^b \in M$, $r = \eth\bar{\eth}Z(x^a,\zeta,\bar{\zeta}) - \eth\bar{\eth}Z(x_0^a,\zeta,\bar{\zeta})$ [refer to (C.1)–(C.3)], then a null geodesic on this cone is characterized by a fixed value of $(\zeta,\bar{\zeta})$ (a generator of $\mathscr{I}^+$). Neighboring rays around this fixed null geodesic, corresponding to different values of ζ, are described by the geodesic deviation vector $M^a \equiv \partial x^a$. As shown in Appendix C [see (C.6)], M^a can be written as

$$M^a \equiv (1+\zeta\bar{\zeta})\,\frac{\partial x^a(x_0,\zeta,\bar{\zeta},r)}{\partial\zeta} = (r-r_0)\hat{M}^a + (\Lambda-\Lambda_0)\hat{\hat{M}}^a - \eth(r-r_0)\hat{L}^a, \tag{II.15}$$

where the subindex 0 means that the corresponding scalar is evaluated at the apex x_0.

Finally, we present the explicit form of the dual of the light cone 2-surface element $\hat{L}_{[a}M_{b]}$ in our basis since this result is later used in the derivation of the field equations. We start by noting from (II.9), (II.14) and (II.8) that

$$\begin{gathered} Z^a = g^{ab}u_{,b} = g^{01}\hat{L}^a, \\ g^{ab}\omega_{,b} = g^{1+}\hat{L}^a + g^{01}\Lambda_{,r}\hat{\hat{M}}^a + g^{01}\hat{M}^a. \end{gathered} \tag{II.16}$$

The determinant of (II.14) yields the volume form

$$\epsilon = \frac{-i}{\sqrt{p}(g^{01})^2}\, du\wedge d\omega\wedge d\bar{\omega}\wedge dr, \quad p = 1 - \Lambda_{,r}\bar{\Lambda}_{,r}, \tag{II.17}$$

which, together with (II.8), (II.2) and (II.16), gives

$$*\hat{L}_{[a}\hat{M}_{b]} = \frac{-i}{\sqrt{p}(g^{01})^2}\, u_{[,a}\omega_{,b]} = \frac{-i}{\sqrt{p}}\,[\hat{L}_{[a}\hat{M}_{b]} + \Lambda_{,r}\hat{L}_{[a}\hat{\hat{M}}_{b]}]. \tag{II.18}$$

(II.15) and (II.18) yield

$$*\hat{L}_{[a}M_{b]} = -i[\hat{L}_{[a}M_{b]} + 2\bar{f}\hat{L}_{[a}\hat{M}_{b]} + 2g\hat{L}_{[a}\hat{\hat{M}}_{b]}], \tag{II.19}$$

where

$$\begin{gathered} f = \tfrac{1}{2}[(r-r_0)h - (1+h)(\bar{\Lambda}-\bar{L}_0)\Lambda_{,r}], \\ g = \tfrac{1}{2}[(1+h)(r-r_0)\Lambda_{,r} - (2+h)(\Lambda-\Lambda_0)], \end{gathered} \tag{II.20}$$

and $h \equiv 1/\sqrt{p} - 1$. Alternatively, using (II.3),

$$\hat{L}_{[a}M^{+}_{b]} = -[\bar{f}\hat{L}_{[a}\hat{M}_{b]} + g\hat{L}_{[a}\hat{\hat{M}}_{b]}]. \tag{II.21}$$

Note that in the $\Lambda = 0$ limit $\hat{L}_{[a}M_{b]}$ is ASD. Thus f and g represent the departure of $\hat{L}_{[a}M_{b]}$ from being an ASD 2-form.

III. A NON-LOCAL VARIABLE FOR MAXWELL FIELDS

A. Definitions and kinematical relations

Let A_a be a Maxwell potential whose projection to $\mathscr{I}^+$ has vanishing contraction with vectors tangent to the generators, $F_{ab} = 2\nabla_{[a}A_{b]}$ be its field strength. We define our basic variable F as

$$F(x,\zeta,\bar{\zeta}) = -\int_x^\infty A_a Z^a ds = -\int_x^\infty A_a \hat{L}^a dr, \tag{III.1}$$

where the integral is taken along the null geodesic $l_x(\zeta,\bar{\zeta})$ that connects the point x with the generator $(\zeta,\bar{\zeta})$ of $\mathscr{I}^+$, s is an affine length and the second equality follows directly from (II.16). F is a non-local function on the space of null directions of space–time with a simple geometrical meaning: $G \equiv e^{iF}$ gives the parallel transport along $l_x(\zeta,\bar{\zeta})$ between x and the point at which $l_x(\zeta,\bar{\zeta})$ intersects $\mathscr{I}^+$. As A_a is a regular potential with a smooth extension to $\mathscr{I}^+$, the integrand of (III.1) behaves as s^{-2} when $s \to \infty$, thus giving a finite F. Conversely, if F is assumed to be regular, then the integrand "peels" appropriately.

We consider now the problem of obtaining A_a from a given regular F, i.e., inverting (III.1). This will be particularly important when we impose field equations on F since the potential A_a will then be a derived object. It follows from (III.1) and (II.9) that F satisfies

$$Z_{,a}(\nabla^a F - A^a(x)) = 0. \tag{III.2}$$

Taking a sufficient number of ð and $\bar{ð}$ derivatives of (III.2) we obtain the components of $A^a(x)$ in the θ_i^a basis:

$$\begin{aligned}
&Z_{,b}(A^b - \nabla^b F) = 0,\\
&ðZ_{,b}(A^b - \nabla^b F) = Z_{,b}\nabla^b ðF,\\
&\bar{ð}Z_{,b}(A^b - \nabla^b F) = Z_{,b}\nabla^b \bar{ð}F,\\
&ð\bar{ð}Z_{,b}(A^b - \nabla^b F) = \bar{ð}Z_{,b}\nabla^b ðF + ðZ_{,b}\nabla^b \bar{ð}F + Z_{,b}\nabla^b ð\bar{ð}F,
\end{aligned} \tag{III.3}$$

from where

$$A^a = \nabla^a F + (\theta_-^a \theta^0_{,b} + \theta_1^a \theta^+_{,b})\nabla^b \bar{ð}F + (\theta_+^a \theta^0_{,b} + \theta_1^a \theta^-_{,b})\nabla^b ðF + \theta_1^a \theta^0_{,b}\nabla^b ð\bar{ð}F. \tag{III.4}$$

Note that (III.4) naturally induces a gauge transformation giving a new potential,

$$A'_b = A_b - \nabla_b F.$$

Consider now the 2-surface $\Delta_x(\zeta,\bar{\zeta})$ swept by $l_x(\zeta,\bar{\zeta})$ as ζ moves in $[\zeta,\zeta+d\zeta]$ with $\bar{\zeta}$ fixed, denote by $\partial\Delta_x(\zeta,\bar{\zeta})$ its closed boundary constructed from two neighboring null geodesics $l_x(\zeta,\bar{\zeta})$ and $l_x(\zeta+d\zeta,\bar{\zeta})$, closed at $\mathscr{I}^+$ by the connecting vector $M^a d\zeta/P$. Using the phase of the holonomy operator,

$$\mathscr{H} = \int_{\partial\Delta_x(\zeta,\bar{\zeta})} A_a dx^a = \int_{\Delta_x(\zeta,\bar{\zeta})} F_{ab} dx^a dx^b, \tag{III.5}$$

one can find a useful relationship between F, the field strength F_{ab} and the free Maxwell data at $\mathscr{I}^+$.

We recall that source free Maxwell fields are uniquely determined by the data $(A(u,\zeta,\bar{\zeta}),\bar{A}(u,\zeta,\bar{\zeta}))$ on the "initial value" surface $\mathscr{I}^+$ [12]. These functions are defined by the following equations:

$$A(u,\zeta,\bar{\zeta}) \equiv \lim_{\mathscr{I}^+} A_a M^a, \quad \bar{A}(u,\zeta,\bar{\zeta}) \equiv \lim_{\mathscr{I}^+} A_a \bar{M}^a, \tag{III.6}$$

where the limit is taken along $l_x(\zeta,\bar{\zeta})$, x is any point satisfying $Z(x,\zeta,\bar{\zeta})=u$, and M^a and $\bar{M}^a$, given in (C.6) and c.c., are geodesic deviation vectors of the future null cone of x (see Appendix C). A and $\bar{A}$ are, respectively, associated to the SD and ASD parts of the field by the following equations (obtained from Ref. 13):

$$\lim_{\mathscr{I}^+} F^-_{ab} M^a \bar{M}^b = \eth \bar{A}, \quad \lim_{\mathscr{I}^+} F^+_{ab} M^a \bar{M}^b = -\bar{\eth} A. \tag{III.7}$$

Thus, $\bar{A}=0$ [$A=0$] for SD [ASD] fields. For real fields, $\bar{A}$ is the complex conjugate of A, the two degrees of freedom of the radiation fields are contained in a single complex function.

Defining the differential holonomy as

$$H(x,\zeta,\bar{\zeta}) \equiv \int_x^\infty F_{ab} \hat{L}^a M^b dr, \tag{III.8}$$

i.e. $\mathscr{H}=H d\zeta/P$, it follows directly from (III.5) and (III.6) that

$$\eth F(x,\zeta,\bar{\zeta}) + A_R(x,\zeta,\bar{\zeta}) = H(x,\zeta,\bar{\zeta}), \tag{III.9}$$

where the complex scalar $A_R(x,\zeta,\bar{\zeta})$ is the restriction of the free data $A(u,\zeta,\bar{\zeta})$ at $\mathscr{I}^+$ to the cut $u=Z(x,\zeta,\bar{\zeta})$, i.e.,

$$A_R(x,\zeta,\bar{\zeta}) \equiv A(u=Z_1\zeta,\bar{\zeta}). \tag{III.10}$$

In an analogous way, using the holonomy phase $\bar{\mathscr{H}}$ around the loop $\partial\bar{\Delta}_x(\zeta,\bar{\zeta})[\Delta_x(\zeta,\bar{\zeta})$ being the 2-surface swept by $l_x(\zeta,\bar{\zeta})$ as $\bar{\zeta}$ moves in $[\bar{\zeta},\bar{\zeta}+d\bar{\zeta}]$ with ζ fixed] and defining

$$\bar{H}(x,\zeta,\bar{\zeta}) \equiv \int_x^\infty F_{ab} \hat{L}^a \hat{M}^b dr, \tag{III.11}$$

one obtains

$$\bar{\eth} F(x,\zeta,\bar{\zeta}) + \bar{A}_R(x,\zeta,\bar{\zeta}) = \bar{H}(x,\zeta,\bar{\zeta}), \tag{III.12}$$

with $\bar{A}_R(x,\zeta,\bar{\zeta})$ the restriction of $\bar{A}(u,\zeta,\bar{\zeta})$ to the cut

$$\bar{A}_R(x,\zeta,\bar{\zeta}) = \bar{A}(u=Z,\zeta,\bar{\zeta}) \tag{III.13}$$

(we will omit the subindex R from now on).

Equations (III.8) and (III.9) give $\eth F$ as a functional of the field strength F_{ab} and the free data $A(u,\zeta,\bar{\zeta})$. One can invert (III.8), using (III.9), to obtain F_{ab} in terms of $\eth F$ and c.c. This is done in Appendix A. The desired relationship is

$$\hat{\phi} \equiv F_{ab}(x) \hat{L}^a \hat{M}^b = \left[\frac{(\eth F)_{,r} - \Lambda_{,r} (\bar{\eth} F)_{,r}}{1 - \Lambda_{,r} \bar{\Lambda}_{,r}} \right]_{,r} \equiv \hat{\Phi}[F] \quad \text{and c.c.} \tag{III.14}$$

Note that F_{ab} does not depend on $(\zeta,\bar{\zeta})$, whereas $\hat{L}^a \hat{M}^b$ does. Thus, evaluating (III.14) for different values of $(\zeta,\bar{\zeta})$ yields different components of $F_{ab}(x)$. Alternatively, we can take a sufficient number of $\eth$ and $\bar{\eth}$ derivatives of (III.14) [as was done in (III.3)] to obtain the components of F_{ab} in the $\theta^i_{\ ,a}$ basis.

B. The field equations for F

We now want to find the equation satisfied by F when the field strength F_{ab} is a (closed) SD 2-form. The fact that F_{ab} is closed was already used in the definition of F and in eq. (III.9) as an application of the Stokes theorem.

A SD field satisfies

$$F_{ab}=F^{+}_{ab}. \tag{III.15}$$

Thus, integrating (III.15) on $\Delta_x(\zeta,\bar{\zeta})$ yields

$$H=\int_x^{\infty} F_{ab}\hat{L}^{[a}M^{b]}dr=\int_x^{\infty} F^{+}_{ab}(\hat{L}^{[a}M^{b]})dr=\int_x^{\infty} F^{ab}(\hat{L}_{[a}M_{b]})^{+}dr, \tag{III.16}$$

where the last equality follows from the vanishing contraction between SD and ASD 2-forms. Inserting (II.21) in (III.16) and using (III.9) and (III.14) yields the following integro-differential equation for F in the SD case (the $+$ sign is used to indicate that the associated field is SD),

$$\eth F^{+}+A+\int_x^{My}(\bar{f}\bar{\Phi}[F^{+}]+g\hat{\bar{\Phi}}[F^{+}])dr=0. \tag{III.17}$$

Equation (III.17) is the desired generalization of the Sparling equation[8,9] to curved space-times.

A similar calculation yields the equation for F in the ASD case,

$$\bar{\eth} F^{-}+\bar{A}+\int_x^{\infty}(f\hat{\bar{\Phi}}[F^{-}]+\bar{g}\hat{\Phi}[F^{-}])dr=0. \tag{III.18}$$

A few remarks follow.

1. The kernel of the eth operator-acting on spin weight 0 functions- is the set of constant functions on the sphere (Appendix B). It then follows from (III.17) and (III.14) that the solutions of eq. (III.17) have the ambiguity of an arbitrary additive function $h(x)$. The origin of this term can be traced back to the definition of F.[14] If we integrate (III.17) using the Green function $G_{0,1'}(\zeta,\bar{\zeta};\zeta',\bar{\zeta}')$ for $\eth$ given in (B.8) (subindices indicate the spin weight for each variable), we obtain the following compact form for (III.17):

$$F-\mathscr{G}[F]=\mathscr{F}, \tag{III.19}$$

where

$$\mathscr{F}\equiv-\int_{S^2}G_{0,-1'}(\zeta,\bar{\zeta};\zeta',\bar{\zeta}')A_R(x,\zeta',\bar{\zeta}')dS' \tag{III.20}$$

and

$$\mathscr{G}[\cdot]\equiv-\int_{S^2}G_{0,-1'}(\zeta,\bar{\zeta};\zeta',\bar{\zeta}')\left(\int(\bar{f}\bar{\Phi}[\cdot]+g\hat{\bar{\Phi}}[\cdot])dr'\right)dS'. \tag{III.21}$$

In (III.21), the line integral is along $l_x(\zeta',\bar{\zeta}')$, and extends from x to $\mathscr{I}^{+}$. By omitting the term $h(x)$ on the r.h.s. of (III.19) we have selected the particular solution (i.e., chosen the gauge[14]) satisfying the condition $\int_{S^2}F(x,\zeta,\bar{\zeta})dS=0$. This follows from the fact that $\int_{S^2}G_{0,1'}(\zeta,\bar{\zeta};\zeta',\bar{\zeta}')dS=0$.[15] Analogous comments to this and the following remarks apply to the ASD case.

2. We know that there is a one to one correspondence between the free data $A(u,\zeta,\bar{\zeta})$ and the solution of the source free Maxwell's equations. Therefore it is appropriate to ask if the regular solutions to (III.19) uniquely correspond to a given $A(u,\zeta,\bar{\zeta})$. Although this issue is very involved due to the non-local nature of the equation, in those cases where a norm $\|\cdot\|$ can be defined on the linear space V of admissible functions $F(x,\zeta,\bar{\zeta})$, such that the linear operator $\mathscr{G}$ is continuous and has a norm less than 1 (meaning $\sup_V \|\mathscr{G}[F]\|/\|F\|<1$), the uniqueness of the solution of (III.19) is easily proved. In this case the operator $I-\mathscr{G}$ is known to be invertible, with its inverse given by the -convergent- power series,

$$[I-\mathscr{G}]^{-1}=I+\mathscr{G}+\mathscr{G}^2+\mathscr{G}^3+\dots . \tag{III.22}$$

The definition of $(V,\|\cdot\|)$ is a technically involved problem we defer for future work. Note however that, as $\mathscr{G}=0$ when $\Lambda=0$, any natural norm will satisfy the condition $\|\mathscr{G}\|<1$ when $\Lambda\simeq 0$ [this limit corresponds to a small deviation from -conformal- Minkowski space (Appendix B)].

From now on, we will assume that the linear operator $\mathscr{G}$ is such that (III.19) has a unique solution.

3. Although we have only proved that SD Maxwell's equations imply (III.19), the converse is also true. The equivalence of both sets of equations follows from the uniqueness of the solution of (III.19). A is the data for a unique SD Maxwell field F_{ab}. If B_a is the potential satisfying the conditions in Ref. 14 and

$$\int_{S^2}\left[\int_x^\infty B_a Z^a ds\right]dS=0,$$

then (III.1) constructed from B_a is a solution of (III.19) (remark 1). Now, F constructed in this way is *the only* solution of (III.19). It follows from remark 1 that, when applied to a solution of (III.17), (III.4) yields a potential $A_a(x)=B_a(x)+\nabla_a h(x)$ for the SD field $F_{ab}(x)$.

4. The equation satisfied by F in the general case is derived in Appendix A. It is shown that we can first solve for the SD and ASD parts and then add them up, i.e., if

$$F\equiv F^+ + F^-, \tag{III.23}$$

where F^+ and F^- are solutions of (III.17) and (III.18), respectively, then F yields -via (III.4) or (III.14)- a regular solution of the full source free-Maxwell's equations with "initial data" $(A,\bar{A})$ on the characteristic surface $\mathscr{I}^+$.

5. The function Z plays two distinct roles in the field equations (III.17), (III.18). It determines the restriction of the free data $A(\bar{A})$ to the cut $u=Z$ and it also enters the integral term via (II.20) and (III.14).

Two different perturbative methods to solve (III.17) may be suggested based on the information we have about Z:

(a) If the conformal geometry of the space–time is fully known (Z is given) we can construct the linear operator (III.21), and then apply (III.22) to solve (III.19),

$$F=[1-\mathscr{G}]^{-1}\mathscr{F}=[1+\mathscr{G}+\mathscr{G}^2+\mathscr{G}^3+\dots]\mathscr{F}. \tag{III.24}$$

Note that if F_n is the sum of the first n terms in (III.24) then

$$F=\lim_{n\to\infty} F_n, \quad F_{n+1}=\mathscr{G}[F_n]+\mathscr{F}, \tag{III.25}$$

an iterative process that could have been suggested directly from (III.19).

(b) If the space–time is a small deviation from -conformal- Minkowski space, then $\Lambda\simeq 0$ (Appendix B) and we can expand F around $\Lambda=0$. The expansion is obtained by rearranging

(III.24) guided by the observation that f only contains terms Λ^n with $n \geqslant 2$, g contains terms with $n \geqslant 1$ (II.20) and $\mathscr{G}$ contains terms $n \geqslant 1$ (III.21). Slightly modifying (III.25) by keeping track of the different powers of Λ we generate an iterative method to solve (III.17) in nearly Minkowskian backgrounds. The first two terms are explicitly calculated in part B of the following section.

6. The leading order term $\mathscr{F}(x,\zeta,\bar{\zeta})$ in the expansion series (III.24) represents the Huygens' part of the field, i.e., the contribution of A_R (the restriction of the data A to the light cone cut of x). It is clear that the other terms are non-Huygens. As an example, to calculate $\mathscr{G}\mathscr{F}$ at x we need to know $\mathscr{F}$ in the future cone of x [this follows from (III.21) and (III.14)]. From (III.20), the restriction of the data to the whole region of $\mathscr{I}^+$ enclosed by the light cone cut of x is needed, not only its value at the cut. A similar reasoning shows that to calculate all the other terms in (III.24) we only need the data in this region, as expected from general principles (the field at x can not affect its value out of the cut on $\mathscr{I}^+$). This shows that, on a general background, Maxwell fields do not obey the Huygens' principle. In the perturbation scheme (III.24) one assumes that the background geometry is such that the Huygens' part is dominant. The (finite) perturbation series will therefore be a good approximation to the solution whenever the support of the Green function for Maxwell's equations lies mainly on the characteristic surfaces.

IV. APPLICATIONS

A. Propagation of Maxwell fields on self-dual space–times

It is well known that SD and ASD Maxwell fields have zero stress-energy tensor and thus are solutions of the Einstein–Maxwell equations on a vacuum space–time. If the vacuum space–time is self-dual these solutions represent the (classical) interaction of photons with non-linear gravitatons.[16] Before proceeding further with our formalism, we present a brief review of the geometry of asymptotically flat self-dual space–times, the so called H-spaces. For a detailed account the reader is referred to Ref. 16.

Given an arbitrary null hypersurface whose intersection with $\mathscr{I}^+$ is described by the "cut" $u=Z(\zeta,\bar{\zeta})$, the condition for it to have asymptotically vanishing shear is that it satisfies the so called "good cut equation":[16]

$$\eth^2 Z=\sigma_B(Z,\zeta,\bar{\zeta}), \tag{IV.1}$$

where $\sigma_B(u,\zeta,\bar{\zeta})$ is the asymptotic shear associated with the Bondi cut u=constant. Note that (IV.1) is a non-linear second order p.d.e for a function Z on the sphere. Since σ_B is complex and Z real, in general eq. (IV.1) has no real solutions.

However, if we allow Z to be complex and if $\sigma_B(u,\zeta,\bar{\zeta})$ can be extended to a holomorphic function of *three independent complex variables* $(u,\zeta,\bar{\zeta})$, one can show that the good cut equation (IV.1) has a four-parameter family of (complex) solutions $Z(x,\zeta,\bar{\zeta})$. The space of solutions $\{Z(x,\zeta,\bar{\zeta}), x \in C^4\}$ is called H-space and can be given the structure of a complex manifold. One can also show that the function Z induces a holomorphic metric g_{ab} on H that satisfies the vacuum equations and has a self-dual Weyl tensor.[16]

Since σ_B represents (from the standard formulation of GR) outgoing gravitational waves, one has available a natural linear structure of superposition of incoming waves that yields a non-linear superposition of self-dual metrics [eq. (IV.1) is intrinsically non-linear]. Moreover, despite all the non-linearity of the theory, the scattering of self-dual gravitational waves is trivial. The solutions of (IV.1) are usually called "non-linear gravitons." Although self-dual solutions of Einstein's equations seem to be a mathematical device with no physical interest, they play a key role in the canonical approach to quantum gravity.[5]

To obtain the general solution of Maxwell's equations on H-spaces we will assume the Z function, which is obtained by solving (IV.1), is given and repeat the steps leading to (II.14). However, in this case the conformal factor is chosen as $g^{01}=1$.[16] From this condition and the fact that $\nabla_a \Lambda=(d\sigma/du)Z_a \equiv \sigma_B Z_a$ [see (IV.1)], we can readily write down the metric components g^{ij},

$$g^{ij}(x,\zeta,\bar{\zeta})=\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & -\bar{\Lambda}_{,r} & g^{1-} \\ 1 & 0 & g^{1-} & g^{11} \end{pmatrix}, \tag{IV.2}$$

with

$$g^{1-}=-\tfrac{1}{2}\,\eth\bar{\Lambda}_{,r} \quad g^{11}=-2+\bar{\Lambda}_{,r}\dot{\sigma}-\tfrac{1}{2}\,\eth^2\bar{\Lambda}_{,r}. \tag{IV.3}$$

We also observe from (II.20) and (IV.1) that, for H-spaces,

$$f=\bar{f}=g=0, \quad \bar{g}=\tfrac{1}{2}\,[(r-r_0)\bar{\Lambda}_{,r}-2(\bar{\Lambda}-\bar{\Lambda}_0)], \tag{IV.4}$$

i.e., in H-spaces the SD part of $\hat{L}_{[a}M_{b]}$ vanishes. Therefore, for self-dual space–times (III.17) adopts the simple form

$$\eth F=-A(Z(x,\bar{\zeta},\bar{\zeta}),\zeta,\bar{\zeta}). \tag{IV.5}$$

The solution of eq. (IV.5) is

$$F(x,\zeta,\bar{\zeta})=-\int_{S^2} G_{0,-1'}(\zeta,\bar{\zeta};\zeta',\bar{\zeta}')A(Z(x,\zeta',\bar{\zeta}'),\zeta',\bar{\zeta}')dS'. \tag{IV.6}$$

Note from (IV.5) that two terms on the r.h.s. of (III.4) vanish. The remaining term is simplified by the explicit form of the metric, eq. (IV.2), giving

$$A_a(x)=2(\nabla^b\bar{\eth}F)Z_{[a}\eth Z_{b]}+\nabla_a F. \tag{IV.7}$$

Introducing the normalized null tetrad $\{\mathscr{L}^a,\mathscr{M}^a,\bar{\mathscr{M}}^a,\mathscr{N}^a\}$, where

$$\begin{gathered}\mathscr{L}_a=\nabla_a u, \quad \mathscr{M}_a=\nabla_a\omega, \quad \bar{\mathscr{M}}_a=\nabla_a\bar{\omega}+\tfrac{1}{4}\,(\eth\bar{\Lambda}_{,r})\nabla_a u-\tfrac{1}{2}\,\bar{\Lambda}_{,r}\nabla_a\omega, \\ \mathscr{N}_a=\nabla_a u+\nabla_a r+(\tfrac{1}{4}\,\eth^2\bar{\Lambda}_{,r}-\tfrac{1}{2}\,\bar{\Lambda}_{,r}\dot{\sigma})\nabla_a u-\tfrac{1}{4}\,(\eth\bar{\Lambda}_{,r})\nabla_a\omega,\end{gathered} \tag{IV.8}$$

and performing the natural gauge transformation $A_a\rightarrow A_a-\nabla_a F$, we can rewrite (IV.7) in a more convenient form. The details are given in Appendix D. The final result is

$$A_a(x)=\frac{1}{4\pi}\int_{S^2}dS\dot{A}(Z(x,\zeta,\bar{\zeta}),\zeta,\bar{\zeta})\bar{\mathscr{M}}_a(x,\zeta,\bar{\zeta}), \tag{IV.9}$$

$$F_{ab}(x)=\frac{1}{2\pi}\int_{S^2}dS[\ddot{A}\mathscr{L}_{[a}\bar{\mathscr{M}}_{b]}+\dot{A}\nabla_{[a}\bar{\mathscr{M}}_{b]}], \tag{IV.10}$$

where $\dot{A}\equiv(\partial/\partial u)A$.

To check that F_{ab} satisfies the field equations, we only have to prove that it is a SD field. Since $\mathscr{L}_{[a}\bar{\mathscr{M}}_{b]}$ is, by construction, SD (IV.8), we should only show that $\nabla_{[a}\bar{\mathscr{M}}_{b]}$ is SD, that is, it has vanishing contraction with any ASD 2-form. In particular, if we introduce the ASD 2-form basis,

$$\mathscr{L}_{[a}\mathscr{M}_{b]}, \quad \mathscr{L}_{[a}\mathscr{N}_{b]}+\bar{\mathscr{M}}_{[a}\mathscr{M}_{b]}, \quad \bar{\mathscr{M}}_{[a}\mathscr{N}_{b]},$$

we should check that

$$\mathscr{L}^a \mathscr{M}^b \nabla_{[a} \bar{\mathscr{M}}_{b]} = (\mathscr{L}^a \mathscr{N}^b + \bar{\mathscr{M}}^a \mathscr{M}^b) \nabla_{[a} \bar{\mathscr{M}}_{b]} = \bar{\mathscr{M}}^a \mathscr{N}^b \nabla_{[a} \bar{\mathscr{M}}_{b]} = 0,$$

where

$$\nabla_{[a} \bar{\mathscr{M}}_{b]} = \mathscr{M}_{[a} \nabla_{b]} (\bar{\Lambda} \bar{\Lambda}_{,r}).$$

The first equality is immediate, whereas the second and third follow from the following relationships in H-space,[17]

$$\bar{\Lambda}_{,\bar{\omega}} = -\tfrac{1}{2}\, \eth \bar{\Lambda}_{,r}, \quad \bar{\Lambda}_{,\omega} = \tfrac{1}{2}\, \bar{\eth} \bar{\Lambda}_{,r},$$
$$\bar{\Lambda}_{,u} = \tfrac{1}{2}\, \bar{\Lambda}_{,r} - \tfrac{1}{4}\, \bar{\Lambda}^2_{,r} \dot{\sigma}_B - \tfrac{1}{4}\, \bar{\eth}\eth \bar{\Lambda}_{,r} + \tfrac{1}{8}\, \bar{\Lambda}_{,r} \eth^2 \bar{\Lambda}_{,r} - \tfrac{1}{16}\, (\eth \bar{\Lambda}_{,r})^2. \tag{IV.11}$$

The solution to the ASD field equation (III.18) can be obtained following a similar approach to the one outlined above. In this case, however, the equations are more involved since the integral term in (III.18) is not trivial. (Note that the ASD fields are not just the complex conjugate of SD fields, as the underlying manifold is complex in this case.) There is an alternative approach based on the observation from (IV.8) that for each $(\zeta, \bar{\zeta}) \in S^2$, the 2-form,

$$\mathscr{L}_{[a} \mathscr{M}_{b]} = \nabla_{[a} u \nabla_{b]} \omega \tag{IV.12}$$

is closed and ASD and thus, it is a particular solution of Maxwell equations.

If we multiply (IV.12) with *any* complex scalar function of $(u, \zeta, \bar{\zeta})$ and then integrate on the sphere we should obtain the general ASD field. Taking eqs. (IV.9) and (IV.10) as a reference, we make the following ansatz for the general solution of the ASD field equations:

$$A_a(x) = \frac{1}{4\pi} \int_{S^2} dS \dot{A}(Z(x, \zeta, \bar{\zeta}), \zeta, \bar{\zeta}) \mathscr{M}_a(x, \zeta, \bar{\zeta}), \tag{IV.13}$$

$$F_{ab}(x) = \frac{1}{2\pi} \int_{S^2} dS \ddot{A} \mathscr{L}_{[a} \mathscr{M}_{b]}. \tag{IV.14}$$

It only remains to check that $\bar{A}$ in (IV.13) is in fact the asymptotic value of the Maxwell connection, as had previously been defined. This is done in Appendix D.

As pointed out in Appendix B, the function (B.5) $Z^{(0)}(x, \zeta, \bar{\zeta})$ with $x \in C^4$ is the general regular solution of the good cut equation (IV.1) when $\sigma_B = 0$. This function describes the light cone cuts of complex Minkowski space–time. Complex Minkowski space–time is therefore a particular case of H space where (IV.10) and (IV.14) reduce to D'Adamard [18] solutions of Maxwell equations. Thus, (IV.10) and (IV.14) are the generalizations of D'Adamard formula to H spaces.

B. Small deviations from Minkowski space

When discussing the propagation of light on vacuum space–times, it is often a good approximation to consider the background geometry as a small deviation from Minkowski space. In those cases we can replace F by the first terms in the expansion $F = \Sigma_n F^{(n)}$ in powers of Λ, as Minkowski space is characterized by the equation $\Lambda = 0$. We will apply method b of remark 5 in section III to calculate the first two terms of this perturbative solution. This method is based on the facts that $\mathscr{S}$ in (III.24) contains no $\mathscr{O}(\Lambda^0)$ term and that Z is a linear functional of Λ,

$$Z=Z^{(0)}+\int_{S^2}G_{0,-2'}\Lambda dS'\equiv Z^{(0)}+Z^{(1)}. \tag{IV.15}$$

In (IV.15) $G_{0,-2'}$ is the Green function (B.9) for $\eth^2$ and $Z^{(0)}$ the general solution of the equation $\eth^2 Z=0$, given in inertial coordinates in (B.5).

Order 0

At $\mathcal{O}(\Lambda^0)$, $Z=Z^{(0)}$ (IV.15), $f=g=0$ (II.20) and (III.17) reduces to the Sparling equation (I.3), as expected. The solution is given by the first term in (III.24), with $\mathcal{F}$ calculated using $Z^{(0)}$. We can readily write down the field strength from (IV.10) and (B.7). This is just the D'Adamard formula,

$$F_{ab}^{(0)}(x)=\frac{1}{2\pi}\sum_{S^2}\ddot{A}(x^c l_c',\zeta',\bar{\zeta}')l_{[a}(\zeta',\bar{\zeta}')\bar{m}_{b]}(\zeta',\bar{\zeta}')dS'. \tag{IV.16}$$

Order 1

As pointed out above, $\mathcal{G}$ in (III.24) contains no order zero term. Therefore, the first order expansion of F agrees with that of $(1+\mathcal{G})\mathcal{F}$. From (II.20) we obtain the first order approximations for f and g,

$$f=0,\quad g=\tfrac{1}{2}[(r-r_0)\Lambda_{,r}-2(\Lambda-\Lambda_0)]. \tag{IV.17}$$

Thus, the linear expansion of F is

$$F^{(0)}+F^{(1)}=-\int_{S^2}G_{0,-1'}\left[A(Z^{(0)},\zeta',\bar{\zeta}')+\dot{A}(Z^{(0)},\zeta',\bar{\zeta}')Z^{(1)}+\int_x^\infty g\hat{\phi}^{(0)}dr'\right]dS', \tag{IV.18}$$

with $\hat{\phi}^0=F_{ab}^{(0)}l^a\bar{m}^b$ and g in (IV.17).

Although in principle the Z function is assumed to be given, the background geometry is usually described in terms of the metric and its associated tensors. In those cases, the process of obtaining Z or Λ is involved (one must solve the geodesic deviation). However, it can be shown[2] that in the linear gravity approximation

$$\Lambda\simeq\sigma_B(Z^{(0)},\zeta,\bar{\zeta})-\int_r^\infty(r'-r)\Psi_0(Z^{(0)},\eth Z^{(0)},\bar{\eth}Z^{(0)},r',\zeta,\bar{\zeta})dr'. \tag{IV.19}$$

If the space–time is a small deviation from Minkowski space and if the Weyl tensor, rather than Z or Λ, is given, one can then use (IV.19) in (IV.15) and (IV.18) to obtain the linear gravity approximations of Z and F.

V. SUMMARY AND CONCLUSIONS

We have presented a manifestly conformally invariant formulation of Maxwell theory on asymptotically flat space–times. The field equations for our basic variable, the phase of the parallel propagator associated with self-dual (or antiself-dual) Maxwell fields, contain the free data $A(u,\zeta,\bar{\zeta})$ (or $\bar{A}$) at $\mathcal{I}^+$ as a source term and the solution of those equations is the generalization of D'Adamard's formula to curved space–times. A method to reconstruct the Maxwell field was given. In particular, adding a solution of (III.17) and its complex conjugate yields a real field that satisfies the source free Maxwell's equations. Since the free data has a well defined physical interpretation, namely, it represents incoming radiation, the solutions to the field equations represent the scattering of light due to the background curvature.

Exact solutions of the field equations were obtained when the background geometry was self-dual. A Huygens perturbation series was presented. The terms in the series can be given a physical interpretation. The zeroth order term represents a Huygens propagation on curved spaces. The first and higher order term yield the tails of the electromagnetic wave produced by the scattering of the field in the non-trivial geometry.

Since this formalism is specially adapted to discuss the evolution of electromagnetic radiation on a curved space–time we now address a few questions that can be answered without involved calculations or too many technical details.

We start with a question that was partially answered in Ref. 19; do SD or ASD Maxwell fields have trivial scattering in self-dual space–times? Since for H spaces σ_B is the same at $\mathscr{I}^+$ and $\mathscr{I}^-$, it follows from (IV.1) that cuts Z at $\mathscr{I}^+$ or at $\mathscr{I}^-$ are the same. Inserting this condition in (IV.10) and (IV.14) implies that the free data A at $\mathscr{I}^+$ or $\mathscr{I}^-$ are also the same. Thus, we conclude that electromagnetic radiation does not interact with self-dual gravitational waves.

A question that arises in the context of gravitational lensing is: does the curvature of space-time rotate the polarization vector of an electromagnetic wave? The difficulty with this question is that one should assign a meaning to the word ''rotate'' since it implies a comparison of vectors at different locations. An analogous, question is: can the curvature of space–time change the helicity on an electromagnetic wave?

To answer this question we first discuss the notion of helicity in our formalism. In Minkowski space one can show that self-dual Maxwell fields yield positive and negative helicity states via the positive or negative frequency decomposition of the free data $A(u,\zeta,\bar{\zeta})$ at $\mathscr{I}^+$.[20] Since asymptotically flat space–times share the same null boundary with Minkowski space, it is possible to define the notion of asymptotic helicity for the electromagnetic radiation. The Fourier transform of the data, $A(\omega,\zeta,\bar{\zeta})$, defines, for positive or negative frequencies ω, the corresponding asymptotic helicity states for the incoming radiation. The scattering of Maxwell fields is then studied by giving the initial data $A_{\mathscr{I}^-}$ at $\mathscr{I}^-$, solving the field equations in an entirely analogous way to what was done in section III, and then projecting the field to $\mathscr{I}^+$ as in (III.7) to obtain $A_{\mathscr{I}^+}$. By analyzing the frequency content of $A_{\mathscr{I}^+}$ we can determine whether or not helicity has been conserved. Our equations clearly suggest that helicity is not conserved.

The problem of scattering and its applications to gravitational lensing is currently being studied. A thorough discussion using this formalism will be presented in a following work.

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APPENDIX A: THE FIELD EQUATION FOR THE *F* FUNCTION

We derive here an equation for F equivalent to the full set of source free Maxwell's equations. To do this consider the 3-volume V limited by (i) $\Delta_x(\zeta,\bar{\zeta})$, (ii) $\Delta_x(\zeta,\bar{\zeta}+d\bar{\zeta})$, (iii) $\bar{\Delta}_x(\zeta,\bar{\zeta})$, (iv) $\bar{\Delta}_x(\zeta+d\zeta,\bar{\zeta})$ and (v) the cap on $\mathscr{I}^+$; the corresponding surface elements being $\hat{L}^{[a}M^{b]}(x,\zeta,\bar{\zeta})(d\zeta dr/P)$, etc.[21] If the source free Maxwell's equations,

$$\partial^*_{[a}F_{bc]}=0,$$

are integrated in V we obtain (the i factor is put for later convenience)

$$0=\int_V i3!\,\partial^*_{[a}F_{bc]}dx^a dx^b dx^c=\int_{\partial V} i^*F_{bc}dx^b dx^c. \tag{A1}$$

The surface term is the sum of five integrals. As an example we calculate the surface integral over regions (i) and (ii):

$$\int_{(\mathrm{i})+(\mathrm{ii})} i^*F_{ab}dx^a dx^b = \frac{C(x,\zeta,\bar{\zeta})}{P(\zeta,\bar{\zeta})}\, d\zeta' - \frac{C(x,\zeta,\bar{\zeta}+d\bar{\zeta})}{P(\zeta,\bar{\zeta}+d\bar{\zeta})}\, d\zeta = -\bar{\eth}C\,\frac{d\zeta d\bar{\zeta}}{P^2}, \tag{A2}$$

where

$$C \equiv \int i^*F_{ab}\hat{L}^{[a}M^{b]}dr = \int F^{ab}i^*\hat{L}_{[a}M_{b]}ds. \tag{A3}$$

Analogously

$$\int_{(\mathrm{iii})+(\mathrm{iv})} i^*F_{ab} = -\eth\bar{C}\,\frac{d\zeta d\bar{\zeta}}{P^2}. \tag{A4}$$

Using (II.19), (A3) takes the following form:

$$C = H + 2\int (\bar{f}\hat{\phi} + g\hat{\bar{\phi}})dr \tag{A5}$$

From (A1), (A2) and (A4) we get

$$\bar{\eth}C + \eth\bar{C} + (i^*F_{ab}M^a\bar{M}^b)_{,7^+} = 0. \tag{A6}$$

Using (A5), (III.7), (III.9) and (III.12), we can rewrite (A6) as

$$\eth\bar{\eth}F + \eth\bar{A} + \bar{\eth}A + \eth\int_{r_0}^{\infty}(f\hat{\bar{\phi}} + \bar{g}\hat{\phi})dr + \bar{\eth}\int_{r_0}^{\infty}(\bar{f}\hat{\phi} + g\hat{\bar{\phi}})dr = 0. \tag{A7}$$

Note that for SD [ASD] fields (A7) reduces to (III.17) [(III.18)].

We now show how to recover the Maxwell field F_{ab} from $F(x,\zeta,\bar{\zeta})$. Using eq. (C6) we can rewrite (III.3) as

$$\begin{aligned} H(x,\zeta,\bar{\zeta}) = \int_r^{\infty} F_{ab}(r')\hat{L}^a(r',\zeta,\bar{\zeta})[(r'-r)\hat{M}^b(r',\zeta,\bar{\zeta}) + (\Lambda(r',\zeta,\bar{\zeta}) \\ -\Lambda(r,\zeta,\bar{\zeta}))\hat{\bar{M}}^b(r',\zeta,\bar{\zeta})]dr'. \end{aligned} \tag{A8}$$

The integral is along $l_x(\zeta,\bar{\zeta})$, parametrized as in Appendix C, i.e., a point x' in the geodesic is given by $x' = x'(u,\omega,\bar{\omega},r';\zeta,\bar{\zeta})$. From (A8)

$$\hat{L}^a\nabla_a H = H_{,r} = -\int_r^{\infty}\hat{\phi}(r',\zeta,\bar{\zeta})dr' - \Lambda_{,r}\int_r^{\infty}\hat{\bar{\Phi}}(r',\zeta,\bar{\zeta})dr'.$$

Using this equation and its complex conjugate one can algebraically solve for $\int_r^{\infty}\hat{\phi}(r',\zeta,\bar{\zeta})dr'$. Taking one more radial derivative yields

$$\hat{\phi} = \left[\frac{(\eth F)_{,r} - \Lambda_{,r}(\bar{\eth}F)_{,r}}{1 - \Lambda_{,r}\bar{\Lambda}_{,r}}\right]_{,r} \equiv \hat{\Phi}[F] \quad \text{and c.c.,} \tag{A9}$$

i.e., eq. (III.14).

Putting together (A7) and (A9) we obtain the following integro-differential equation for F:

$$\eth\bar{\eth}F+\eth\left[\bar{A}+\int_{r_0}^{\infty}(f\hat{\bar{\Phi}}[F]+\bar{g}\hat{\Phi}[F])dr\right]+\bar{\eth}\left[A+\int_{r_0}^{\infty}(\bar{f}\hat{\Phi}[F]+g\hat{\bar{\phi}}[F])dr\right]=0. \tag{A10}$$

(A10) is our version of the source free Maxwell equations on curved spaces. It is not hard to see from (A10) that F is linear in the data $(A,\bar{A})$, from where it follows that we can first solve for the SD part of the field (by setting $\bar{A}=0$), and then add the solution for the ASD part ($A=0$), as in (III.23).

APPENDIX B: GREEN FUNCTIONS FOR THE ETH OPERATOR

The action of the eth operator on a spin weight s function f_s is given by[15]

$$g_{s+1}=\eth f_s\equiv P^{1-s}\frac{\partial}{\partial\zeta}(P^s f_s), \quad P\equiv 1+\zeta\bar{\zeta}. \tag{B1}$$

Similarly, the action of the eth-bar operator is

$$h_{s-1}=\bar{\eth}f_s\equiv P^{1+s}\frac{\partial}{\partial\bar{\zeta}}(P^{-s}f_s), \tag{B2}$$

therefore the commutator is

$$\bar{\eth}\eth-\eth\bar{\eth})f_s=2sf_s. \tag{B3}$$

From (B1), it follows that the only *regular* solution of the equation $\partial f_0=0$ is a constant. As the linear operator $\hat{\Phi}[\cdot]$ (III.14) acting on an S^2 constant gives zero, the solution F to the field equation (III.17) has the indeterminacy of an additive function of the space–time coordinates $f(x)$. As explained in remark 2 of Section III and in Ref. 14 this is a gauge term. Also from (B1), the general (regular) solution $Z^{(0)}$ of the "good cut equation" (IV.1) with $\sigma_B=0$,

$$\Lambda=\eth^2 Z^{(0)}=0, \tag{B4}$$

is found to be a linear combination of 1, ζ/P, $\bar{\zeta}/P$ and $\zeta\bar{\zeta}/P$.[16] The choice of parametrization of the solution space gives a coordinate system for this H space (section IV). In particular, by choosing

$$Z^{(0)}(x,\zeta,\bar{\zeta})=x^a l_a(\zeta,\bar{\zeta}), \tag{B5}$$

$$l_a(\zeta,\bar{\zeta})=\frac{1}{\sqrt{2}P}(-1-\zeta\bar{\zeta},\zeta+\bar{\zeta},i(\bar{\zeta}-\zeta),-1+\zeta\bar{\zeta}), \tag{B6}$$

it is easily shown that this H space is (complex) Minkowski space–time (equivalently, $Z^{(0)}$ gives the family of space–times conformal to Minkowski space), and that x are inertial coordinates. This is proven by first obtaining the metric tensor (IV.2) and then transforming to the x coordinates using (II.7) and (B5). The result is $g^{ab}=\text{diag}(-1,1,1,1,)$. Furthermore, the null tetrad (IV.8) is covariantly constant. In the x^a coordinates,

$$\mathscr{L}^a(x,\zeta,\bar{\zeta})=l^a(\zeta,\bar{\zeta})=g^{ab}l_a(\zeta,\bar{\zeta}), \quad \mathscr{M}^a(x,\zeta,\bar{\zeta})=\eth l^a(\zeta,\bar{\zeta})\equiv m^a(\zeta,\bar{\zeta}),$$

$$\bar{\mathscr{M}}^a(x,\zeta,\bar{\zeta})=\bar{\eth}l^a(\zeta,\bar{\zeta})\equiv\bar{m}^a(\zeta,\bar{\zeta}), \quad \mathscr{N}^a(x,\zeta,\bar{\zeta})=(1+\eth\bar{\eth})l^a(\zeta,\bar{\zeta})\equiv n^a(\zeta,\bar{\zeta}), \tag{B7}$$

where the usual notation for this tetrad in Minkowski space was introduced.

In a guide to construct Green functions for arbitrary combinations of ð and ð in terms of contractions of the tetrad (B7) is given. In particular, it is easy to verify from the information given there that

$$G_{0,-1'}(\zeta,\bar{\zeta};\zeta',\bar{\zeta}_,)\equiv\frac{1}{4\pi}\frac{l(\zeta,\bar{\zeta})\cdot\bar{m}(\zeta',\bar{\zeta}')}{l(\zeta,\bar{\zeta})\cdot l(\zeta',\bar{\zeta}')} \tag{B8}$$

is the Green function for the ð operator acting on spin-weight 0 functions. This function is used in the definition of $\mathscr{S}$ in the solution of the field equation (III.17)

Similarly, the Green function of $\eth^2$ acting on spin-weight zero functions is

$$G_{0,-2'}(\zeta,\bar{\zeta};\zeta',\bar{\zeta}_,)\equiv\frac{1}{4\pi}\frac{(l\cdot\bar{m}')^2}{l\cdot l'}. \tag{B9}$$

This is used in (IV.15) to obtain $Z^{(1)}$.

APPENDIX C: GEODESIC DEVIATION VECTOR

Inverting (II.7) with $(\zeta,\bar{\zeta})$ fixed we get

$$x^a=x^a(u,\omega,\bar{\omega},r;\zeta,\bar{\zeta}). \tag{C1}$$

If $x\in l_{x_0}(\zeta,\bar{\zeta})$, the null geodesic with end points at x_0 and the $(\zeta,\bar{\zeta})$ generator of $\mathscr{I}^+$, then

$$u(x;\zeta,\bar{\zeta})=u(x_0,\zeta,\bar{\zeta}),\quad \omega(x,\zeta,\bar{\zeta})=\omega(x_0;\zeta,\bar{\zeta}),\quad \bar{\omega}(x,\zeta,\bar{\zeta})=\bar{\omega}(x_0;\zeta,\bar{\zeta}), \tag{C2}$$

as can be easily seen from (II.9), (II.10), (II.11) and the fact that $Z^a(x,\zeta,\bar{\zeta})$ is tangent to the geodesic at x. Thus, for $x\in C_{x_0}$, the future null cone of x_0, it follows from (C1) and (C2) that

$$x^a=x^a(u=Z(x_0;\zeta,\bar{\zeta}),\quad \omega=\eth Z(x_0,\zeta,\bar{\zeta}),\quad \bar{\omega}=\bar{\eth}Z(x_0;\zeta,\bar{\zeta}),\quad r+\eth\bar{\eth}Z(x_0,\zeta,\bar{\zeta});\zeta,\bar{\zeta}. \tag{C3}$$

In (C3), r has been redefined in order to obtain a coordinate system $(r,\zeta,\bar{\zeta})$ for C_{x_0}. The geodesic deviation vector $M^a\equiv\eth x^a$ can be obtained from (C3) using (B1) and (B2),

$$\begin{aligned}P\frac{d}{d\zeta}x^a={}&\hat{N}^a\omega(x_0^b,\zeta,\bar{\zeta})-\hat{\bar{M}}^a[\Lambda(x_0^b,\zeta,\bar{\zeta})-\bar{\zeta}\omega]\\&-\hat{M}^a[r(x_0^b,\zeta,\bar{\zeta})+\bar{\zeta}\bar{\omega}]+\hat{L}^a\partial r(x_0,\zeta,\bar{\zeta})+P\frac{\partial x^a}{\partial\zeta}.\end{aligned} \tag{C4}$$

The last term in (C4) is the partial derivative of (C1) with respect to ζ. To calculate this derivative, we note from (II.7), (B1) and (B2) that, if x and $(\zeta,\bar{\zeta})$ moves in such a way that $u,\omega,\bar{\omega},r$ remain fixed, then

$$0=\delta u=\delta x^a\nabla_a u+\frac{\delta\zeta}{P}\,\omega,\quad 0=\delta\omega=\delta x^a\nabla_a\omega+\frac{\delta\zeta}{P}[\Lambda-\bar{\zeta}\omega],$$

$$0=\delta\bar{\omega}=\delta x^a\nabla_a\bar{\omega}+\frac{\delta\zeta}{P}[r+\bar{\zeta}\bar{\omega}],\quad 0=\delta r=\delta x^a\nabla_a r+\frac{\delta\zeta}{P}\,\bar{\eth}r.$$

Using (B3) and the fact that $\nabla_a u,\nabla_a r,\nabla_a\omega,\nabla_a\bar{\omega}$ is the dual of the basis $\hat{N}^a,\hat{L}^a,-\hat{\bar{M}}^a,-\hat{M}^a$ we get

$$P\frac{\partial}{\partial\zeta}x^a=P\frac{\delta x^a}{\delta\zeta}=-\hat{L}^a\eth r+\hat{\hat{M}}^a[\Lambda-\bar{\zeta}\omega]+\hat{M}^a[r+\bar{\zeta}\bar{\omega}]-\omega\hat{N}^a. \tag{C5}$$

From (C4) and (C5) we obtain the following formula for the geodesic deviation vector:

$$M^a=P\frac{dx^a}{d\zeta}=(\Lambda-\Lambda_0)\hat{\hat{M}}^a(r-r_0)\hat{M}^a-\hat{L}^a\partial(r-r_0). \tag{C6}$$

APPENDIX D:

In this Appendix we complete the steps that lead to Eq. (IV.10) and we prove that in the ASD solution (IV.14), $\bar{A}$ is the data at $\mathscr{I}^+$ as had previously been defined.

Since the r.h.s of (IV.7) does not depend on $(\zeta,\bar{\zeta})$ one can perform an integration on the sphere without changing the result. Thus,

$$A_a(x)=2\frac{1}{4\pi}\int_{S^2}[(\nabla^b\bar{\eth}F)Z_{[a}\eth Z_{b]}]dS+\nabla_a\frac{1}{4\pi}\int_{S^2}F(x,\zeta,\bar{\zeta})dS. \tag{D1}$$

The last integral on the right vanishes if the -c.c. of-the solution (III.24) of (III.17) is chosen (see remark 1 in Section III). If not, it merely reduces to an unimportant gauge term that we will omit. Using (IV6) we can rewrite the above expression as

$$A_a(x)=-2\frac{1}{4\pi}\int_{S^2\times S^2}dSdS'Z_{[a}(x,\zeta,\bar{\zeta})\eth Z_{b]}(x,\zeta,\bar{\zeta})\bar{\eth}G_{0,-1'}$$
$$\times(\zeta,\bar{\zeta};\zeta',\bar{\zeta}')\dot{A}(Z',\zeta',\bar{\zeta}')Z'^b, \tag{D2}$$

where $\dot{A}(u,\zeta,\bar{\zeta})\equiv(\partial/\partial u)A(u,\zeta,\bar{\zeta})$. (B3) and $\eth(Z_{[a}\eth Z_{b]})=0$ [which follows from (IV.1)] yield

$$\eth\bar{\eth}(Z_{[a}\bar{\eth}Z_{b]})=\bar{\eth}\eth(Z_{[a}\bar{\eth}Z_{b]})-2(Z_{[a}\eth Z_{b]})=-2Z_{[a}\eth Z_{b]},$$

which allows us to rewrite the dS integral in (D2) as

$$-2\int_{S^2}(\bar{\eth}G)Z_{[a}\bar{\eth}Z_{b]}dS=\int_{S^2}(\bar{\eth}G_{0,-1'})\eth\bar{\eth}(Z_{[a}\eth Z_{b]})dS$$
$$=\int_{S^2}(\eth G_{0,-1'})\bar{\eth}^2(Z_{[a}\eth Z_{b]})dS$$
$$=\bar{\eth}'^2(Z'_{[a}\eth Z'_{b]}), \tag{D3}$$

where we have used the fact that $G_{0,-1'}$ is the Green function of the $\eth$ operator. Inserting this result in (D2) gives

$$A_a(x)=\frac{1}{4\pi}\int_{S^2}dS'\dot{A}(Z',\zeta',\bar{\zeta}')Z'^b\bar{\eth}'^2(Z'_{[a}\eth' Z'_{b]}). \tag{D4}$$

Using (IV.2) the contraction can be calculated giving (IV.9).

We now show that the free data in (IV.13) is the asymptotic value of the connection at $\mathscr{I}^+$. From (III.18), the (restricted) free data is given by the following limit, taken along $l_{x_0}(\zeta,\bar{\zeta})$:

$$\bar{A}(x_0,\zeta,\bar{\zeta})=-\lim_{x\to\mathscr{I}^+}\bar{\eth}F(x,\zeta,\bar{\zeta}). \tag{D5}$$

If (III.1) is calculated using (IV.13) then

$$F(x,\zeta,\bar{\zeta})=\frac{1}{4\pi}\int_{S^2}dS'\int_{r(x,\zeta,\bar{\zeta})}^{\infty}dr'\dot{A}(Z'(x'),\zeta',\bar{\zeta}')Z^b(x')\eth Z_b'(x'),\tag{D6}$$

where again the natural coordinates of section II are used and

$$x'=x'(u,\omega,\bar{\omega},r';\zeta,\bar{\zeta}),\quad u=Z(x;\zeta,\bar{\zeta}),\quad \omega=\eth Z(x;\zeta,\bar{\zeta}),\quad \bar{\omega}=\bar{\eth}Z(x;\zeta,\bar{\zeta}).\tag{D7}$$

Let us change coordinates and view a point x' in $l_x(\zeta,\bar{\zeta})$ as the intersection of $l_x(\zeta,\bar{\zeta})$ with the past cone of $(u,\zeta',\bar{\zeta}')$ at $\mathscr{I}^+$ [i.e., $(u,\zeta',\bar{\zeta}')$ is the point at which $l_{x'}(\zeta',\bar{\zeta}')$ intersects $\mathscr{I}^+$]. Keeping $(\zeta',\bar{\zeta}')$ fixed, u parametrizes $l_x(\zeta,\bar{\zeta})$, thus

$$x'=x'(x,\zeta,\bar{\zeta};u,\zeta',\bar{\zeta}'),\quad \frac{du}{dr'}=Z^c(x')Z_c'(x')\tag{D8}$$

and

$$\eth F=\frac{1}{4\pi}\int_{S^2}dS'\int_{Z(x,\zeta',\bar{\zeta})}^{\infty}du\dot{A}(u,\zeta',\bar{\zeta}')\eth\left[\frac{Z^b(x')\eth' Z_b'(x')}{Z^c(x')Z_c'(x')}\right],\tag{D9}$$

where $\eth'$ is calculated without taking into account the $(\zeta',\bar{\zeta}')$ dependence of x' in (D7). Note that[22]

$$\bar{\eth}\left[\frac{Z^b(x')\bar{\partial}' Z_b'(x')}{Z^c(x')Z_c'(x')}\right]=\frac{\mathscr{L}^a(x')\ \bar{\mathscr{M}}^b(x')\mathscr{L}_{[a}'(x')\mathscr{M}_{b]}'(x')+\mathscr{O}(\bar{\Lambda},r)}{(\mathscr{L}^c(x')\mathscr{L}_c'(x'))^2}.\tag{D10}$$

Being $\mathscr{L}_{[a}'\mathscr{M}_{b]}'$ ASD and $\mathscr{L}_{[a}\bar{\mathscr{M}}_{b]}$ SD, their contraction is zero, so it is clear that in the $x\to\mathscr{I}^+$ limit (D10) will be zero in the $\zeta\neq\zeta'$ case, and a 0/0 indeterminacy if $\zeta=\zeta'$. Note that, if the H space is asymptotically flat, then $\lim_{\mathscr{I}^+}\bar{\Lambda}_{,r}(x,\zeta,\bar{\zeta})=0$, the limit taken along $l_x(\zeta,\bar{\zeta})$. To study the $\zeta\simeq\zeta'$ case, we express Z_c' as a second order Taylor expansion of Z_c around $(\zeta,\bar{\zeta})$. To do this, we obtain from (B.1) and (B.2) the relationship between $\eth Z_c,\bar{\eth}Z_c,\eth\bar{\eth}Z_c,\eth^2 Z_c,\bar{\eth}^2 Z_c$ and the first and second partial derivatives of Z_c, and then use (IV.1) and (IV.2). The result is

$$\begin{aligned}Z^cZ_c'&=\left[(\zeta-\zeta')(\bar{\zeta}-\bar{\zeta}')+\frac{1}{2}(\bar{\zeta}-\bar{\zeta}')^2\bar{\Lambda}_{,r}\right]/P^2+\mathscr{O}((\zeta-\zeta')^3)\\&=\left[(\zeta-\zeta')(\bar{\zeta}-\bar{\zeta}')+\frac{1}{2}(\bar{\zeta}-\bar{\zeta}')^2\bar{\Lambda}_{,r}\right]/[PP']+\mathscr{O}((\zeta-\zeta')^3)\\&=l\cdot l'+\frac{(\bar{\zeta}'-\bar{\zeta})^2}{2PP'}\bar{\Lambda}_{,r}+\mathscr{O}((\zeta-\zeta')^3),\end{aligned}\tag{D11}$$

where the notation of Appendix B has been used. In the $x\to\mathscr{I}^+$ limit, $\bar{\Lambda}_{,r}\to 0$. From (D11) and (B8),

$$-\lim_{x\to\mathscr{I}^+}\frac{1}{4\pi}\bar{\eth}\left[\frac{Z^b(x')\bar{\eth}' Z_b'(x')}{Z^c(x')Z_c'(x')}\right]=-\frac{1}{4\pi}\bar{\eth}\bar{\eth}'\ \ln(l\cdot l')=\delta_{-1,1}(\zeta,\bar{\zeta};\zeta',\bar{\zeta}').\tag{D12}$$

From (D9) and (D12),

$$-\lim_{x\to\mathscr{I}^+}\eth F(x,\zeta,\bar{\zeta})=-\int_{Z(x,\zeta,\bar{\zeta})}^{\infty}du\dot{\bar{A}}(u,\zeta,\bar{\zeta})=\bar{A}(Z(x_0,\zeta,\bar{\zeta}),\zeta,\bar{\zeta}), \tag{D13}$$

and so $\bar{A}$ in (IV.13) is the (restricted) free data, as we wanted to show.

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